1. Let $F$, $G$ and $H$ be formulas and let $S$ be a set of formulas. Which of the following statements are true? Justify your answer.

(a) If $F$ is unsatisfiable, then $\neg F$ is valid.
(b) If $F \rightarrow G$ is satisfiable and $F$ is satisfiable, then $G$ is satisfiable.
(c) $P_1 \rightarrow (P_2 \rightarrow (P_3 \rightarrow \ldots (P_n \rightarrow P_1) \ldots))$ is valid.
(d) $S \models F$ and $S \models \neg F$ cannot both hold.
(e) If $S \models F \lor G$, $S \cup \{F\} \models H$ and $S \cup \{G\} \models H$, then $S \models H$.

2. Let $F$ and $G$ be two formulas.

(a) Explain the difference between $F$ and $G$ being equisatisfiable and them being logically equivalent.
(b) Explain very briefly the difference between $F \leftrightarrow G$ and $F \equiv G$.

3. Give an equational proof of the following equivalence, justifying each step with reference to the Boolean algebra axioms and the Substitution Rule as appropriate.

$$\neg((\neg P \lor Q) \land P) \lor Q \equiv \text{true}$$

4. Suppose that $F$ and $G$ are formulas such that $F \models G$.

(a) Show that if $F$ and $G$ have no variable in common then either $F$ is unsatisfiable or $G$ is valid.
(b) Now let $F$ and $G$ be arbitrary formulas. Show that there is a formula $H$, mentioning only propositional variables common to $F$ and $G$, such that $F \models H$ and $H \models G$.

**Hint.** Recall that every truth table is realised by some propositional formula and consider what the truth table of $H$ ought to look like: under which assignments must $H$ be true and under which assignments must $H$ be false?

5. A **perfect matching** in an undirected graph $G = (V, E)$ is a subset of the edges $M \subseteq E$ such that every vertex $v \in V$ is an endpoint of exactly one edge in $M$. Given a finite graph $G$, describe how to obtain a propositional formula $\varphi_G$ such that $\varphi_G$ is satisfiable if and only if $G$ has a perfect matching. The formula $\varphi_G$ should be computable from $G$ in time polynomial in $|V|$.

6. Fix a non-empty set $U$. A **$U$-assignment** is a function from the collection of propositional variables to the power set of $U$, that is, $\mathcal{A}$ maps each propositional variable to a subset of $U$. Such an assignment is extended to all formulas as follows:
• \( A[\text{false}] = \emptyset \) and \( A[\text{true}] = U \);
• \( A[\neg F] = U \setminus A[F] \).

Say that a formula \( F \) is \textbf{U-valid} if \( A[F] = U \) for all \textit{U}-assignments \( A \).

(a) Show that if \( F \) is \textit{U}-valid then \( F \) is valid with respect to the standard semantics defined in the lecture notes.
\textbf{Hint:} Show that each standard assignment \( A \) can be “simulated” by a certain \textit{U}-assignment \( A' \).

(b) Show that if \( F \) is valid then \( F \) is \textit{U}-valid.
\textbf{Hint:} Fix an arbitrary \( u \in U \) and argue that \( u \in A[F] \).

7. (a) Write down a \textbf{DNF}-formula equivalent to \((P_1 \lor Q_1) \land (P_2 \lor Q_2) \land \cdots \land (P_n \lor Q_n)\).

(b) Prove that any \textbf{DNF}-formula equivalent to the above formula must have at least \( 2^n \) clauses.