Non-Newtonian Fluids and Finite Elements

Janice Giudice

Oxford University Computing Laboratory
Keble College

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Definitions

Fluid: a substance that continually deforms (flows) under an applied shear stress



Definitions

Viscosity: a measure of the resistance of a fluid to deform under shear stress. It is commonly perceived as "thickness", or resistance to pouring.

Definitions

Sir Isaac Newton

Newtonian Fluid: The concept was first deduced by Isaac Newton and is directly analogous to Hooke's law for a solid. A fluid that flows like water and whose stress at each point is proportional to its strain rate at that point.

Definitions

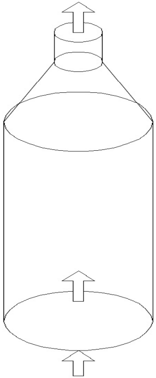
Non-Newtonian A fluid that is not Newtonian! That is, the stress and the strain are no longer linearly related.

Shear-Thinning Viscosity decreases with increasing applied stress

Shear-Thickening Viscosity increases with increasing applied stress.

The Physical Problem

Tightly confined flow of a non-Newtonian fluid.



Cauchy's Equation of Motion

Equates the rate of change of momentum of a selected fluid element and the sum of all forces acting on that fluid element.

$$\begin{aligned} \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right) &= \nabla\cdot\mathbf{T} + \rho\mathbf{f} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

- $\mathbf{u}$ velocity of fluid,
- $\mathbf{T}$ stress tensor (internal forces),
- $\mathbf{f}$ forcing function (external forces),
- ρ density.

A Viscous Fluid

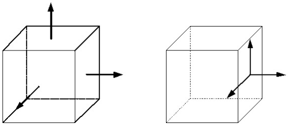


Figure: Two elemental volumes.

The stress tensor for a viscous fluid is

$$\mathbf{T} = -p\mathbf{I} + \tau = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

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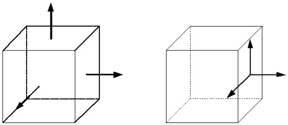

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- $\tau = \mu \mathbf{e}(\mathbf{u})$ Deviatoric stress tensor.
- μ is the apparent viscosity.
- $\mathbf{e}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right)$ is the rate of strain tensor.

Navier-Stokes Equations

The stress tensor is $\mathbf{T} = -\mathbf{p}\mathbf{I} + \frac{\mu}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right)$

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Navier-Stokes Equations

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\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} &= -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f} \\
\nabla\cdot\mathbf{u} &= 0
\end{aligned}
$$

- $\nu = \frac{\mu}{\rho}$ Kinematic viscosity.

Newtonian Fluids

The Newtonian property is encoded in the stress tensor

$$\mathbf{T} = -\mathbf{pI} + \tau$$

Newtonian Fluids

The Newtonian property is encoded in the stress tensor

$$\mathbf{T} = -\mathbf{pI} + \tau$$

with

$$\tau = \mu \mathbf{e}(\mathbf{u}).$$

The apparent viscosity, μ, is a constant and so the stress and rate of strain are proportional – linearly related. This models a Newtonian fluid.

Non-Newtonian Fluids

Non-linear stress tensor

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{k}\left(\mathbf{x}, |\mathbf{e}(\mathbf{u})|\right)\mathbf{e}(\mathbf{u})$$

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$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} &= -\frac{1}{\rho}\nabla p + 2\nu\nabla\cdot\left(k\left(\mathbf{x}, |\mathbf{e}(\mathbf{u})|\right)\mathbf{e}(\mathbf{u})\right) + \mathbf{f} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

Simplifying the System

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Assuming that the flow is tightly confined and slow, we can drop the non-linear term and neglect inertial effects. Also we assume steady flow.

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$$

Under such restrictions, the governing equations are

$$
\begin{aligned}
-\mu\nabla \cdot \left(k\left(\mathbf{x}, |\mathbf{e}(\mathbf{u})|\right)\mathbf{e}(\mathbf{u})\right) + \frac{1}{\rho}\nabla p &= \mathbf{f} \\
\nabla \cdot \mathbf{u} &= 0
\end{aligned}
$$

The Power-Law

Observation: For typical shear-thinning fluids, μ falls with increasing shear rate

Experimental Data: The log-log plot of shear stress to rate of shear is often found to be linear with a slope between zero and one:

$$\begin{aligned} \log k\left(|\mathbf{e}(\mathbf{u})|\right) &= (r-2)\log|\mathbf{e}(\mathbf{u})| + \log 2\mu \\ \Rightarrow k\left(|\mathbf{e}(\mathbf{u})|\right) &= 2\mu\,|\mathbf{e}(\mathbf{u})|^{r-2} \end{aligned}$$

Power-Law

Choice of Constitutive Relation

Possible choices are:

- Stokes flow: $k(|\mathbf{e}(\mathbf{u})|) \equiv 1$
- Power-law model: $k(|\mathbf{e}(\mathbf{u})|) = |\mathbf{e}(\mathbf{u})|^{r-2}, \quad 1 < r < \infty$
- Ladyzhenskaya model:
  $k(|\mathbf{e}(\mathbf{u})|) = \mu_0 + \mu_1 |\mathbf{e}(\mathbf{u})|^{r-2}, \quad \mu_0, \mu_1 > 0$

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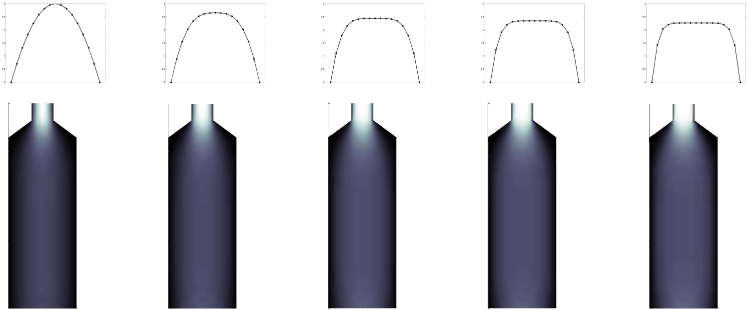
Of greater generality and of more practical significance is the

- Carreau model:
  $k(|\mathbf{e}(\mathbf{u})|) = \mu_\infty + (\mu_0 - \mu_\infty)(1 + \lambda |\mathbf{e}(\mathbf{u})|^2)^{(\theta-2)/2}$,
  $\mu_0 > \mu_\infty \geq 0$, $\lambda > 0$, $\theta \in (0, \infty)$

Some Preliminary Numerical Results



$r = 2$ $\quad$ $r = 1 + \frac{1}{2}$ $\quad$ $r = 1 + \frac{1}{4}$ $\quad$ $r = 1 + \frac{1}{8}$ $\quad$ $r = 1 + \frac{1}{16}$

What do we Want?

- Numerical simulation of incompressible, viscous extrusion flows for shear-thinning power-law fluids.

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Adaptive Finite Element Methods!!

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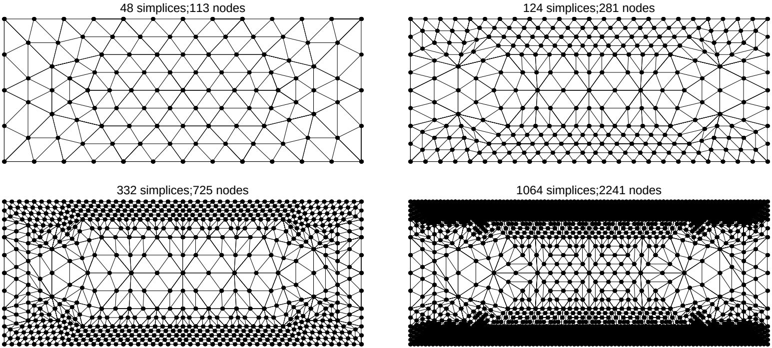
Adaptive Finite Element Methods!!

SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE

Adaptivity in a Channel



Weak Formulation

Find $\mathbf{u} \in V = [\mathrm{W}_0^{1,r}(\Omega)]^d$ and $p \in Q = \mathrm{L}_0^{r'}(\Omega) = \mathrm{L}^{r'}(\Omega)/\mathbf{R}$

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in V \\ b(q, \mathbf{u}) &= 0 \qquad \forall q \in Q. \end{aligned}$$

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Here

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega k(x,|e(\mathbf{u})|)e(\mathbf{u}) : e(\mathbf{v})\,\mathrm{d}\Omega$$

$$b(q,\mathbf{v}) = -\int_\Omega (\nabla \cdot \mathbf{v}) q\,\mathrm{d}\Omega.$$

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Inf-sup condition [Amrouche & Girault (1990)]: $\exists c_0 > 0$ s.t.

$$\inf_{q \in Q} \sup_{\mathbf{v} \in V} \frac{b(q,\mathbf{v})}{\|q\|_Q \|\mathbf{v}\|_V} \geq c_0 \qquad \forall q \in Q.$$

Finite Element Approximation

Let $V_h \subset V$ and $Q_h \subset Q$ be finite-dimensional spaces consisting of p.w. polynomial functions, defined on a triangulation $\mathcal{T}_h = \{T\}$ of the computational domain Ω.

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Find $\mathbf{u}_h \in V_h \quad$ and $\quad p_h \in Q_h$ such that

$$
\begin{aligned}
a(\mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in V_h \\
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Discrete inf-sup condition: there exists $c_0 > 0$, s.t.

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{b(q_h, \mathbf{v}_h)}{\|q_h\|_Q \|\mathbf{v}_h\|_V} \geq c_0 \qquad \forall q_h \in Q_h.$$

A Posteriori Error Analysis

How to quantify the size of the error

$$\mathbf{u} - \mathbf{u}_h, \qquad p - p_h$$

in terms of a computable bound?

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We define the residual functionals $\mathbf{S}_1 \in V'$ and $S_2 \in Q'$ by

$$\begin{aligned} \langle \mathbf{S}_1, \mathbf{w} \rangle &= (\mathbf{f}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) - b(p_h, \mathbf{w}) && \forall \mathbf{w} \in V \\ \langle S_2, q \rangle &= -b(q, \mathbf{u}_h) && \forall q \in Q. \end{aligned}$$

Our aim is to bound $\mathbf{u} - \mathbf{u}_h$ and $p - p_h$ in terms of norms of $\mathbf{S}_1$ and S_2.

Residual functionals

Here $\mathbf{S}_1 \in V'$ and $S_2 \in Q'$ are defined by:

$$\begin{aligned} \langle \mathbf{S}_1, \mathbf{w} \rangle &= (\mathbf{f}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) - b(p_h, \mathbf{w}) && \forall \mathbf{w} \in V \\ \langle S_2, q \rangle &= -b(q, \mathbf{u}_h) && \forall q \in Q. \end{aligned}$$

Note the error representation formula:

$$\begin{aligned} a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) + b(p - p_h, \mathbf{w}) &= \langle \mathbf{S}_1, \mathbf{w} \rangle \\ b(q, \mathbf{u} - \mathbf{u}_h) &= \langle S_2, q \rangle \end{aligned}$$

for all $\mathbf{w} \in V$ and all $q \in Q$.

Bounding the residual functionals

We have that

$$\begin{aligned}\langle \mathbf{S}_1, \mathbf{w} \rangle \;&=\; \sum_{T \in \mathcal{T}_h} \int_T \left(\mathbf{f} + \nabla \cdot \left(k(|e(\mathbf{u}_h)|)e(\mathbf{u}_h)\right) - \nabla p_h\right) \cdot (\mathbf{w} - I_h \mathbf{w}) \,\mathrm{d}T \\ &\quad - \sum_{T \in \mathcal{T}_h} \int_{\partial T} [k(|e(\mathbf{u}_h)|)e(\mathbf{u}_h)\mathbf{n}_T - p_h \mathbf{n}_T] \cdot (\mathbf{w} - I_h \mathbf{w}) \,\mathrm{d}s\end{aligned}$$

and

$$\langle S_2, q \rangle = \sum_{T \in \mathcal{T}_h} \int_T (\nabla \cdot \mathbf{u}_h)\, q \,\mathrm{d}T$$

Bounding the residual functionals

$$\|\mathbf{S}_1\|_{V'} \leq C \left[\left(\sum_{T\in\mathcal{T}_h} h_T^{r'} \|\mathbf{R}_1\|^{r'}_{\mathrm{L}^{r'}(T)} \right)^{1/r'} + \left(\sum_{T\in\mathcal{T}_h} \sum_{e\subset\partial T\cap\Omega} h_T \|\mathbf{R}_2\|^{r'}_{\mathrm{L}^{r'}(e)} \right)^{1/r'} \right].$$

and

$$\|\mathrm{S}_2\|_{Q'} \leq \left(\sum_{T\in\mathcal{T}_h} \|\mathrm{R}_3\|^{r}_{\mathrm{L}^{r}(T)} \right)^{1/r}$$

$$\mathbf{R}_1 = \mathbf{f} + \nabla\cdot(k(|e(\mathbf{u}_h)|)e(\mathbf{u}_h)) - \nabla p_h, \qquad \mathrm{R}_3 = \nabla\cdot\mathbf{u}_h$$

$$\mathbf{R}_2 = \tfrac{1}{2} [\![\, \sigma_h \mathbf{n} \,]\!], \qquad \sigma_h = -\left(k(|e(\mathbf{u}_h)|)e(\mathbf{u}_h) - p_h I\right).$$

A posteriori error bound

Theorem. Let $(\mathbf{u}, p) \in V \times Q$ denote the solution to b.v.p., and let $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ denote its finite element approximation. Then, there is a positive constant $C = C(K_1, K_2, c_0, c_0', r, \|\mathbf{f}\|_{V'})$ s.t.

$$\|\mathbf{u} - \mathbf{u}_h\|_V^{\mathrm{R}} + \|p - p_h\|_Q^{\hat{\mathrm{R}}} \leq C \left(\|\mathbf{S}_1\|_{V'}^{\mathrm{R}'} + \|\mathrm{S}_2\|_{Q'}^{\hat{\mathrm{R}}'} \right),$$

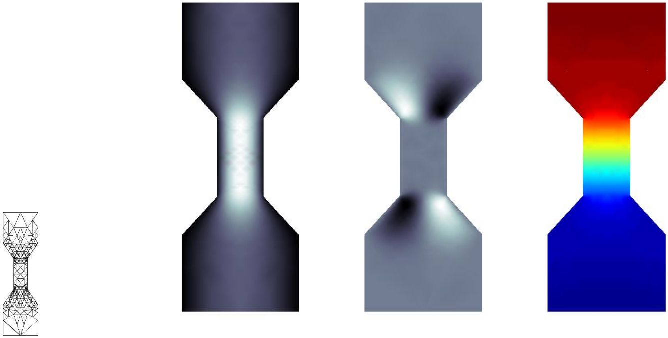
where

$$\mathrm{R} = \max\{r, 2\}, \; \hat{\mathrm{R}} = \max\{r', 2\}, \; 1/\mathrm{R} + 1/\mathrm{R}' = 1, \; 1/\hat{\mathrm{R}} + 1/\hat{\mathrm{R}}' = 1,$$

and $\mathbf{S}_1$ and S_2 residual functionals which are computably bounded.

[Barrett, Robson, Süli (2004)]

Some Numerical Results



$r = 2$ $\quad$ U $\quad$ V $\quad$ P

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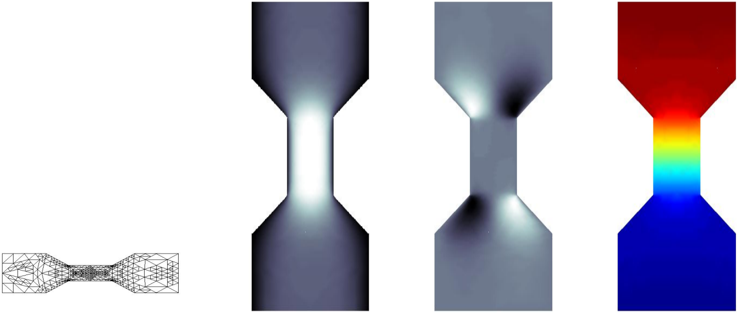
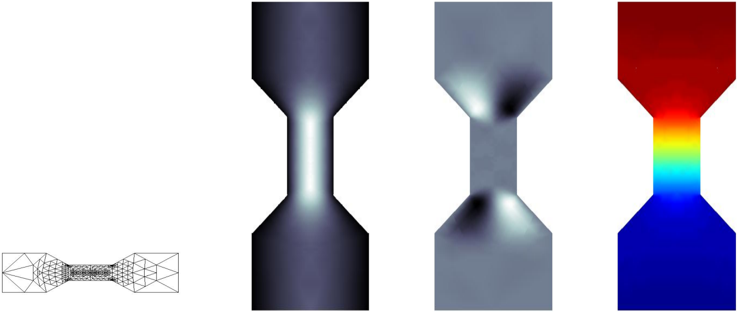


$r = 1.3$ | U | V | P

Some Numerical Results



$r = 3.3$ $\quad U \quad V \quad P$

Conclusions, ongoing and future research

- We developed the a posteriori error analysis of finite element approximations to a class on non-Newtonian flows.
- Ongoing research: implementation into an adaptive finite element method in 2D.
- Future work: application to multiple fluids, time-dependent problems in time-dependent geometries.