Note Title 14/04/2005

Algorithmic Problem-Solving

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Knockout Tournament

In a knockout tournament there are 1234 players.
How many games are played ?

Solution Introduce variables:

p no. of players

g no. of games .

Playing a game: $g,p := g+1,p$

Invariant :

$$(p+g)[p,g := p-1, g+1]$$

$=$ { substitution rule }

$$(p-1)+(g+1)$$

$=$ { arithmetic }

$$p+g \ .$$

Jealous Couples

3 couples (husband and wife) want to cross a river.

1 boat which can carry at most 2 people

A wife may not be at the same bank as a man, unless her husband is present.

Problem Decomposition

- Introduce notation to express state.
- Exploit symmetry between banks.

3C ||

S

C|C|C

2

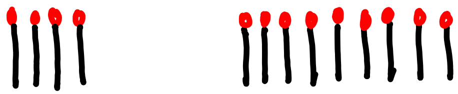
|| 3C

S and 2

are mirror images

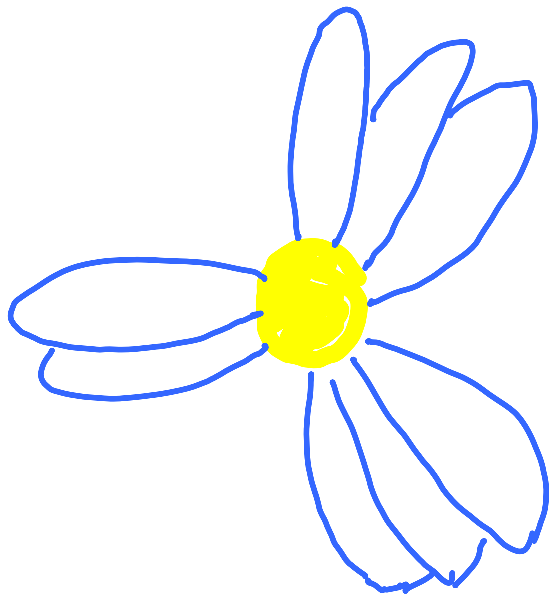
Combinatorial Games

Match Game



Remove at least one match from one pile.

Daisy Game



Remove one or two adjacent petals.

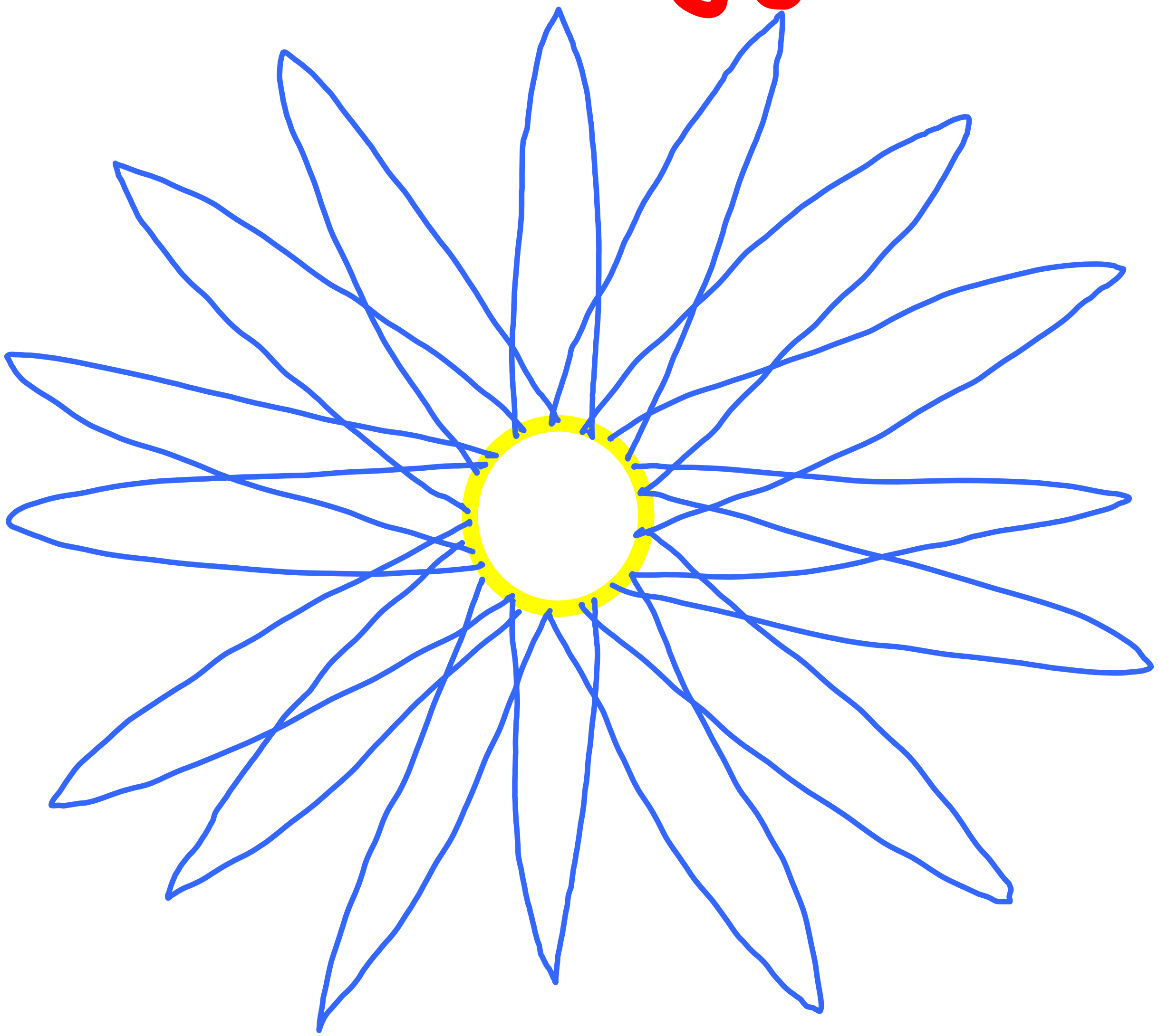
<u>Winner</u> : player who makes the last move.

How to win.

Identify a property of positions (the *strategy invariant*) such that

- all end positions satisfy the property
- *every* move from a position satisfying the property *falsifies* the property.
- for every position that does not satisfy the property *there is* a move that *truthifies* the property.

Strategy Invariant



Move

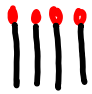
Remove one or two adjacent petals

Strategy

Maintain symmetry

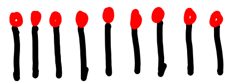
Game Sum

Left game



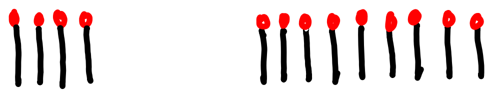
remove at least one match

Right game



remove at least one match

Left + Right



Choose a pile. Remove at least one match.

Deriving a Specification

- Evaluate game positions, so that:

Key:	loser
	winner

{ value.left = value.right }

move in left or right game

{ value.left ≠ value.right }

if value.left < value.right → move in right game

□ value.left > value.right → move in left game

fi

{ value.left = value.right }

Proof

by contradiction

goal-directed

calculational

Knights and Knaves

- Knights always tell the truth.
- Knaves always lie.

Rule: If a native makes a statement, then

$$A = S$$

where A is the truth value of "the native is a knight", and S is the truth value of the statement.

NB: Equality, not "if and only if".

Equality

- reflexive
- symmetric
- transitive
- substitutive

of booleans

- associative

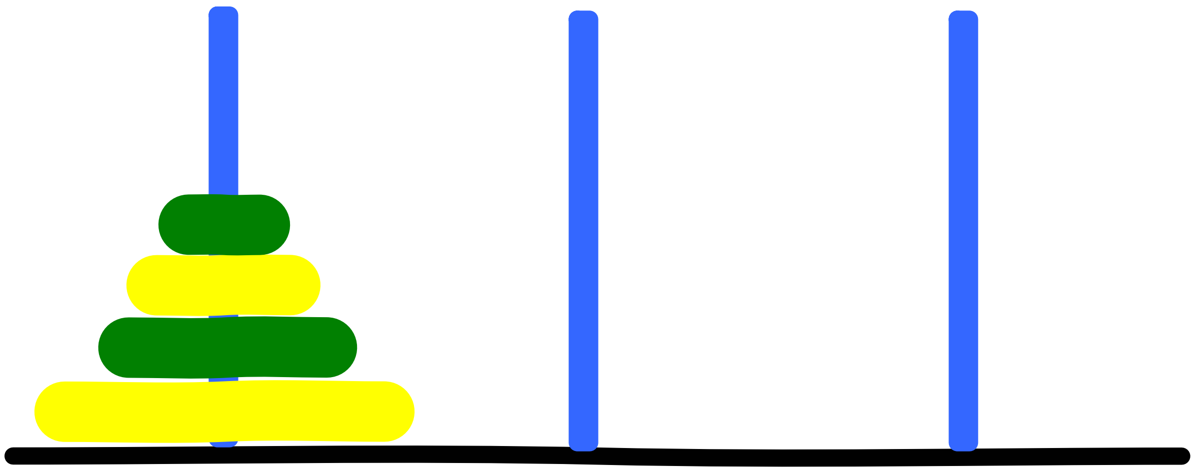
Rule: If a native is asked a question, their response is the truth value of $A = Q$ where A is the truth value of "the native is a knight", and Q is the answer to the question.

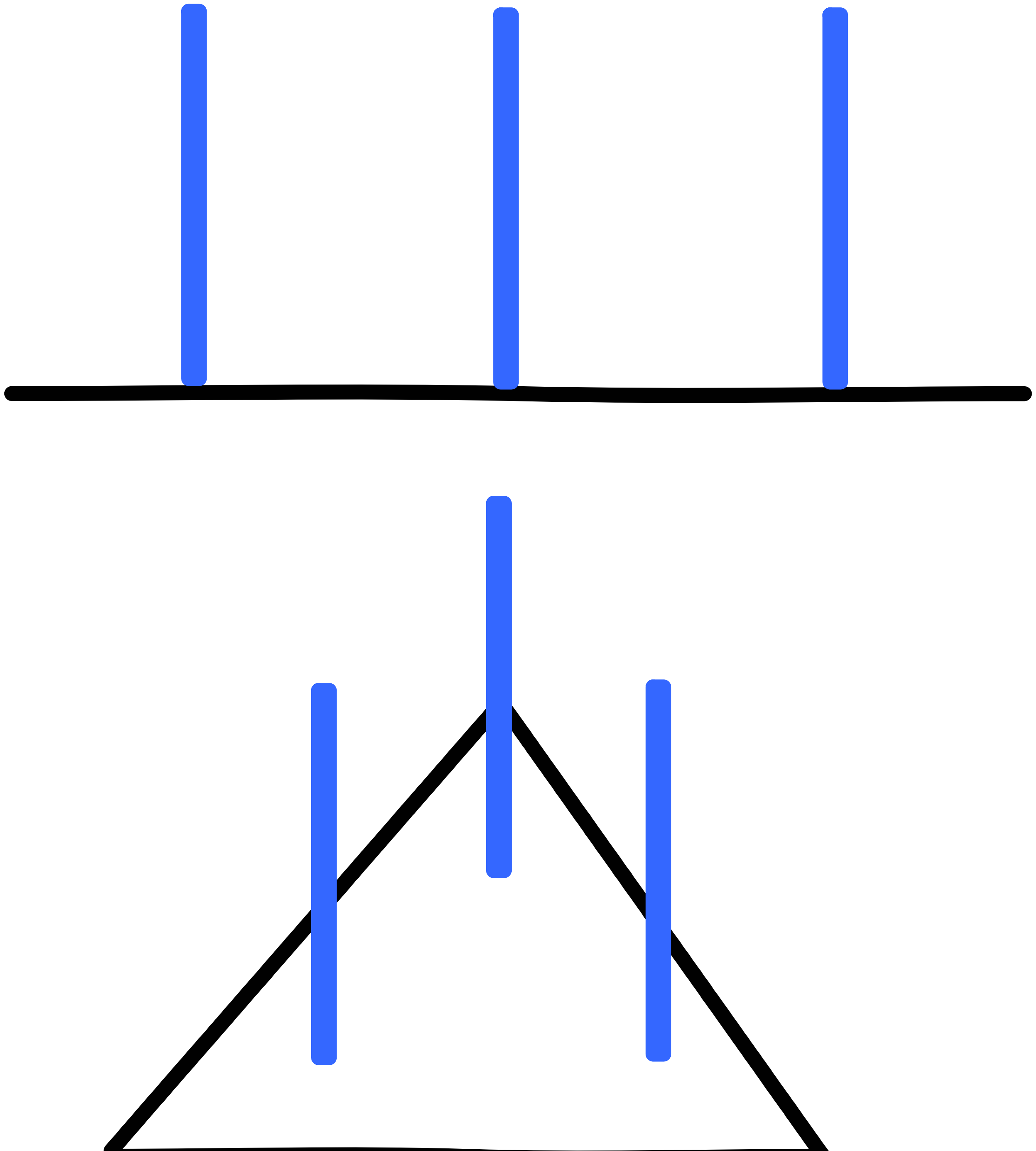
Getting the right answer:

Pose the question $A = Q$.

$$
\begin{array}{ll}
 & A = (A = Q) \\
= & \quad \{ \text{ equality is associative } \} \\
 & (A = A) = Q \\
= & \quad \{ \text{ } (A = A) = \mathit{true}\text{, substitution } \} \\
 & \mathit{true} = Q \\
= & \quad \{ \text{ } \mathit{true} \text{ is unit of equality } \} \\
 & Q \, .
\end{array}
$$

Towers of Hanoi





n	no. of disks
d	direction of movement
H	sequence of moves

Inductive solution

$$H.0.d \quad = \quad skip$$

$$H.(n+1).d \quad = \quad H.n.(\neg d) \;;\; (n+1, d) \;;\; H.n.(\neg d)$$

(Why not

$$H.0.d \quad = \quad skip$$

$$H.(n+1).d \quad = \quad (1, \neg d) \;;\; H.n.d \;;\; (1, \neg d) \quad ?)$$

$$H.(n+1).d \;=\; H.n.(\neg d)\;;\;(n+1,d)\;;\;H.n.(\neg d)$$

Invariant:

$$
\begin{aligned}
& even.(n+1) = d \\
= \quad & \{\ even \text{ distributes through addition } \} \\
& (even.n = even.1) = d \\
= \quad & \{\ even.1 = false\ \} \\
& (even.n = false) = d \\
= \quad & \{\ \text{associativity of boolean equality } \} \\
& even.n = (false = d) \\
= \quad & \{\ (false = d) = \neg d\ \} \\
& even.n = \neg d \quad .
\end{aligned}
$$

WHY?

- correct-by-construction programming discipline is largely unknown.
- computing has changed the art of effective reasoning.

HOW?

- simple games and puzzles
- emphasise method (construction, not verification)

CONCLUSION

- Easily understood, non-mathematical problems encourages interaction.
- Emphasis on construction minimises demotivation.
- Benefits of mathematical calculation introduced by back door.