On the Determinacy of Concurrent Games on Event Structures with Infinite Winning Sets

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Abstract

We consider nondeterministic concurrent games played on event structures and study their determinacy problem—the existence of winning strategies. It is known that when the winning conditions of the games are characterized by a collection of finite winning sets/plays, a restriction (called race-freedom) on the boards where the games are played guarantees determinacy. However the games may no longer be determined when the winning sets are infinite. This paper provides a study of concurrent games and nondeterministic winning strategies by analysing conditions that ensure determinacy when infinitely many events are played, that is, when the winning sets are infinite. The main result is a determinacy theorem for a class of games with a bounded concurrency property and infinite winning sets shown to be finitely decidable.

Keywords: Mathematical logic, Concurrency theory, Games

1. Introduction

One of the most fundamental questions when studying games with winning conditions is determinacy, that is, the existence of winning strategies (see [8, 12, 11, 15] for some examples particularly relevant in informatics). It is well known that even the simplest setting where two players, which we call Eve ($\exists$) and Adam ($\forall$) hereafter, play against each other (possibly concurrently) without taking turns leads to games where determinacy fails.

Here we study conditions for which nondeterministic concurrent games, i.e. games where the players are allowed to use nondeterministic concurrent strategies, are determined. We consider concurrent games played on event structures, a model of concurrent computation where the causal dependencies between the events of a system are modelled as partially ordered structures. This paper, in particular, studies concurrent games on event structures where the winning conditions allow winning sets/plays that are infinite.

Concurrent games [13] form a model of interactive behaviour where nondeterministic strategies are formalized as certain maps of event structures. This games model, as first introduced in [13], did not allow for the definition of winning conditions. In order to overcome this limitation, in [7], the initial concurrent games model was extended with winning conditions and a determinacy result was given for games that satisfy two properties: firstly, a structural condition, *race-freedom*, which prevents a player from interfering with the moves available to the other; secondly, a restriction to winning conditions where only *finite* winning sets are allowed. This paper extends the work on concurrent games, mainly, by providing a new determinacy result.

The paper starts with a very general study of properties of concurrent games and strategies; in particular, operations on concurrent games which preserve the existence of nondeterministic winning strategies and a study relating strategies seen as maps of events structures to strategies seen as certain kinds of closure operators on the boards where the games are played.

Then, the main result of the paper is presented, namely a determinacy theorem for concurrent games where the winning conditions allow infinite winning sets. The theorem holds on a class of games which satisfies two properties: firstly, a structural property called *bounded concurrency* which ensures that no event (or move) of either player is concurrent with infinitely many events of the other player; and secondly, a restriction to concurrent games where the winning conditions determine infinite winning sets/plays (or winning configurations in the terminology of event structures) which can be regarded as the elements of a class of *concurrent open sets*. Such sets, which due to bounded concurrency can be shown to be finitely decidable, are a characteristic feature of the winning conditions of our concurrent games.

Structure of the paper: Sections 2 and 3 introduce event structures and concurrent games on event structures with winning conditions.[1] Sections 4 to 6 contain the main contributions of the paper, as described above. Finally, Section 7 presents related work, conclusions, and ideas for future work.

2. Preliminaries

Here we introduce event structures and concurrent games on them. The material in this preliminary section can be found in [7, 13].

2.1. Event structures

An *event structure* $(E, \leq, \mathrm{Con})$ comprises a set E of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$\begin{aligned}
&\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \mathrm{Con} \text{ for all } e \in E,\\
&Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}\\
&X \in \mathrm{Con} \;\&\; e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.
\end{aligned}$$

[1]Some of the notations used in this paper differ from the ones used in [7, 13], where concurrent games are also studied. Instead, we sometimes use the terminology used in [9]; the differences are insubstantial with respect to the work being presented.

The *configurations*, $\mathcal{C}(E)$, of E consist of those subsets $x \subseteq E$ that are

$$\begin{aligned} &\textit{Consistent: } \forall X \subseteq x.\ X \text{ is finite} \implies X \in \mathrm{Con}, \text{and} \\ &\textit{Down-closed: } \forall e, e'.\ e' \leq e \in x \implies e' \in x. \end{aligned}$$

Write $\mathcal{C}^{\omega}(E)$ for the set of *finite* configurations of E and $\mathcal{C}^{\infty}(E)$ for the set of *infinite* configurations of E. Two events which are both consistent and incomparable with respect to $\leq$ are regarded as *concurrent*.

Notation 1. We use $\rightarrowtriangle$ for the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between; also, we write $e \,co\, e'$ when e and e' are concurrent. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note that if $X \in \mathrm{Con}$ then $[X]$ is a configuration; in particular, for singletons we write $[e]$ instead of $[\{e\}]$. We also write $[e)$ for the configuration $[e] \setminus \{e\}$ which contains the finite set of events that e depends on. Moreover, we write $x \overset{e}{-\!\!\subset} x'$ if $x \cup \{e\} = x'$ or simply $x -\!\!\subset x'$ if e is irrelevant. Finally we often refer to an event structure $(E, \leq, \mathrm{Con})$ by referring to its set of events E and may use subscripts, e.g. as in $\leq_E$ or Con_E, when necessary. □

Maps of event structures. Let E and E' be event structures. A *(partial) map* of event structures $f : E \to E'$ is a partial function on events $f : E \to E'$ such that for all $x \in \mathcal{C}(E)$ its direct image fx is in $\mathcal{C}(E')$ and

$$\text{if } e_1, e_2 \in x \text{ and } f(e_1) = f(e_2) \text{ (with both defined), then } e_1 = e_2.$$

Partial maps of event structures compose as partial functions, with identity maps given by identity functions. For any event e a map $f : E \to E'$ must send the configuration $[e]$ to the configuration $f[e]$. A map is *total* if f is total. A total map f is *locally injective* in the sense that with respect to any configuration x of the domain the restriction of f to a function from x is injective; the restriction of f to a function from x to fx is thus bijective.

Event structures are rich in useful constructions. For instance, the category of event structures has products and pullbacks (both forms of synchronised composition) and coproducts (nondeterministic sums). Also, event structures support a restriction operation and a simple form of hiding, called projection, associated with a factorization system. Both such operations are defined next. Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ if $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ if $X \in \mathrm{Con}$ & $X \subseteq V$. Consider a partial map of event structures $f : E \to E'$. Let $V =_{\mathrm{def}} \{e \in E \mid f(e) \text{ is defined}\}$. Then f clearly factors into the composition $E \xrightarrow{f_0} E{\downarrow}V \xrightarrow{f_1} E'$ of f_0, a partial map of event structures taking $e \in E$ to itself if $e \in V$ and undefined otherwise, and f_1, a total map of event structures acting like f on V. On the other hand, the *restriction* of $(E, \leq, \mathrm{Con})$ to a set of events $R \subseteq E$, written $E \upharpoonright R$, is the event structure with events the set $\{e \in E \mid [e] \subseteq R\}$ and causal dependency and consistency relations induced by E. More about event structures operations can be found in [7, 13].

2.2. Concurrent games and strategies

A game (also known as board or arena) and a strategy in a game are both represented using event structures with polarity, which are defined next.

An *event structure with polarity* comprises (E, pol) where E is an event structure and $pol : E \to \{+, -\}$ is a function ascribing a polarity $+$ (Eve) or $-$ (Adam) to events in E; the events correspond to moves. Maps of event structures with polarity are maps of event structures which preserve polarity. Hereafter, by event structures we mean event structures with polarity.

Event structures support two key operations: the *dual* $E^\perp$ of an event structure E is a copy of E where the polarities are reversed. Write $\overline{e} \in E^\perp$ for the event complementary to $e \in E$ and vice-versa. The *simple parallel composition* $E \| E'$ forms the disjoint juxtaposition of E, E', two event structures; a finite subset of events is consistent if its intersection with each component is consistent. The empty event structure $\varnothing$ is the unit with respect to $\|$.

2.2.1. Pre-strategies, strategies, and composition

Let A be an event structure representing a *game*; its events are possible moves of Eve and Adam and its causal dependency and consistency relations are the constraints imposed by the game. A *pre-strategy* in A is a total map $\sigma : S \to A$ from an event structure S. All moves in S are mapped to moves allowed in A and obey the constraints of the game (e.g. causality, choices, concurrency, etc.). A *strategy* is a pre-strategy that satisfies:

Receptivity. A pre-strategy σ is *receptive* if and only if
$\sigma x \overset{a}{\longrightarrow\!\!\subset}\ \&\ pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{\longrightarrow\!\!\subset}\ \&\ \sigma(s) = a\,.$

Innocence. A pre-strategy σ is *innocent* if and only if
$s \rightarrowtriangle s'\ \&\ (pol(s) = + \vee pol(s') = -) \Longrightarrow \sigma(s) \rightarrowtriangle \sigma(s').$

From Eve's viewpoint, receptivity ensures an openness to all possible moves of Adam. Innocence restricts the behaviour of Eve; she may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ in S beyond those imposed by the game A. Thus, innocence gives Eve the freedom to await Adam's moves before making her moves, but prevents her from having any influence on Adam's moves beyond those already in A.

Example 2. Consider the event structure A comprising three consistent and concurrent events $\oplus_a$, $\oplus_b$, and $\ominus_c$ with the obvious polarities. The maps of event structures σ_1, σ_2, σ_3, and σ_4 (depicted below) fail to be strategies.

$$\begin{tikzcd}
S_1 \arrow[d, "\sigma_1"'] & & & s_c \arrow[d, dotted] \arrow[r, squiggly, no head] & s'_c \arrow[dl, dotted] \\
A & \oplus_a & \oplus_b & \ominus_c &
\end{tikzcd}
\qquad
\begin{tikzcd}
S_2 \arrow[d, "\sigma_2"'] & & & \\
A & \oplus_a & \oplus_b & \ominus_c
\end{tikzcd}$$

$$\begin{tikzcd}
S_3 \arrow[d, "\sigma_3"'] & s_a \arrow[d, dotted] & s_b \arrow[l, -{Triangle[open]}] \arrow[d, dotted] & s_c \arrow[d, dotted] \\
A & \oplus_a & \oplus_b & \ominus_c
\end{tikzcd}
\qquad
\begin{tikzcd}
S_4 \arrow[d, "\sigma_4"'] & & s_b \arrow[r, -{Triangle[open]}] \arrow[d, dotted] & s_c \arrow[d, dotted] \\
A & \oplus_a & \oplus_b & \ominus_c
\end{tikzcd}$$

The $\rightarrowtriangle$ arrow denotes immediate causal dependency and the wiggly line between s_c and s'_c denotes binary conflict. The dotted arrows between the events in each S_i and the events in A are the maps under σ_i, e.g. $\sigma_3(s_b) = \oplus_b$.

The reasons for which each σ_i fails to be a strategy are as follows: on the one hand, σ_1 and σ_2 are not receptive; whereas σ_1 fails to have a unique event in S which maps to $\ominus_c$, the map σ_2 does not have an negative event $s_c \in S$ such that $\sigma(s_c) = \ominus_c$. On the other hand, σ_3 and σ_4 are not innocent; the map σ_3 introduces a forbidden $\oplus_b \rightarrowtriangle \oplus_a$ causal dependency and the map σ_4 introduces a forbidden $\oplus_b \rightarrowtriangle \ominus_c$ causal dependency. Note that if permitted the former would allow Eve to wait for own moves, and the latter would allow Eve to make Adam wait for her moves—two situations that are not permitted in S unless such causal dependencies were already in A.

On thing Eve can actually do is, for instance, to wait until Adam plays his event—in case he decided to do so. The strategy σ_5 is one such map.

$$\begin{tikzcd}
S_5 \arrow[d, "\sigma_5"'] & & s_b \arrow[d, dotted] & s_c \arrow[l, -Triangle] \arrow[d, dotted] \\
A & \oplus_a & \oplus_b & \ominus_c
\end{tikzcd}$$

Then, σ_5 is a strategy ($\sigma_5 : S_5 \to A$ is receptive and innocent) which does not play $\oplus_a$ and waits until Adam plays $\oplus_c$; if he does so, then Eve plays $\oplus_b$. □

An important concept in this games framework is that of nondeterministic strategies $\sigma : S \to A$, whose definition depends on whether S is deterministic or not. Say an event structure E is *deterministic* if and only if

$$\forall X \subseteq_{\mathrm{fin}} E.\ \mathit{Neg}[X] \in \mathrm{Con}_E \implies X \in \mathrm{Con}_E\,,$$

where $\mathit{Neg}[X] =_{\mathrm{def}} \{e' \in E \mid \mathit{pol}(e') = - \ \&\ \exists e \in E.\ e' \leq e\}$. Naturally, E is *non-deterministic* if it fails to be deterministic. Thus, a strategy $\sigma : S \to A$ is deterministic (resp. nondeterministic) if the event structure S is deterministic (resp. nondeterministic). Roughly, a strategy (for Eve) is deterministic if it does not contain choices—necessarily between positive events—in S.

Example 3. Let A be the event structure in Example 2 but where the events $\oplus_a$ and $\oplus_b$ are in conflict, i.e. represent a choice for Eve to make. The two maps of event structures depicted below illustrate two strategies for Eve: a deterministic one (on the left) and a nondeterministic one (on the right).

$$\begin{tikzcd}
S \arrow[d, "\sigma"'] & s_a \arrow[d, dotted] & & s_c \arrow[d, dotted] \\
A & \oplus_a \arrow[r, no head, squiggly] & \oplus_b & \ominus_c
\end{tikzcd}
\qquad
\begin{tikzcd}
S' \arrow[d, "\sigma'"'] & s_a \arrow[d, dotted] \arrow[r, no head, squiggly] & s_b \arrow[d, dotted] & s_c \arrow[d, dotted] \\
A & \oplus_a \arrow[r, no head, squiggly] & \oplus_b & \ominus_c
\end{tikzcd}$$

The strategy on the left plays $\oplus_a$ only. On the other hand, the strategy on the right nondeterministically chooses between playing $\oplus_a$ and playing $\oplus_b$. In both cases, the strategies play the positive events regardless of the behaviour of Adam as no causal dependencies with respect to $\ominus_c$ were introduced. □

Composition via pullbacks. We can define strategy composition via pullbacks. Given two (pre-)strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ and ignoring polarities, we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A\|B$ and $\tau : T \to B\|C$. Viewed this way we can form their pullback as shown below

$$\begin{tikzcd}
 & P \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
S\|C \arrow[dr, "\sigma\|\mathrm{id}_C"'] & & A\|T \arrow[dl, "\mathrm{id}_A\|\tau"] \\
 & A\|B\|C\,. &
\end{tikzcd}$$

There is an obvious partial map of event structures $A\|B\|C \to A\|C$ which acts as identity on A and C and is undefined on B. The partial map from P to $A\|C$ given by the diagram above (either way round the pullback square) factors as the composition of the partial map $P \to P\downarrow V$, where V is the set of events of P at which the map $P \to A\|C$ is defined, and a total map $P\downarrow V \to A\|C$. The resulting total map gives us the desired composition of (pre-)strategies $\tau\odot\sigma : P\downarrow V \to A^\perp\|C$ once we reinstate polarities.

Very often, however, one is interested in the results of the interaction between σ and τ in the game B, for instance, when playing, morally, in the "same" board, i.e. when $A = \varnothing = C$. In such a case we would have a strategy $\sigma : S \to B$ for Eve and a (counter-)strategy $\tau : T \to B^\perp$ for Adam.[2]

In order to define the results of playing two strategies against each other, suppose that $\sigma : S \to A$ is a strategy *in* a game A. A *counter-strategy* is a strategy of Adam, so a strategy $\tau : T \to A^\perp$ in the dual game. Then (ignoring polarities) we have total maps $\sigma : S \to A$ and $\tau : T \to A$ whose pullback,

$$\begin{tikzcd}
 & P \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
S \arrow[dr, "\sigma"'] & & T \arrow[dl, "\tau"] \\
 & A\,, &
\end{tikzcd}$$

produces P, the event structure resulting from the interaction $\tau\odot\sigma$ of σ and τ. Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}(P)$. A configuration z images to a configuration $\sigma\Pi_1 z = \tau\Pi_2 z$ in $\mathcal{C}(A)$. Define the set of *results* of playing σ against τ to be

$$\langle\sigma,\tau\rangle =_{\mathrm{def}} \{\sigma\Pi_1 z \mid z \text{ is maximal in } \mathcal{C}(P)\}\,.$$

[2]We say 'morally, in the "same" board' since in fact when playing a game, say in a board B, formally it is Eve who plays in B whereas Adam plays in its dual, $B^\perp$.

The results of a game are the *complete plays* of the game. Plays are, therefore, configurations of A; a subconfiguration of a result is a *partial play.*

Example 4 (Composition). Let $\sigma_i : S_i \to A$ be a strategy in $A = \oplus \; co \; \ominus$

$$\begin{tikzcd}
S_0 \arrow[d, "\sigma_0"'] & \ominus \arrow[d, dotted] & & S_1 \arrow[d, "\sigma_1"'] & \ominus \arrow[d, dotted] & \oplus \arrow[d, dotted] & S_2 \arrow[d, "\sigma_2"'] & \ominus \arrow[r] \arrow[d, dotted] & \oplus \arrow[d, dotted] \\
A & \ominus & \oplus & A & \ominus & \oplus & A & \ominus & \oplus
\end{tikzcd}$$

Similarly, there are three counter-strategies τ_j for Adam—the duals of Eve. The results of playing each σ_i against each τ_j are as follows:

$$\langle \sigma_i, \tau_j \rangle = \begin{cases} \{\varnothing\} & \text{if } i \in \{0,2\} \;\&\; j \in \{0,2\}, \\ \{\{\oplus\}\} & \text{if } i = 1 \;\&\; j = 0, \\ \{\{\ominus\}\} & \text{if } i = 0 \;\&\; j = 1, \\ \{\{\oplus, \ominus\}\} & \text{otherwise.} \end{cases}$$

Note that Eve/Adam can try to force *some* plays to happen sequentially by adding causal dependencies, e.g. when using σ_2/τ_2. This situation may lead to a deadlock since, when using σ_2/τ_2, Eve/Adam would stay waiting for Adam/Eve to play first—something that may never happen. □

3. Concurrent games with winning conditions

We now introduce the concurrent games model that will be used in the reminder of this document. This section is based on [7].

A *game with winning conditions* comprises (A, W), where A is an event structure and $W \subseteq \mathcal{C}(A)$ consists of the *winning configurations* or *winning sets* for Eve. Let $L = \mathcal{C}(A) \setminus W$ be the losing conditions for Eve; then, W and L are, respectively, the losing and winning conditions for Adam.

Notation 5. Let x and x' be configurations. Write $x \subseteq^- x'$ to mean $x \subseteq x'$ and $pol(x' \setminus x) \subseteq \{-\}$, i.e. x' extends x solely by events of negative polarity, which we call $\ominus$-events. Similarly we call $\oplus$-events such moves with the dual polarity property. We often write $\exists$-strategy to mean a strategy for Eve and $\forall$-strategy to mean one for Adam. Moreover, a configuration x is $\oplus$-*maximal* if $x \overset{s}{\longrightarrow\!\!\subset}$ implies $pol(s) = -$. Moreover, a configuration x is $\oplus$-*maximal within a configuration* z, with $x \subseteq z$, if $x \overset{s}{\longrightarrow\!\!\subset}$ and $pol(s) = +$ implies $(x \cup \{s\}) \notin z$. Finally, define $\ominus$-*maximality* by changing the polarity of the event s. □

The concept of strategy is further refined. A $\exists$-strategy $\sigma : S \to A$ in a game (A, W) is winning for Eve if $\sigma x \in W$, for all $\oplus$-maximal configurations of S. A winning strategy for Adam, i.e. a $\forall$-strategy, is defined dually.

One can also define a winning strategy as a strategy that ensures winning the game regardless of the counter-strategy it is played against. Let $\sigma : S \to A$ be a $\exists$-strategy and $\tau : T \to A^\perp$ a $\forall$-strategy it is played against. It can be shown [7], that a $\exists$-strategy σ of Eve is a winning if and only if all the results of the interaction $\langle \sigma, \tau \rangle$ lie within W, for any $\forall$-strategy $\tau : T \to A^{\perp}$ of Adam. We say that a $\exists$-strategy $\sigma : S \to A$ *dominates* a $\forall$-strategy $\tau : T \to A^{\perp}$ if and only if $\langle \sigma, \tau \rangle \subseteq W$. Similarly, τ dominates σ if and only if $\langle \sigma, \tau \rangle \subseteq L$. Write $\sigma \gg \tau$ whenever σ dominates τ. Note that given two strategies σ and τ it may be the case that none of them dominates the other.

Example 6. Consider the event structure A with two inconsistent events $\oplus$ and $\ominus$ with the obvious polarities and winning conditions $W = \{\{\oplus\}\}$. In the game (A, W) no strategy for either player dominates all other counter-strategies of the other player. In particular, let σ be the unique map of event structures that contains $\oplus$ and τ a particular counter-strategy for Adam:

$$\text{Eve:} \quad \begin{tikzcd} S \arrow[d, "\sigma"'] \\ A \end{tikzcd} \quad \begin{tikzcd} \oplus \arrow[r, no head, squiggly] \arrow[d, dotted] & \ominus \arrow[d, dotted] \\ \oplus \arrow[r, no head, squiggly] & \ominus \end{tikzcd} \qquad \text{Adam:} \quad \begin{tikzcd} T \arrow[d, "\tau"'] \\ A^{\perp} \end{tikzcd} \quad \begin{tikzcd} \ominus \arrow[r, no head, squiggly] \arrow[d, dotted] & \oplus \arrow[d, dotted] \\ \ominus \arrow[r, no head, squiggly] & \oplus \end{tikzcd}$$

Then, neither $\langle \sigma, \tau \rangle \subseteq W$ nor $\langle \sigma, \tau \rangle \subseteq L$ since $\{\{\oplus\}, \{\ominus\}\} \subseteq \langle \sigma, \tau \rangle$. □

Then, in a game (A, W) a strategy σ is winning if σ dominates all counter-strategies it can be played against. This statement provides the usual intuition of what a winning strategy is and leads to the definition of *determinacy* of a class of games. A game is determined if either Eve or Adam has a winning strategy, i.e. if there is a strategy that dominates all counter-strategies.

Before we study the determinacy problem for concurrent games with winning conditions it is useful to state some facts about concurrent games.

4. Operational properties

Often, it is useful to think "operationally" of a strategy $\sigma : S \to A$ as an operator that associates to a configuration of A another configuration of A that can, potentially, be played next. Since, in general, a concurrent strategy can be nondeterministic then that operator may not be a function between the configurations of A, but rather a relation between them. Nevertheless, when one is restricted to *deterministic* concurrent strategies, as it was first shown in [13] (and presented summarily here) a concurrent strategy $\sigma : S \to A$ corresponds to a closure operator on the configurations of A.

Notation 7. Recall that given a set of events x we write $[x]$ for the down-closure of x, i.e. for $\{e \in E \mid \exists e' \in x.\ e \leq e'\}$. Then, write $Pos[x]$ for the set of $\oplus$-events in $[x]$ (as $Neg[x]$ is used for the set of $\ominus$-events in $[x]$). □

A deterministic concurrent strategy $\sigma : S \to A$ determines a closure operator φ on possibly infinite configurations $\mathcal{C}(S)$: for $x \in \mathcal{C}(S)$, define

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = + \ \&\ Neg[s] \subseteq x\}\,.$$

The closure operator φ on $\mathcal{C}(S)$ induces a *partial* closure operator φ_p on $\mathcal{C}(A)$. This in turn determines a closure operator $\varphi_p^{\top}$ on $\mathcal{C}(A)^{\top}$, where configurations are extended with a top element $\top$: take $y \in \mathcal{C}(A)^\top$ to the least fixpoint of φ_p above y, if such exists, and $\top$ otherwise. An earlier definition of concurrent strategies as closure operators was studied in [1]. Also, closure operators on partial orders were used as concurrent strategies in [9]; however, the concurrent games model in this paper generalizes the one in [9], amongst other reasons, because nondeterministic strategies are considered instead.

Finally, the following structural property, called *race-freedom*—which first appeared in [13] for the definition of the deterministic concurrent *copy-cat* strategy—will be used again, this time, in order to define classes of determined concurrent games. Formally, we say that a game A is *race-free* if

$$\forall y \in \mathcal{C}(A). \\ y \xrightarrow{a}\subset \ \&\ y \xrightarrow{a'}\subset \ \&\ pol(a) \neq pol(a') \implies y \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad \textbf{(Race-free)}$$

Informally speaking, race-freedom is a *structural* condition that prevents one player from interfering with the moves available to the other player.

Properties of games with winning conditions. We shall present a number of simple but useful properties of games with winning conditions. Hereafter by a game we mean a concurrent game with winning conditions (A, W).

Proposition 8 (Existence of a winner)**.** *Every play of a game has a winner.*

Proof. Given any pair of strategies $\sigma : S \to A$ and $\tau : T \to A^\perp$, the set of results of plays $\langle \sigma, \tau \rangle$ is not empty. And, since every configuration in A has assigned a (uniquely defined) winner, then every play must have a winner. □

Definition 9 (Dual games)**.** *The dual of a game (A, W), denoted by $(A, W)^\perp$ is the game $(A^\perp, L)$.*

Proposition 10 (Closure under dual games)**.** *If there is a winning strategy in (A, W) for one of the players, then there is a winning strategy in $(A, W)^\perp$ for the other player.*

Proof. Suppose Eve has a winning strategy $\sigma : S \to A$ and recall that Adam's strategies are in $A^\perp$. Then, in the dual game $(A, W)^\perp$ the strategy σ becomes a $\forall$-strategy and only maps to losing configurations by the definition of the dual game. Then, it is a winning strategy for Adam in $(A, W)^\perp$. By duality the argument also applies when Adam has a winning strategy in (A, W). □

Definition 11 (Subgames)**.** *Let (A, W) be a game. Then for every $y \in \mathcal{C}(A)$ define a residual subgame $(A, W)_y =_{\mathrm{def}} (A_y, W_y)$ where:*

$$\begin{array}{lcl} A_y & = & \{a \in A \setminus y \mid \exists y' \in \mathcal{C}(A).\ y \subseteq y' \ \&\ a \in y'\} \\ W_y & = & \{y' \subseteq A_y \mid y \cup y' \in W\} \end{array}$$

with $\leq_{A_y}$ the restriction of $\leq_A$ to A_y and $\mathrm{Con}_{A_y} = \{Y \subseteq_{\mathrm{fin}} A_y \mid Y \in \mathrm{Con}_A\}$.

The notion of residual subgames is key to analyse concurrent games as well as their determinacy problem. Intuitively, a residual subgame is the remainder of a concurrent game once a certain position (or configuration) has been reached. Thus, they can be understood as the residual of a concurrent game with respect to a particular configuration, whether finite or infinite.

Residual subgames are well behaved with respect to winning strategies in the sense that they preserve the winner of the games they belong to—because winning (sub-)strategies can always be constructed. More precisely:

Proposition 12 (Closure under residual subgames)**.** *For each game (A, W), if $\sigma : S \to A$ is a winning strategy then for every configuration $y \in \mathcal{C}(A)$ such that $y = \sigma x$ for some $x \in \mathcal{C}(S)$, there is a winning strategy in $(A, W)_y$.*

Proof. Firstly, note that in the strategy σ_{S_x} defined by the event structure

$$S_x \quad =_{\text{def}} \quad \{s \in S \setminus x \mid \exists x' \in \mathcal{C}(S).\ x \subseteq x' \ \&\ s \in x'\}$$

with $\leq_{S_x}$ the restriction of $\leq_S$ to S_x and $\text{Con}_{S_x} = \{X \subseteq_{\text{fin}} S_x \mid X \in \text{Con}_S\}$, if x_m is a $\oplus$-maximal configuration in S_x then $x \cup x_m$ is a $\oplus$-maximal configuration in S. And since all $\oplus$-maximal configurations in S map under σ to configurations in W, then, because of the definition of W_y, all $\oplus$-maximal configurations in S_x map under σ_{S_x} to configurations in W_y.

Therefore σ_{S_x} is winning in $(A, W)_y$. □

We are now ready to study the determinacy of concurrent games.

5. On undetermined games

In general a concurrent game is undetermined, as shown in Example 6. The issue illustrated in the example is that there is a *race* between Eve and Adam when trying to play their own moves as they are inconsistent with one another (or in *conflict*) in A and therefore cannot be played together. Although being race-free is not sufficient to ensure that a game is determined, whenever the configurations of A are finite a determinacy result holds:

Theorem 13 (from [7])**.** *Let A be a well-founded game, i.e. all configurations in $\mathcal{C}(A)$ are finite. Then (A, W) is determined for all W iff A is race-free.*

However, if the game is not well founded then determinacy cannot be guaranteed. For instance, as shown in the following example.

Example 14. Let A be the event structure consisting of one positive event $\oplus$ which is concurrent with an infinite chain of alternating negative and positive events (and let $i \in \mathbb{N}$), i.e. for each i we have both $\oplus$ *co* $\oplus_i$ and $\oplus$ *co* $\ominus_i$:

$$A = \qquad \oplus \qquad \ominus_1 \rightarrowtriangle \oplus_1 \rightarrowtriangle \ominus_2 \rightarrowtriangle \oplus_2 \rightarrowtriangle \cdots$$

and winning conditions (for Eve) given by

$$W = \{\varnothing, \{\ominus_1, \oplus_1\}, \{\ominus_1, \oplus_1, \ominus_2, \oplus_2\}, \ldots, \{\ominus_1, \oplus_1, \ldots, \ominus_i, \oplus_i\}, \ldots, A\}.$$

Intuitively, Eve wins if (i) no event is played, or (ii) the event $\oplus$ is not played and the play is finite and finishes in some $\oplus_i$, or (iii) all of the events in A are played. Otherwise, Adam wins the game—recall that $L = \mathcal{C}(A) \setminus W$.

First, Eve does not have a winning strategy because Adam has an infinite family of $\forall$-strategies which cannot all be dominated by a single $\exists$-strategy. Let $\tau_\infty : T_\infty \to A^\perp$ and $\tau_i : T_i \to A^\perp$ be $\forall$-strategies, with $i \in \mathbb{N}$, such that

$$\begin{array}{lll} T_\infty^\perp & =_{\text{def}} & A \text{ , and} \\ T_i^\perp & =_{\text{def}} & A \setminus \{e' \in A \mid \ominus_i \leq e' \text{ for some finite } i\}. \end{array}$$

Any $\exists$-strategy that plays $\oplus$ is dominated by some $\forall$-strategy τ_i; likewise, any $\exists$-strategy that does not play $\oplus$ and plays only finitely many positive events $\oplus_i$ is also dominated by some $\forall$-strategy τ_i. Moreover, a $\exists$-strategy that does not play $\oplus$ and plays all of the events $\oplus_i$ in A is dominated by τ_∞. Then, Eve does not have a winning strategy in this game.

Similarly, Adam does not have a winning strategy in A because Eve has two $\exists$-strategies that cannot be both dominated by any $\forall$-strategy. Let $\sigma_{\overline{\oplus}} : S_{\overline{\oplus}} \to A$ and $\sigma_\oplus : S_\oplus \to A$ be $\exists$-strategies such that

$$\begin{array}{lll} S_{\overline{\oplus}} & =_{\text{def}} & A \setminus \{\oplus\} \text{ , and} \\ S_\oplus & =_{\text{def}} & A \end{array}$$

On the one hand, any $\forall$-strategy that plays only finitely many (possibly zero) negative events $\ominus_i$ is dominated by $\sigma_{\overline{\oplus}}$; on the other hand, any $\forall$-strategy that plays all of the negative events $\ominus_i$ in A is dominated by $\sigma_\oplus$. Thus, neither Eve nor Adam has a winning strategy in this game! □

Notation 15. Note that in order to define a $\forall$-strategy $\tau : T \to A^\perp$ we actually describe the dual of T, as if Adam was to play in A instead of $A^\perp$. We will also say that Adam plays a $\ominus$-event in A to mean that using τ he plays the dual event in $A^\perp$, so that we can use τ and avoid referring to $A^\perp$. □

Note that the main issue in order to build a winning strategy for Eve is that $\oplus$ cannot causally depend on *infinitely many* events. Then, one could try to achieve determinacy in the infinite case—i.e. when the games are not well founded—by requiring a *structural* condition to be satisfied by the boards where the games are played; for instance, a structural condition that disallows configurations where an event is concurrent with infinitely many events of the opposite polarity. Such a property is called *bounded-concurrency*:

$$\forall y \in \mathcal{C}^\infty(A).\ \forall e \in y.\ \{e' \in y \mid e \mathrel{co} e' \ \&\ pol(e) \neq pol(e')\} \text{ is finite.}$$

(Bounded-concurrency)

Bounded-concurrency is a key property to achieve determinacy. In fact, as shown by Lemma 17, whenever A fails to satisfy bounded-concurrency, one can define winning conditions W that make a game (A, W) not determined.

Notation 16. We write $max_+(y', y)$ if and only if y' is $\oplus$-maximal in y, i.e. $y' \overset{e}{\longrightarrow\!\!\subset}\ \&\ pol(e) = + \Longrightarrow e \notin y$; in a dual way, we write $\overline{max}_+(y', y)$ if and only if y' is not $\oplus$-maximal in y. We also use max_- when $pol(e) = -$ instead. □

In order to show that if a race-free event structure A fails to have bounded-concurrency, then there are winning conditions W so that the game (A, W) is not determined, we shall use the following general schema (a set of rules) for defining the winning conditions/sets of the game.

Suppose A fails to satisfy bounded-concurrency. Then, we know that there is $y \in \mathcal{C}^\infty(A)$ and $e \in y$ such that e is concurrent with infinitely many events $e_i \in y$ of opposite polarity. Without loss of generality, assume that $pol(e) = +$ and based on y define W using the following rules (let $y' \in \mathcal{C}(A)$):

1. $y' \supseteq y \Longrightarrow y' \in W$;
2. $y' \subset y \ \&\ e \in y' \Longrightarrow y' \in L$;
3. $y' \subset y \ \&\ e \notin y'\ \&$
   $max_+(y', y \setminus \{e\})\ \&\ \overline{max}_-(y', y \setminus \{e\}) \Longrightarrow y' \in W$;
4. $y' \subset y \ \&\ e \notin y'\ \&$
   $\overline{max}_+(y', y \setminus \{e\})$ or $max_-(y', y \setminus \{e\}) \Longrightarrow y' \in L$;
5. $y' \not\subset y \ \&\ y' \not\supseteq y \ \&\ (y' \cap y) \subset^- y' \Longrightarrow y' \in W$;
6. $y' \not\subset y \ \&\ y' \not\supseteq y \ \&\ (y' \cap y) \subset^+ y' \Longrightarrow y' \in L$;
7. otherwise assign (arbitrarily) y' to W.

The rules assign a winner to every configuration (because of rule 7). In addition, no configuration y' is assigned as winning to both Eve and Adam: the antecedents of all implications are pair-wise mutually exclusive.[3]

Lemma 17. *Let (A, W) be a race-free game. If A does not have bounded-concurrency, then there is W such that the game (A, W) is not determined.*

Proof. Define the winning conditions W using the general schema (set of rules) given above. Without loss of generality, assume that $y \setminus \{e\}$ is a configuration and that $y = [e] \cup [\{e_i \in y \mid pol(e_i) = -\}]$.

First, let us show that Eve does not have a winning strategy. Consider the following infinite family of $\forall$-strategies, namely $\tau_\infty : T_\infty \to A^\perp$ and $\tau_i : T_i \to A^\perp$ (for $i \in \mathbb{N}$ and recall that for each $e_i \in y$ we have that $e\ co\ e_i$), such that:

$$\begin{array}{lll} T^\perp_\infty & =_{\text{def}} & \{e' \in A \mid e' \in y \ \vee\ pol(e) = +\}\text{ , and} \\ T^\perp_i & =_{\text{def}} & \{e' \in A \mid e' \in y \setminus \{e_i\} \ \vee\ pol(e') = +\}. \end{array}$$

Then each $\forall$-strategy only plays $\ominus$-events contained in y; moreover, each $\forall$-strategy τ_i does not play a $\ominus$-event e_i which is concurrent with e.

To get a contradiction, suppose Eve has a winning strategy $\sigma : S \to A$.

Since σ is a winning strategy then $\sigma \gg \tau_\infty$, i.e. $y' \in \langle \sigma, \tau_\infty \rangle \Longrightarrow y' \in W$. Note that because of the definition of τ_∞ we know that for all $y' \in \langle \sigma, \tau_\infty \rangle$, we have that $y' \supseteq y$ (Eve only wins using rule 1); rules 3 and 5 cannot be used to win the game, respectively, because (for 3) τ_∞ always plays $\ominus$-maximally in y—hence in $y \setminus \{e\}$ too—and (for 5) τ_∞ never plays $\ominus$-events not in y. Then:

$$Pos[y] \subseteq \sigma S.$$

[3]Note that W in Example 14 is an instance of the use of this set of winning rules.

We also have that $\sigma \gg \tau_i$, for every $i \in \mathbb{N}$. As every τ_i does not play some $\ominus$-event in y then Eve cannot win using rule 1 when playing against every τ_i. And, as for τ_∞, each τ_i never plays $\ominus$-events not in y; then Eve cannot win using rule 5 either. As a consequence, Eve can only win using rule 3.

Winning with rule 3 requires that

$$\forall \tau_i.\ y' \in \langle \sigma, \tau_i \rangle \Longrightarrow e \notin y'.$$

But we know that there is $s_e \in S$ such that $\sigma(s_e) = e$ (because $Pos[y] \subseteq \sigma S$). Since $[e]$ is finite then $[s_e]$ is finite too—hence $Neg[s_e]$ is also finite. And because $Neg[y]$ is infinite, then there are infinitely many τ_i such that

$$\exists y' \in \langle \sigma, \tau_i \rangle.\ y' \subset y \ \&\ e \in y',$$

i.e. infinitely many τ_i with which Adam wins using rule 2—which contradicts that Eve wins using rule 3 when playing against every τ_i (formally, a contradiction with the statement above, namely, that $\forall \tau_i.\ y' \in \langle \sigma, \tau_i \rangle \Longrightarrow e \notin y'$).

Then, we conclude that $\sigma : S \to A$ is not a winning strategy, i.e. that Eve does not have a winning strategy in the concurrent game (A, W).

Now, we show that Adam does not have a winning strategy either. Consider the following two $\exists$-strategies, $\sigma_{\overline{\oplus}} : S_{\overline{\oplus}} \to A$ and $\sigma_{\oplus} : S_{\oplus} \to A$, where:

$$\begin{array}{lll} S_{\oplus} & =_{\text{def}} & \{e' \in A \mid e' \in y \ \vee\ pol(e) = -\} \text{ , and} \\ S_{\overline{\oplus}} & =_{\text{def}} & \{e' \in A \mid e' \in y \setminus \{e\} \ \vee\ pol(e) = -\}. \end{array}$$

Thus, $\sigma_{\oplus}$ and $\sigma_{\overline{\oplus}}$ only play $\oplus$-events in y; moreover, $\sigma_{\oplus}$ plays $\oplus$-maximally in y—hence in $y \setminus \{e\}$ too—and $\sigma_{\overline{\oplus}}$ plays $\oplus$-maximally in $y \setminus \{e\}$. And, while $\sigma_{\oplus}$ plays e as long as Adam plays $Neg[e]$, the strategy $\sigma_{\overline{\oplus}}$ never plays e.

Again, in order to get a contradiction, suppose that Adam has a strategy $\tau : T \to A^{\perp}$ which is winning; in particular, so that both $\tau \gg \sigma_{\oplus}$ and $\tau \gg \sigma_{\overline{\oplus}}$.

Because of the definitions of $\sigma_{\oplus}$ and $\sigma_{\overline{\oplus}}$ and the set of winning rules there are two ways how τ can win (see rules 2 and 4), namely when:

(i) $y' \subset y \ \&\ e \in y'$, or

(ii) $y' \subset y \ \&\ e \notin y' \ \&\ max_{-}(y', y \setminus \{e\})$,

for any result y'.

The first observation is that since both $\sigma_{\oplus}$ and $\sigma_{\overline{\oplus}}$ play $\oplus$-maximally in $y \setminus \{e\}$, then every result y' of playing τ against either $\sigma_{\oplus}$ or $\sigma_{\overline{\oplus}}$ satisfies that

$$max_{+}(y', y \setminus \{e\}).$$

The second observation is that since $y \setminus y' \neq \varnothing$ and $max_{+}(y', y \setminus \{e\})$, then it follows that for all $e' \in y$ such that $y' \xrightarrow{e'} \!\subset$ we have that

$$pol(e') = + \Longrightarrow e' = e \quad \text{and} \quad e \in y' \Longrightarrow pol(e') = -.$$

Let $y' \in \langle \sigma_{\oplus}, \tau \rangle$. Since $max_{+}(y', y)$ holds (because $\sigma_{\oplus}$ plays $\oplus$-maximally in y—rather than only in $y \setminus \{e\}$) then it follows that $pol(y \setminus y') \subseteq \{-\}$, i.e. all events in the non-empty set $y \setminus y'$ have negative polarity. Formally, that

$$\forall e' \in y.\ y' \xrightarrow{e'} \!\subset \Longrightarrow pol(e') = -.$$

Thus, there are two options: either $e \notin y'$ or $e \in y'$. The former is impossible because in such a case Adam would have to win using rule 4, and hence y' would satisfy (ii), but y' fails to satisfy $max_{-}(y', y \setminus \{e\})$. Therefore, we have that $e \in y'$ and hence Adam wins using rule 2, i.e. y' satisfies (i). Since y' is $\oplus$-maximal in y, we know, in particular, that τ does not play all negative events in A, that is, we have that

$$Neg[y] \nsubseteq (\tau T)^{\perp},$$

as otherwise there would be a result where Eve would win using rule 1.

Now, let $y' \in \langle \sigma_{\overline{\oplus}}, \tau \rangle$. In this case, $max_{+}(y', y \setminus \{e\})$ holds (as $\sigma_{\overline{\oplus}}$ plays $\oplus$-maximally in $y \setminus \{e\}$) and hence $\forall e' \in y.\ y' \xrightarrow{e'} \subset\ \&\ pol(e') = + \Longrightarrow e' = e$.

Necessarily $e \notin y'$ (because $\sigma_{\overline{\oplus}}$ does not play e) and Adam can only win using rule 4, that is, so that y' satisfies (ii) above. This implies that $max_{-}(y', y \setminus \{e\})$ must hold and we know that $max_{+}(y', y \setminus \{e\})$ holds too. As y' is both $\oplus$-maximal and $\ominus$-maximal in $y \setminus \{e\}$ and $y \setminus y' \neq \emptyset$, then there is only one event that y' enables, namely e; formally $\exists e' \in y.\ y' \xrightarrow{e'} \subset\ \&\ \forall e' \in y.\ y' \xrightarrow{e'} \subset \Longrightarrow e' = e$. Thus, it necessarily is the case that $y \setminus y' = \{e\}$ and hence that

$$Neg[y] \subseteq (\tau T)^{\perp},$$

which leads to a contradiction.

As a consequence, Adam does not have a strategy that dominates both $\sigma_{\oplus}$ and $\sigma_{\overline{\oplus}}$, i.e., Adam does not have a winning strategy either.

Thus, we finally conclude that neither player has a winning strategy. □

Remark 18. Note that the reason why Adam does not have a winning strategy has nothing to do with the fact that y is infinite. Indeed, if one restricts to a setting where y is finite, then a winning strategy for Eve can be defined: simply use $\sigma_{\oplus}$ but make e causally depend on all $\ominus$-events in y. □

6. Determinacy and infinite winning sets

In a seminal paper on (sequential, two-player, and perfect information) infinite games by Gale and Stewart [8] it was shown that when the pay-off sets of the game—the winning conditions—are open sets, then the game is determined. Playing in an open set implied the following property:

Property 19. Given an infinite play $p = < m_1, m_2, m_3, ... >$, seen as an infinite sequence of moves, if Adam (resp. Eve) does not have a winning strategy in any sub-game $p(i) = < m_i, m_{i+1}, m_{i+2}, ... >$, for all i in $\mathbb{N}$, then it follows that Adam (resp. Eve) does not win in such a particular play p. □

Important pay-off sets for sequential games in logic and computer science are open, e.g. some ω-regular winning sets studied in formal verification.

Definition 20 (concurrent open games)**.** *Let (A, W) be a concurrent game. We say that (A, W) is open when for all $y \in \mathcal{C}^{\infty}(A)$ the following holds: If Adam (Eve) does not have a winning strategy in any subgame $(A, W)_{y'}$, for all $y' \subseteq_{\text{fin}} y$, then Adam (Eve) does not have a winning strategy in $(A, W)_y$.*

Then, instead of defining open sets in our framework, we will require that concurrent games satisfy what sequential games on open sets satisfy.

In the reminder of this section we show that if a concurrent game (A, W) has winning conditions whose winning sets (i.e. configurations) are allowed to be infinite but (A, W) is open and has bounded-concurrency, then the game is determined. Note that well-founded games (as in [7]) trivially satisfy bounded-concurrency and openness since in that case $\mathcal{C}^{\infty}(A) = \varnothing$.

Example 21. The game (A, W) in Example 14 is not open. Take the infinite configuration $y = \{\ominus_1, \oplus_1, \ominus_2, \oplus_2, ...\}$; even though Eve does not have a winning strategy in any subgame $(A, W)_{y'}$, with $y' \subseteq_{\text{fin}} y$, she has a winning strategy in the subgame $(A, W)_y$, namely in the subgame $(\{\oplus\}, \{\oplus\})$. The winning strategy is obvious: $\sigma : \{s_\oplus\} \to \{\oplus\}$, i.e. where $\sigma(s_\oplus) = \oplus$. □

6.1. *Concurrent defensive strategies*

Another seminal idea introduced by Gale and Stewart in [8] was that the class of *defensive strategies*—strategies which try to avoid losing—was complete for (sequential perfect-information) open games in the sense that if a player has a winning strategy, then it has a defensive winning strategy.

For our determinacy theorem we use strategies of a similar kind; we call them *concurrent defensive* strategies, since they always try to avoid losing but in a concurrent setting. The technique to build concurrent defensive strategies—together with an example—is given in the appendix.

Not all concurrent defensive strategies are winning, since it is not always possible to extend losing configurations to winning ones, e.g. when a player does not have a winning strategy. However, all defensive strategies have the following, intuitively simple, property: namely that if the strategy $\sigma : S \to A$ in an even structure A is in fact a concurrent defensive strategy, then

$$\begin{aligned} \forall x \in \mathcal{C}(S). \quad & \sigma x \in L \;\&\; (\exists y \in W.\ \sigma x \subset^+ y) \\ & \Longrightarrow \\ & \exists x' \in \mathcal{C}(S).\ x \subset^+ x' \;\&\; \sigma x' \in W. \end{aligned}$$

6.2. *Determinacy of concurrent open games*

We now prove the main result of the paper, namely the determinacy theorem for concurrent open games on event structures. Hereafter we sometimes will require that the games are *race-free*, *open*, or have *bounded-concurrency*.

Lemma 22. *Let (A, W) be a race-free game with bounded-concurrency and y a finite configuration in $\mathcal{C}^{\omega}(A)$. If $y \subseteq^+ y'$, and Eve does not have a winning strategy in $(A, W)_y$, and the set $y' \smallsetminus y$ is infinite, then $y' \in L$.*

Proof. Since Eve does not have a winnings strategy in $(A, W)_y$, then because (A, W) is race-free

$$y' \in W \Longrightarrow (\exists y'' \in L.\ y' \subseteq^- y'');$$

and since the set $y' \smallsetminus y$ is infinite and the configuration y is finite, then y' is infinite and contains infinitely many $\oplus$-events (because $y \subseteq^+ y'$).

Now, in order to get a contradiction, suppose that $y' \in W$. If $y' \in W^\infty$, and y' contains infinitely many $\oplus$-events, and $y' \subseteq^- y''$, then there exists a $\ominus$-event $e'' \in y'' \setminus y'$ such that $\{e' \in y' \mid e'' \; co \; e' \;\&\; pol(e'') \neq pol(e')\}$ is infinite; contradiction with A having bounded-concurrency. Then, $y' \in L$. $\square$

Corollary 23. *Let (A, W) be a race-free game with bounded-concurrency and y a finite configuration in $\mathcal{C}^\omega(A)$. If $y \subseteq^+ y'$ and Eve does not have a winning strategy in $(A, W)_y$ then*

- *$y' \in W \Longrightarrow y'$ is finite, and*
- *y' is infinite $\Longrightarrow y' \in L$.*

Lemma 24. *If $y \in L^\infty$, an infinite configuration of a race-free game with bounded-concurrency (A, W), and y contains infinitely many $\ominus$-events, then Adam has a winning strategy in $(A, W)_y$.*

Proof. If y contains infinitely many $\ominus$-events and the game (A, W) has bounded-concurrency, then

$$y \xrightarrow{e}\!\!\subset \;\Longrightarrow pol(e) = -,$$

as otherwise e would be concurrent with infinitely many events with opposite polarity—i.e. the event structure A would fail to be bounded-concurrent. It then immediately follows that the empty $\forall$-strategy on $(A, W)_y$ is winning. Thus, not only has Adam a winning strategy but also such a strategy is the unique (receptive) empty $\forall$-strategy on $(A, W)_y$. $\square$

Informally speaking, the proof of Lemma 24 says that Adam does not need to do anything, i.e. play any $\ominus$-move, to win the subgame $(A, W)_y$.

Lemma 25. *If y is an infinite configuration in a race-free game (A, W) with bounded-concurrency and Eve does not have a winning strategy in $(A, W)_y$ then Adam has a winning strategy in $(A, W)_y$.*

Proof. There are four cases depending on whether y is in W^∞ or in L^∞ and on whether y has finitely many or infinitely many $\ominus$-events.

1. Suppose that $y \in W^\infty$ has finitely many $\ominus$-events. Since Eve does not have a winning strategy then there is y' such that $y \subseteq^- y' \in L^\infty$. And as (A, W) has bounded-concurrency then no $\ominus$-event $e \in y' \setminus y$ is concurrent with infinitely many $\oplus$-events in y. Then, y has finitely many $\oplus$-events and infinitely many $\ominus$-events; contradiction. Then, this case cannot occur (as if it could then Eve would have a winning strategy—the empty $\exists$-strategy in $(A, W)_y$ as only $\oplus$-extensions of y would be possible).
2. Suppose that $y \in W^\infty$ has infinitely many $\ominus$-events. Then, we have that only $\ominus$-extensions of y are possible. As before, since Eve does not have a winning strategy then there is y', with $y \subseteq^- y' \in L^\infty$, where Adam wins the game since no $\oplus$-extension of y' is possible. Then Adam has a winning strategy in $(A, W)_{y'}$, and therefore also in all games $(A, W)_{y''}$, with $y'' \in \{y'' \in \mathcal{C}^\infty(A) \mid y'' \subseteq^- y'\}$; in particular, Adam has a winning strategy in the game $(A, W)_y$, where $y'' = y$.

3. Suppose that $y \in L^\infty$ has infinitely many $\ominus$-events. By Lemma 24, Adam has a winning strategy in $(A, W)_y$.
4. Suppose that $y \in L^\infty$ has finitely many $\ominus$-events. Then, y has infinitely many $\oplus$-events and only $\oplus$-extensions of y are possible. As Eve does not have a winning strategy then each $\oplus$-extension of y must be in L^∞. Hence, again, Adam has a winning strategy in $(A, W)_y$.

Therefore, in all cases, Adam has a winning strategy in $(A, W)_y$. □

The next two lemmas are key to defining winning strategies, which are realised as concurrent defensive strategies (see Appendix).

Lemma 26. *Let (A, W) be a race-free game with bounded-concurrency and $y \in \mathcal{C}(A)$. If Eve does not have a winning strategy in $(A, W)_y$ then*

$$\forall y' \in W.\ y \subseteq^+ y'$$
$$\Longrightarrow$$
$$\exists y'' \in L.\ y' \subseteq^- y'' \ \&\ \textit{Eve does not have a winning strategy in } (A, W)_{y''}$$

Proof. Consider two cases: either y is finite or y is infinite.

If y is infinite and Eve does not have a winning strategy in $(A, W)_y$ then, due to Lemma 25, Adam has a winning strategy in $(A, W)_y$ and therefore also a winning strategy in every $\oplus$-extension y' of y, i.e. in every game $(A, W)_{y'}$. Now, as (A, W) has bounded-concurrency, either y cannot be $\oplus$-extended to $y' \in W^\infty$ or y' cannot be $\ominus$-extended to $y'' \in L^\infty$. But, since Adam has a winning strategy in both $(A, W)_y$ and $(A, W)_{y'}$ then no $\oplus$-extension y' of y can be in W^∞. Hence, the statement is trivially satisfied when y is infinite.

Now suppose that y is finite and that Eve does not have a winning strategy in $(A, W)_y$. If y is $\oplus$-extended to y' and y' is infinite then, due to Corollary 23, y' is in L; hence, if y is finite and $\oplus$-extended to $y' \in W$ then y' must be finite too. Recall that as Eve does not have a winning strategy in $(A, W)_y$ then she does not have a winning strategy in any $(A, W)_{y'}$ either (with $y \subseteq^+ y'$).

Then, in order to get a contradiction, suppose that for some $y' \in W$ every $\ominus$-extension y'' of y' defines a game $(A, W)_{y''}$ where Eve has a winning strategy. Suppose that y'' in infinite. Since (A, W) has bounded-concurrency, y' is finite, y'' is infinite, and y'' extends y' only with $\ominus$-events then y'' contains infinitely many $\ominus$-events. Then, due to Lemma 24, Adam has a winning strategy in $(A, W)_{y''}$; contradiction. As a consequence, y'' cannot be infinite. Note, in particular, that since y'' is finite then it has finitely many $\ominus$-events.

Now, let $\sigma_{y''}$ be the winning strategies for Eve at each subgame $(A, W)_{y''}$. Also, let σ_y be the (sub)strategy that takes the game from y to y'. Define σ to be the strategy that at y behaves as σ_y and has every $\oplus$-event of every strategy $\sigma_{y''}$ causally depending on the $\ominus$-events in y''. Clearly, since all y'' are finite (and hence have finitely many $\ominus$-events) the strategy σ can be constructed. But, then, σ is winning since the results of $\langle \sigma, \tau \rangle$, for every τ, are all in W (because it is so for each $\sigma_{y''}$). Therefore, Eve has a winning strategy σ in $(A, W)_y$; contradiction. Thus we conclude that if y is finite, and Eve does not have a winning strategy in $(A, W)_y$, and $y \subseteq^+ y' \in W$ (for some finite y'), then there exists finite $y'' \in L$ such that $y' \subseteq^- y''$ and a game $(A, W)_{y''}$ where Eve does not have a winning strategy. $\square$

Note, in particular, that the previous lemma holds without the requirement that A is well founded, i.e. that the configuration y is finite.

Lemma 27 (Finite decidability)**.** *Let (A, W) be a game such that the game board A is race-free and has bounded-concurrency. Then we have that*

$$\forall y, y' \in \mathcal{C}^\infty(A).\ y \in W^\infty \ \&\ y \subset^- y' \ \&\ y' \in L^\infty \Longrightarrow \exists y'' \subset_{\text{fin}} y.\ y'' \subset^- y'.$$

Proof. Since A has bounded-concurrency, then it follows that

$$y \overset{e}{\longrightarrow\!\subset} \ \&\ pol(e) = - \Longrightarrow Pos[y] \text{ is finite,}$$

as otherwise e would be concurrent with infinitely many events (in y) with opposite polarity. Let y'' be the necessarily finite configuration $[Pos[y]]$. Note that since A is race-free, then $[Pos[y]]$ is $\oplus$-maximal (as well as y), because no $\ominus$-event in $y' \setminus [Pos[y]]$ can be in conflict with any $\oplus$-event in A. Then, it follows that $[Pos[y]] = y'' \subset^-_{\text{fin}} y \subset^- y'$, i.e. that $y'' \subset^-_{\text{fin}} y'$. $\square$

The previous lemma states that the infinite sets/plays of our games can be regarded as *finitely decidable*: if one can avoid a winning infinite configuration (for Eve), then one only needs to avoid some previous finite configuration.

Now, the next lemma is used to construct a game board, i.e. an event structure, where Adam has a winning strategy. The lemma is based on the (repeated) construction of partial game boards which we call *layers*. Layers are event structures in which no chain of events the polarities alternate more than once. By the determinacy result in [7] we know that a winning strategy always exists whenever layers are well founded. The issue may arise when they are not; however, the next lemmas help us show that if the games are open and have bounded-concurrency, then determinacy can be recovered.

Definition 28 ($(\forall/\exists)$-games)**.** *Let (A, W) be a game, $S_\forall$ a subset of $\ominus$-events of A, and $S_\exists$ the set of $\oplus$-events of A. A $\forall$-game $(A_\forall, W_\forall)$ is a game such that $A_\forall = A \upharpoonright (S_\forall \cup S_\exists)$, i.e. $A_\forall$ is the event structure A restricted to the events in $S_\forall \cup S_\exists$, and $W_\forall = W \cap A_\forall$. Define $\exists$-games analogously.*

Note that Adam (resp. Eve) has a winning strategy in (A, W) only if they have a winning strategy in some $\forall$-game (resp. $\exists$-game).

Lemma 29. *Let (A, W) be a race-free, open game with bounded-concurrency. If Eve does not have a winning strategy in (A, W) then there is a $\forall$-game $(A_\forall, W_\forall)$ where*

1. *for every finite $y \in W_\forall$ there is $y' \in L_\forall$ such that $y \subset^- y'$, and*
2. *for every infinite $y' \in \mathcal{C}^\infty(A_\forall)$, if y' is $\ominus$-maximal then $y' \in L_\forall$.*

Proof. We shall construct $A_\forall$, i.e. an event structure $A_\forall \subseteq A$, where every finite configuration in W can be $\ominus$-extended to a configuration in L and all infinite configurations are in L^∞. We use Lemma 26 to do so.

Take the empty configuration $\varnothing$ as well as all $\oplus$-extensions of it, that is, all y such that $\varnothing \subseteq^+ y$. Then, for each *finite* y, if $y \in W$ extend it to a configuration $y' \in L$ where Eve does not have a winning strategy, i.e. which defines a residual subgame $(A, W)_{y'}$ where Eve does not have a winning strategy. Such a configuration always exists: if Eve does not have a winning strategy in $\varnothing$ then she cannot have a winning strategy in any $\oplus$-extension y of $\varnothing$ (otherwise, due to race-freedom, she would have it in $\varnothing$ too); therefore

$$y \in W \Longrightarrow (\exists y' \in L.\ y \subseteq^- y').$$

The process described above constructs one layer (say, layer 0), which may not be well founded. Then, repeat the process of constructing layers for each y' that is *finite*. Let $A_\forall$ be the event structure defined by the configurations y, y' characterized by the recursive process described above.

The first statement to prove, namely that every finite configuration in W can be $\ominus$-extended to a configuration in L, immediately follows from the fact that every layer of $A_\forall$ is constructed using Lemma 26.

In order to show the second statement, namely that all infinite configurations y' are in L, let us consider the following four cases:

1. infinite $\oplus$-extension to y' from finite y, or
2. infinite $\ominus$-extension to y' from finite y, or
3. y' is constructed by an infinite alternation of finite layers, or
4. extension to y' from infinite y.

Case 1: $y \subseteq^+ y'$ and y is finite. Due to Corollary 23, it follows that $y' \in L^\infty$.

Case 2: $y \subseteq^- y'$ and y is finite. Firstly, note that since y is finite, then $Pos[y]$ is finite too and, moreover, that $y' \setminus y$ is set of $\ominus$-events only. Now, due to Lemmas 26 and 27, we know that either y' is $W_\forall$ but it is not $\ominus$-maximal, or it is $\ominus$-maximal and it is in L; thus, $y' \in L$. And since Eve does not have a winning strategy in $(A, W)_{y'}$, (A, W) is race-free and has bounded-concurrency then, because of Lemma 24, it follows that Adam has a winning strategy in $(A, W)_{y'}$. Moreover, since the set $y' \setminus y$ has infinitely many $\ominus$-events, due to bounded-concurrency, there are no $\oplus$-extensions of y'.

Case 3: y' is an infinite configuration and there is no finite configuration y such that either $y \subseteq^+ y'$ or $y \subseteq^- y'$. Since $(A_\forall, W_\forall)$ is constructed using Lemma 26 and (A, W) is a concurrent open game, then Eve does not have a winning strategy in $(A, W)_{y'}$. Suppose, in order to get a contradiction, that $y' \in W^\infty$. Since y' contains infinitely many $\ominus$-events as well as infinitely many $\oplus$-events then it can be extended neither with $\ominus$-events nor with $\oplus$-events. Thus, the property that $y' \in W \Longrightarrow (\exists y'' \in L.\ y' \subseteq^- y'')$ fails to be satisfied; contradiction. As a consequence y' must be in L^∞.

Case 4: y is infinite. Note that the (sub)cases when $y \subseteq y'$, with y infinite, need not be considered because when constructing $A_\forall$ only finite configurations are extended to L at each layer. In particular, note that if y was in W, due to Lemma 27 and the fact that all finite configurations in W are $\ominus$-extended to configurations in L, then it follows (case 2) that $y \in W$ was not $\ominus$-maximal and that $y' \in L$.

Then, all infinite $\ominus$-maximal configurations of $A_\forall$ are in L. $\square$

The previous lemma implies that in a race-free open game with bounded-concurrency, a strategy that does not win at any finite stage, does not win at any infinite stage either (i.e. at any subgame). Now, we show that due to race-freedom and bounded-concurrency—our two structural properties on games—if Eve does not have a winning strategy in (A, W), not only can a $\forall$-game $(A_\forall, W_\forall)$ be constructed but also a winning $\forall$-strategy for Adam.

Lemma 30. *Let (A, W) be a race-free, open game with bounded-concurrency. If Eve does not have a winning strategy in (A, W) then Adam has a winning strategy in (A, W).*

Proof. By the definition of residual subgames, Eve does not have a winning strategy in (A, W) if and only if she does not have a winning strategy in $(A, W)_\varnothing$. Using Lemma 29 we shall construct a game board $A_\forall$, i.e. an event structure $A_\forall \subseteq A$, where Adam has a winning strategy.

Now, let us build a winning strategy $\tau : T \to A_\forall^\perp$ for Adam in the game board $A_\forall$, and hence in A too since any winning strategy in $A_\forall$ does not disallow $\oplus$-events already in A, that is, the receptivity of τ with respect to A makes it a winning strategy also in A. We can use the same techniques to build winning strategies for well-founded games because:

(i) all infinite $\ominus$-maximal configurations of $A_\forall$ are in L and
(ii) no $\ominus$-event in $T^\perp$ needs to depend on infinitely many $\oplus$-events in $T^\perp$.

Item (i) was shown in Lemma 29. Now, let us show item (ii). Without loss of generality, take only $\ominus$-maximal configurations y of $A_\forall$. If y is finite then, obviously, any $\ominus$-event $e \in T^\perp$ would be required to depend only on finitely many $\oplus$-events. Now suppose that y is infinite. As before, there are four cases to consider: (1) infinite $\oplus$-extension to y from some finite y', or (2) infinite $\ominus$-extension to y from some finite y', or (3) y is constructed by an infinite alternation of finite layers, or (4) extension to y from infinite y'.

Case 1: $Pos[y]$ is infinite and $Neg[y]$ is finite. Since $[Neg[y]]$ is finite and $\ominus$-maximal (because, due to bounded-concurrency, y cannot be $\ominus$-extended) then because of Lemma 26 and the way $(A_\forall, W_\forall)$ was constructed, we know that $[Neg[y]] \in L$. As a consequence, any $\oplus$-extension of $[Neg[y]]$ is in L too. Then, for Adam, winning in $[Neg[y]]$ is winning in y itself as well as in any $\oplus$-extension of it. Thus, a winning (sub)strategy that contains $[Neg[y]]$ can be constructed since the set of $\oplus$-events $Pos[Neg[y]]$ is finite, i.e. the infinite set of $\oplus$-events in $y' \setminus Pos[Neg[y]]$ will be taken into account by the receptivity of the $\forall$-strategy to be constructed for Adam.

Case 2: $Pos[y]$ is finite and $Neg[y]$ is infinite. Clearly, in this case a winning (sub)strategy that contains y can easily be constructed since there are only finitely many $\oplus$-events in y.

Case 3: both $Pos[y]$ and $Neg[y]$ are infinite. In this case y is, necessarily, an infinite alternating sequence of the form

$$\emptyset \subseteq^{+} y_{\exists}^{0} \subseteq^{-} y_{\forall}^{0} \subseteq^{+} y_{\exists}^{1} \subseteq^{-} y_{\forall}^{1} \subseteq^{+} \ldots \subseteq y$$

where every $y_{\exists}^{i} \setminus y_{\forall}^{i-1}$ is finite. Then, every event $e \in T$ which is mapped by τ to an event in $y_{\forall}^{i}$ is required to depend (immediately) only on finitely many events of opposite polarity, more precisely, on those events that are mapped to the set of events in $Neg[y_{\exists}^{i} \setminus y_{\exists}^{i-1}]$, which is finite ($\oplus$-events in layer $k = i$).

Case 4: both $Pos[y]$ and $Neg[y]$ are infinite. Already covered by one of the three former cases since we are assuming that y' itself is infinite.

Since (i) in any $\forall$-strategy no $\ominus$-event in required to depend on infinitely many $\oplus$-events and (ii) at every finite stage of the game Adam can avoid losing, because of Lemma 29, a winning strategy can be constructed in $A_{\forall}$. □

Since concurrent games on event structures are closed under dual games, the following result comes almost for free!

Lemma 31. *Let (A, W) be a race-free, open game with bounded-concurrency. If Adam does not have a winning strategy in (A, W) then Eve has a winning strategy in (A, W).*

Proof. Due to Proposition 10 and Lemma 30 there is a dual game $(A^{\perp}, L)$ where Adam has a winning strategy. Since such a winning strategy τ is a strategy for Adam, then it is a strategy in the game dual to $A^{\perp}$, i.e. a strategy $\sigma : S \to A$ (in the game dual to $A^{\perp}$), which is necessarily winning and can be used by Eve to win every play of the game (A, W). □

Thus, it immediately follows from Lemmas 30 and 31 that:

Theorem 32 (Determinacy)**.** *Let (A, W) be a race-free, open game with bounded-concurrency. Then, Eve or Adam has a winning strategy in (A, W).*

7. Related and future work

Determinacy has been a fundamental problem within the games traditionally used in mathematics, especially in mathematical logic and set theory; in this section we describe some related work on concurrent games and determinacy results, and finish by suggesting some avenues for future work.

In *mathematical logic and set theory*, determinacy problems have been studied with respect to different kinds of games for more than a century: e.g. finite games [15], open games [8], Borel games [11], or Blackwell games [12], just to mention a few which are particularly relevant in computer science. Whereas the determinacy theorem in [7] is a generalisation of Zermelo's determinacy theorem for finite sequential games [15] to a concurrent setting, the determinacy theorem we have in this paper generalises the Gale–Stewart determinacy theorem for infinite, open games in [8]. In particular, a key ingredient of the Gale–Stewart determinacy theorem for open games is the fact that the payoff winning sets they consider are finitely decidable.

In *computer science and formal verification* games have also played an important role. Traditionally, most games found in the literature have been defined to be played sequentially, where a great deal of determinacy results are known. However, in order to address some problems in concurrency, logic, programming semantics, and verification, in the last few years, some concurrent game models have been developed. With no intention of providing an exhaustive review of the literature, we now present some games where determinacy theorems (or results of a similar kind) have been studied.

For instance, in [6] a very complete survey of concurrent games as currently used in formal verification can be found. The concurrent games described in [6] are played on so-called 'concurrent game structures', which are graphs where several players can interact concurrently in order to model the behaviour of reactive systems regarded as open ones. These concurrent games, and variants of them, have been used to study ω-regular properties of reactive systems [2, 3] and to give semantics to some logics for multi-agent systems, e.g. [4]. In the cases where winning strategies have been required to exist their solution is to use profiles of *mixed* strategies, i.e. stochastic ones, for which winning strategies exist up to some real value of accuracy. In the case of zero-sum two-player games (as it also is our games model) determinacy follows from the determinacy of Blackwell games [12].

There are two important differences between the games considered in [12] and ours, which make the comparison of determinacy results between both frameworks rather difficult. On the one hand, players do not take turns in our games, i.e. in a concurrent game on an event structure neither player is actually forced to play—as it is the case for (models based on) Blackwell games; moreover, to achieve determinacy in a concurrent game on an event structure, the strategies to be considered must be nondeterministic.

On the other hand, Blackwell games are of imperfect information[4] and turn-based in the following sense: at each turn of the game both players must make their choices, independently of each other, and then share that information with the other player in order to perform a join move and play the next turn of the game. Our determinacy result does not directly apply to Blackwell games—and hence to (some of) the games in [6]—because the games we have considered in this paper are not of imperfect information.

Finally, in [9] a concurrent game on partial orders is defined and a determinacy result is obtained. The present paper is an attempt to obtain similar results but in the more general setting given by event structures. In particular, the games in [9] have properties tailored to be used in verification. Indeed, the main result in [9] was a determinacy theorem for a concurrent logic game (see [5]

[4]Concurrent games need extra structure to model imperfect information. It may be possible that Blackwell games fit in our framework if, for instance, neutral configurations (which are neither winning nor losing) and probabilistic strategies are allowed.

for a survey of logic games) to which various verification problems, including bisimulation and model-checking, can be reduced. Those games provide a simple generalization of previous game-theoretic approaches to verification [14].

Future work. A first avenue for future work is to fully understand which kinds of winning sets/conditions—as used in computer science and formal verification—fit within our games framework. Certainly, some ω-regular winning sets do fit, but a detailed study of this question should be carried out.

Also, as mentioned before, computability issues should be addressed. Note that given a game, a determinacy theorem answers positively only the question 'can one of the players win the game?'. However, this has nothing to do with the fact that the winning strategy is in fact effectively *computable*. Then, there are a number of further questions to be answered, which amount to solving the game in practice. They are: 'is the winner of such a game computable?'; and if so, 'is there an algorithm to compute its winning strategy?'. As one is interested in strategies (and infinite games) with finite representations then, in the quest towards the answer of algorithmic questions, further research is needed to better understand possible finite representations of event structures, e.g. as those afforded by Petri nets.

Acknowledgement. We thank Julian Bradfield, Pierre Clairambault, and Grant Passmore for helpful discussions. This article was written for the JCSS special issue of selected papers presented in WoLLIC'11, where one of the authors studied in [9] (as a part of [10]) determinacy issues for games on concurrent systems. The authors acknowledge the support of the EPSRC Research Grant 'Solving Parity Games and Mu-Calculi' (at Edinburgh) and the ERC Advanced Grants ECSYM (at Cambridge) and RACE (at Oxford).

References

[1] Abramsky, S., Melliès, P.A.: Concurrent games and full completeness. In: LICS. pp. 431–442. IEEE Computer Society (1999)

[2] de Alfaro, L., Henzinger, T.A.: Concurrent omega-regular games. In: LICS. pp. 141–154. IEEE Computer Society (2000)

[3] de Alfaro, L., Henzinger, T.A., Kupferman, O.: Concurrent reachability games. Theoretical Computer Science 386(3), 188–217 (2007)

[4] Alur, R., Henzinger, T.A., Kupferman, O.: Alternating-time temporal logic. J. ACM 49(5), 672–713 (2002)

[5] van Benthem, J.: Logic games, from tools to models of interaction. In: Logic at the Crossroads, pp. 283–317. Allied Publishers (2007)

[6] Chatterjee, K., Henzinger, T.A.: A survey of stochastic ω-regular games. Journal of Computer and System Sciences 78(2), 394–413 (2012)

[7] Clairambault, P., Gutierrez, J., Winskel, G.: The winning ways of concurrent games. In: LICS. pp. 235–244. IEEE Computer Society (2012)

[8] Gale, D., Stewart, F.: Infinite games with perfect information. Annals of Mathematical Studies 28, 245–266 (1953)

[9] Gutierrez, J.: Concurrent logic games on partial orders. In: WoLLIC. LNCS, vol. 6642, pp. 146–160. Springer (2011)

[10] Gutierrez, J.: On Bisimulation and Model-Checking for Concurrent Games with Partial Order Semantics. Ph.D. thesis, University of Edinburgh (2011)

[11] Martin, D.: Borel determinacy. Annals Mathematics 102(2), 363–371 (1975)

[12] Martin, D.: The determinacy of Blackwell games. Journal of Symbolic Logic 63(4), 1565–1581 (1998)

[13] Rideau, S., Winskel, G.: Concurrent strategies. In: LICS. pp. 409–418. IEEE Computer Society (2011)

[14] Stirling, C.: Bisimulation, modal logic and model checking games. Logic J. of the IGPL 7(1), 103–124 (1999)

[15] Zermelo, E.: On an application of set theory to the theory of the game of chess. In: Congress of Mathematicians. pp. 501–504. CUP (1913)

Appendix A. Concurrent defensive strategies

We now will show how to construct a concurrent defensive strategy. A concurrent defensive strategy, say $\sigma : S \to A$, can be constructed progressively, starting from the empty configuration $\varnothing$ of A. The general construction has two parts—detailed in Definitions 33 and 35 presented later on.

One of the parts (Definition 33) consists of adding $\ominus$-events to S so that the resulting map is receptive. The other part of the construction (given by Definition 35) consists of adding $\oplus$-events to S whenever a losing configuration is to be extended to a winning one. In this case, one needs to make sure that innocence is not violated, i.e. that causal dependencies in S that are not already in A are always from $\ominus$-events to positive ones. Since for any event e the set $[e]$ is finite, then one also has to ensure that only finitely many extra causal dependencies are introduced for any event e.

The technique for constructing strategies presented here is different from the method presented in [7]. In particular, the method presented here is not limited to the game A being well founded. An example illustrating how to use Definitions 33 and 35 in order to build a (nondeterministic) concurrent defensive strategy is given at the end.

Definition 33. *Let $\sigma : G \to A$ be a map of event structures with polarity such that $\sigma' : G \to (A \upharpoonright \sigma G)$ is a strategy, i.e. a receptive and innocent map of event structures. In this case, we say that σ is pre-receptive. The map $\sigma^- : H \to A$ is the receptive-closed map of σ such that H is the event structure whose set of events comprises the events in G and in the set*

$$\bigcup_{x \in \mathcal{C}(G), y \in \mathcal{C}(A)} \{ s_e^x \mid x \text{ is maximal in } G \ \& \ \sigma x \subset^- y \ \& \ e \in y \setminus \sigma x \}$$

where

$$s_{e_1}^{x_1} = s_{e_2}^{x_2} \Leftrightarrow (x_1 = x_2 \;\&\; e_1 = e_2)$$

and such that

$$\sigma^-(s_e^x) = e.$$

Moreover, H has causal dependency relation given by:

1. *Case $e_1, e_2 \in E_G$ (causality inherited only from G):*
   if $e_1 \leq_G e_2$ then $e_1 \leq_H e_2$;
2. *Case $s_{e_1}^{x_1}, s_{e_2}^{x_2} \notin E_G$ (causality inherited only from A):*
   if $(x_1 = x_2 \;\&\; e_1 \leq_A e_2)$ then $s_{e_1}^{x_1} \leq_H s_{e_2}^{x_2}$;
3. *Case $e_1 \in E_G \;\&\; s_e^x \notin E_G$ (causality inherited from both A and G):*
   if $\exists e_3 \in E_G.\ e_1 \leq_G e_3 \;\&\; \sigma(e_3) \leq_A e \;\&\; e_3 \in x$ then $e_1 \leq_H s_e^x$.

and consistency relation given by (again based on that of A and G):

$$X \in \mathrm{Con}_H \text{ iff } \begin{cases} \sigma^- X \in \mathrm{Con}_A \;\& \\ ([X] \cap E_G) \in \mathrm{Con}_G \;\& \\ \forall s_{e_1}^{x_1}, s_{e_2}^{x_2} \in (X \setminus E_G).\ s_{e_1}^{x_1} \neq s_{e_2}^{x_2} \Longrightarrow x_1 = x_2 \end{cases}$$

Note that given (i) a pre-receptive map $\sigma : G \to A$ where all $\ominus$-events have been introduced in G as described in Definition 33, and (ii) its receptive-closed map $\sigma^- : H \to A$, if $s_e^x \in E_H$ and $pol(e) = -$ then, because of the definition of receptive-closed maps, every configuration $\sigma^- x$ is both $\ominus$-maximal and $\oplus$-maximal in G as well as $\oplus$-maximal in H—a fact useful to show that:

Lemma 34. *Let $\sigma : G \to A$ be a pre-receptive map of event structures. The receptive-closed map $\sigma^- : H \to A$ of σ is a strategy.*

Proof. Since $\sigma' : G \to (A \upharpoonright \sigma G)$ is innocent, σ is innocent. And since $\leq_H$ does not introduce causal dependencies nor inconsistencies (with respect to G) beyond those already in G and those given by A then σ^- is innocent.

More precisely, take any pair of events in E_H. There are three possibilities depending on whether they are in E_G or $E_H \setminus E_G$. If (i) both events belong to E_G then causal dependency is that given by $\leq_G$; if (ii) both events belong to $E_H \setminus E_G$ then causal dependency is that given by $\leq_A$; finally, if (iii) they belong to different sets, then $\leq_H$ is restricted either by that of $\leq_G$ or by that of $\leq_A$. For similar reasons, inconsistencies that violate innocence are not introduced either.

Now, see that σ^- is receptive. Since $E_H \setminus E_G$ is

$$\bigcup_{x \in \mathcal{C}(G), y \in \mathcal{C}(A)} \{ s_e^x \mid x \text{ is maximal in } G \;\&\; \sigma x \subset^- y \;\&\; e \in y \setminus \sigma x \}$$

then it follows that

$$\sigma^- x' \xrightarrow{e}\!\subset \;\&\; pol(e) = - \Longrightarrow \exists s_e^x \in E_H.\ x' \xrightarrow{s_e^x}\!\subset \;\&\; \sigma^-(s_e^x) = e$$

for at least one $s_e^x \in E_H$, with $x \subseteq^- x'$; note that the case when $(x' \cup \{e\}) \in \mathcal{C}(G)$ need not to be checked because σ' is a strategy, thus a receptive map.

Suppose, in order to get a contradiction, that s_e^x is not unique. Then there exist a configuration x', a $\ominus$-event e, and (at least) two different $\ominus$-events $s_e^{x_1}$ and $s_e^{x_2}$ such that $\sigma^- x' \overset{e}{\longrightarrow\!\!\subset}$, $x' \overset{s_e^{x_1}}{\longrightarrow\!\!\subset}$, $x' \overset{s_e^{x_2}}{\longrightarrow\!\!\subset}$, and $\sigma^-(s_e^{x_1}) = e = \sigma^-(s_e^{x_2})$, with $x_1 \subseteq^- x'$ and $x_2 \subseteq^- x'$. Because the configurations x_1 and x_2 are maximal in G then it follows that $x' = x_1 = x_2$, otherwise x' would not be consistent. But then, we have that $s_e^{x_1} = s_e^{x_2}$, which leads to a contradiction; therefore the $\ominus$-event s_e^x is unique and σ^- is receptive too—thus a strategy. $\square$

Whereas Definition 33 (via Lemma 34) ensures that certain maps of event structures are strategies, Definition 35 ensures that such maps are defensive.

Definition 35. *Let $\sigma^k : G \to A$ be a strategy in an event structure A. The map $\sigma^{k+1} : H \to A$ is a successor map of σ^k if H is an event structure whose set of events comprises the events of G as well as those in the set*

$$\bigcup_{x \in \mathcal{C}(G)} \{ s_e^x \mid \exists y \in \mathcal{C}(A).\ \ \begin{array}{l} x \textit{ is } \oplus\textit{-maximal in } G \ \& \\ \sigma^k x \in L \ \& \ \sigma^k x \subset^+ y \ \& \ y \in W \ \& \ e \in y \setminus \sigma^k x \} \end{array}$$

where

$$s_{e_1}^{x_1} = s_{e_2}^{x_2} \Leftrightarrow (x_1 = x_2 \ \& \ e_1 = e_2)$$

and such that

$$\sigma^{k+1}(s_e^x) = e.$$

Moreover, H has causal dependency relation given by:

1. *Case $e_1, e_2 \in E_G$ (causality inherited only from G): if $e_1 \leq_G e_2$ then $e_1 \leq_H e_2$;*
2. *Case $s_{e_1}^{x_1}, s_{e_2}^{x_2} \notin E_G$ (causality inherited only from A): if $(x_1 = x_2 \ \& \ e_1 \leq_A e_2)$ then $s_{e_1}^{x_1} \leq_H s_{e_2}^{x_2}$;*
3. *Case $e_1 \in E_G \ \& \ s_e^x \notin E_G$ (causality inherited from both A and G): if $((e_1 \in x \ \& \ \sigma^k(e_1) \leq_A e)$ or $e_1 \in [Neg[x]])$ then $e_1 \leq_H s_e^x$*

and consistency relation:

$$X \in \mathrm{Con}_H \ \textit{iff} \ \begin{cases} \sigma^{k+1} X \in \mathrm{Con}_A \ \& \\ ([X] \cap E_G) \in \mathrm{Con}_G \ \& \\ \forall s_{e_1}^{x_1}, s_{e_2}^{x_2} \in (X \setminus E_G).\ s_{e_1}^{x_1} \neq s_{e_2}^{x_2} \Longrightarrow x_1 = x_2 \end{cases}$$

Lemma 36. *Let $\sigma^k : G \to A$ be a strategy in an event structure A. Then, any successor map $\sigma^{k+1} : H \to A$ of σ^k is pre-receptive in A.*

Proof. Observe that $Neg[E_G] = Neg[E_H]$ and $Pos[E_G] \subseteq Pos[E_H]$ because

$$\bigcup_{x \in \mathcal{C}(G)} \{ s_e^x \mid \exists y \in \mathcal{C}(A).\ \ \begin{array}{l} x \text{ is } \oplus\text{-maximal in } G \ \& \\ \sigma^k x \in L \ \& \ \sigma^k x \subset^+ y \ \& \ y \in W \ \& \\ e \in y \setminus \sigma^k x \} \end{array}$$

Since no $\ominus$-events are introduced and σ^k is a strategy then

$$\sigma^{k+1} : H \to A \upharpoonright (\sigma^k G \cup \{e \in E_A \mid pol(e) = +\})$$

is receptive.

Moreover, σ^{k+1} is innocent too: no extra causal dependencies to events in $Neg[E_G]$ are introduced, and any new inconsistencies between events with different polarity are those inherited from A; so σ^{k+1} is $\ominus$-*innocent*, i.e.

$$\forall e, e' \in H.\ e \rightarrowtriangle e' \ \&\ pol(e') = - \Longrightarrow \sigma^{k+1}(e) \rightarrowtriangle \sigma^{k+1}(e')$$

On the other hand, in order to show that σ^{k+1} is $\oplus$-*innocent*, i.e. that

$$\forall e, e' \in H.\ e \rightarrowtriangle e' \ \&\ pol(e) = + \Longrightarrow \sigma^{k+1}(e) \rightarrowtriangle \sigma^{k+1}(e'),$$

take any pair of events $e_1, e_2 \in E_H$ such that $e_1 \leq_H e_2$, with $pol(e_1) = +$. Thus:

1. if $e_1, e_2 \in E_G$ then causal dependency is given by $\leq_G$;
2. if $e_1, e_2 \in E_H \setminus E_G$ then causal dependency is given by $\leq_A$;
3. if $e_1 \in E_G$ and $e_2 \in E_H \setminus E_G$ then $e_2 = s_e^x$ and either

$$(e_1 \in x \ \&\ \sigma^k(e_1) \leq_A e) \text{ or } e_1 \in [Neg[x]];$$

   hence it must be the case that $e_1 \leq_H s_e^x$ too.

Note, additionally, that the case when $e_1 \in E_H$ and $e_2 \in E_H \setminus E_G$ cannot happen since $\leq_H$ respects both $\leq_A$ and $\leq_G$.

Then, σ^{k+1} is innocent and therefore pre-receptive in A. $\square$

Note that for every A there is a map $\sigma_0 : \varnothing \to A$ which is a pre-receptive strategy in A, since $\sigma_0' : \varnothing \to \varnothing$ $(= \sigma_0' : \varnothing \to (A \upharpoonright \sigma_0 \varnothing))$ is a strategy, i.e. a trivially innocent and receptive map of event structures. Then, due to Lemma 34, the receptive-closed map $\sigma_0^- : H \to A$ of σ_0 is a strategy. Let σ_0^- be σ^0. Note that σ^0 is the (necessarily unique) strategy which does nothing but be receptive to $\ominus$-moves reachable from $\varnothing$, the empty configuration of A.

Remark 37 (Construction of concurrent defensive strategies). Starting with $\varnothing$, the empty configuration of A, and by a repeated use of Definitions 33 and 35, one can build strategies in A which always try to avoid losing. In fact, they are winning strategies provided that all newly introduced $\oplus$-maximal configurations of H (when building the successor maps) are in W. $\square$

Definition 38 (Concurrent defensive maps)**.** *Let (A, W) be a concurrent game and $0 \leq n \leq \omega$. Then, a map of event structures $\sigma^k : S^k \to A$ is a concurrent defensive map if k is the largest number such that*

$$\begin{array}{lcl} S^0 & = & \{s_e^{\varnothing} \mid \varnothing \subset^- [e]\}, \\ S^{2n+1} & = & H, \textit{with } G = S^{2n} \textit{ according to Definition 35}, \\ S^{2n+2} & = & H, \textit{with } G = S^{2n+1} \textit{ according to Definition 33}. \end{array}$$

Based on Definition 38 and Lemmas 34 and 36 about strategies, the next result holds for the class of concurrent defensive maps just defined.

Theorem 39. *A concurrent defensive map is a strategy.*

Example 40 (Concurrent defensive strategies). Let (A, W) be a game with winning conditions where A consists of three events $\ominus_1, \oplus_2, \ominus_3$, all of them consistent with each other, and $W = \{\varnothing, \{\ominus_1, \oplus_2\}, \{\oplus_2, \ominus_3\}, \{\ominus_1, \oplus_2, \ominus_3\}\}$; therefore $L = \{\{\ominus_1\}, \{\ominus_3\}, \{\ominus_1, \ominus_3\}, \{\oplus_2\}\}$. In this game there is a winning strategy $\sigma : S \to A$ for Player which we can build as a concurrent defensive strategy using Definitions 33 and 35. The strategy σ is nondeterministic because S, depicted below, is a nondeterministic event structure:

$$
\begin{tikzcd}
s^{x_1}_{\oplus_2} \arrow[r, squiggly, no head] & s^{x_2}_{\oplus_2} \arrow[r, squiggly, no head] & s^{x_3}_{\oplus_2} \\
s^{\varnothing}_{\ominus_1} \arrow[u] \arrow[ur] & & s^{\varnothing}_{\ominus_3} \arrow[u] \arrow[ul]
\end{tikzcd}
$$

where $x_1 = \{s^{\varnothing}_{\ominus_1}\}$, $x_3 = \{s^{\varnothing}_{\ominus_3}\}$, $x_2 = \{s^{\varnothing}_{\ominus_1}, s^{\varnothing}_{\ominus_3}\}$, and S is the only possible total map of event structures; wavy lines are transitive inconsistency. $\square$