Concurrent Logic Games on Partial Orders

Julian Gutierrez

LFCS, School of Informatics, University of Edinburgh

Abstract. Most games for analysing concurrent systems are played on interleaving models, such as graphs or infinite trees. However, several concurrent systems have partial order models rather than interleaving ones. As a consequence, a potentially algorithmically undesirable translation from a partial order setting to an interleaving one is required before analysing them with traditional techniques. In order to address this problem, this paper studies a game played directly on partial orders and describes some of its algorithmic applications. The game provides a unified approach to system and property verification which applies to different decision problems and models of concurrency. Since this framework uses partial orders to give a uniform representation of concurrent systems, logical specifications, and problem descriptions, it is particularly suitable for reasoning about concurrent systems with partial order semantics, such as Petri nets or event structures. Two applications can be cast within this unified approach: bisimulation and model-checking.

1 Introduction

Games form a successful approach to giving semantics to logics and programming languages (semantic games) and to program verification (verification games). Good surveys of some of the most important game-based decision procedures and tools for property and systems verification can be found in [8, 16, 17], and in the references therein. These 'logic games' [5] usually are *sequential* and played on graphs or infinite trees. They offer an elegant approach to studying different properties of sequential processes and of concurrent systems with *interleaving* semantics, e.g., by using Kripke models or (labelled) transition graphs.

However, when dealing with concurrent systems with *partial order* semantics [15], such as Petri nets or event structures (which are semantically richer and more complex), the game-based techniques previously mentioned cannot be directly applied because the explicit notion of independence or *concurrency* in the partial order models is not considered. As a result, one has to construct the graph structures associated with those partial order models—a translation that one would like to avoid since, in many cases, it is algorithmically undesirable.

The reasons to wish to stay in a purely partial order setting are well-known by the concurrency theory community. For instance, partial order models of concurrency can be exponentially smaller than their interleaving counterparts; moreover they are amenable to partial order reductions [10] and are the natural input of the unfolding methods [7] for software and hardware verification—which work very well in practice whenever the systems have high degrees of parallelism.

Then, it is desirable, for several algorithmic reasons, to have a game which can be played directly on the partial order representations of concurrent systems. The main problem is that games played directly on 'noninterleaving structures' (which include Petri nets and event structures) are not known to be *determined* in the general case since they may well turn out to be of *imperfect* information, mainly, due to the information about *locality* and *independence* in such models.

In this paper we study a class of games played on noninterleaving structures (posets in our case) which is *sound* and *complete*, and therefore determined, without using stochastic strategies as traditional approaches to concurrent games [3, 4]. Our framework builds upon two ideas: firstly, the use of posets to give a *uniform* representation of concurrent systems with partial order semantics, logical specifications, and problem descriptions; and secondly, the restriction to games with a *semantic* condition that reduces reasoning on different models and decision problems to the analysis of simpler *local* correctness conditions.

The solution is realised by a new 'concurrent logic game' (CLG) which is shown to be determined—even though it is, locally, of imperfect information. The two players of the game are allowed to make *asynchronous* and *independent* local moves in the board where the game is played. Moreover, the elements of the game are all formalised in order-theoretic terms; as a result, this new model builds a bridge between some concepts in order theory and the more operational world of games. To the best of our knowledge, such an order-theoretic characterisation has not been previously investigated for verification games.

Then our main contribution is the formalization of a concurrent logic game model that generalises the results in [16] to a partial order setting, that is, the games by Stirling for bisimulation and model-checking on interleaving structures (and hence also related tableau-based techniques). The CLG model is inspired by a concurrent semantic game model (for a fragment of Linear Logic [9]) studied by Abramsky and Melliès [2]. However, the mathematics of the original game have been drastically reformulated in the quest towards the answer to algorithmic questions, and only a few technical features were kept.

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2 The Concurrent Game Model

We consider (infinite) logic games played by two players, Eve (∃) and Adam (∀), whose interaction can be used to represent the flow of information when analysing concurrent and distributed systems. The main idea is that by enriching a logic game with the explicit information about local and independent behaviour that comes with any partial order model, the traditional, sequential setting for logic games (usually played on interleaving structures) can be turned into a concurrent one on a partial order. This section studies a 'concurrent logic game' (CLG) played on a poset structure; a simple example is given in the appendix.

Preliminaries and Notations on Partial Orders. A $\bot_{\mathcal{A}}$-*bounded* poset $\mathfrak{A} = (\mathcal{A}, \leq_{\mathcal{A}})$ is a partially ordered set with a *bottom* element $\bot_{\mathcal{A}}$ such that for all $a \in \mathcal{A}$ we have that $\bot_{\mathcal{A}} \leq_{\mathcal{A}} a$. For any $a \in \mathcal{A}$, a *successor* of a is an element a' such that $a <_{\mathcal{A}} a'$ and for all b if $a \leq_{\mathcal{A}} b$ and $b \leq_{\mathcal{A}} a'$ then either $a = b$ or $b = a'$. Write $a \rightarrow a'$ iff a' is a successor of a and call a a *terminal* element iff $a \not\rightarrow$. Given a, a (principal) *ideal* $\downarrow a$ is the downward-closed set $\{b \in \mathcal{A} \mid b \leq_{\mathcal{A}} a\}$; dually, a (principal) *filter* $\uparrow a$ of $\mathcal{A}$ is the upward-closed $\{b \in \mathcal{A} \mid a \leq_{\mathcal{A}} b\}$. Also, for any set $A \subseteq \mathcal{A}$, write $\downarrow A$ for the set $\bigcup_{a \in A}\{b \mid b \in \downarrow a\}$, and likewise, $\uparrow A$ for $\bigcup_{a \in A}\{b \mid b \in \uparrow a\}$; call $\downarrow A$ a *lower* subset and $\uparrow A$ an *upper* subset. We write $\downarrow a$ for the induced poset $(\downarrow a, \leq_{\mathcal{A}})$, and similarly for $\uparrow a$, $\downarrow A$, and $\uparrow A$. Clearly the posets $\downarrow a$ and $\uparrow a$ are $\bot$-bounded if $\mathfrak{A}$ is $\bot$-bounded, since $\bot_{\downarrow a} = \bot_{\mathcal{A}}$ in the former case and $\bot_{\uparrow a} = a$ in the latter. Finally, a function $f : \mathcal{A} \rightarrow \mathcal{A}$ is a *closure operator* iff it is extensive, monotonic, and idempotent, i.e., if satisfies that for all $a, a' \in \mathcal{A}$: $a \leq_{\mathcal{A}} f(a)$; $a \leq_{\mathcal{A}} a'$ implies $f(a) \leq_{\mathcal{A}} f(a')$; and $f(a) = f(f(a))$.

Boards. A board in a CLG is a $\bot$-bounded, well-founded poset $\mathfrak{D} = (\mathcal{D}, \leq_{\mathcal{D}})$. A lower (resp. an upper) sub-board $\mathfrak{B}$ of $\mathfrak{D}$ is a poset $(\mathcal{B}, \leq_{\mathcal{D}})$ such that $\mathcal{B}$ is a lower (resp. an upper) subset of $\mathcal{D}$. Then, a lower sub-board is a $\bot$-bounded poset and an upper sub-board is a union of possibly infinitely large $\bot$-bounded posets; as $\mathfrak{D}$ is well-founded, then all lower sub-boards are also well-founded. We only consider posets (boards) where every chain has a maximal element. Moreover, a *global* position in $\mathfrak{D}$ is an anti-chain $D \subseteq \mathcal{D}$; the *initial* global position of $\mathfrak{D}$ is $\{\bot\}$. Finally, given a global position D of $\mathfrak{D}$, call any $d \in D$ a *local* position.

Notation 1 *Given any $d \in \mathcal{D}$, write $d^{\leftarrow}$ for the set of local positions $\{e \mid e \rightarrow d\}$ and $d^{\rightarrow}$ for the set $\{d' \mid d \rightarrow d'\}$. The sets $d^{\leftarrow}$ and $d^{\rightarrow}$ are, respectively, the 'preset' and 'postset' of local positions of d. Also, let $\texttt{SP}(d)$ be the predicate that holds iff $\mid d^{\leftarrow} \mid > 1$, and call d a 'synchronization point' in such a case.*

Now, let $\nabla : \mathcal{D} \rightarrow \Upsilon$ be a partial function that assigns players in $\Upsilon = \{\exists, \forall\}$ to local positions. More precisely, ∇ is a total function on the set $\mathcal{B} \subseteq \mathcal{D}$ of elements that are not synchronization points—i.e., $\mathcal{B} = \{d \in \mathcal{D} \mid \neg\texttt{SP}(d)\}$; call the pair $(\mathfrak{D}, \nabla)$ a *polarised* board. In the following we only consider polarised boards with the following property (called 'dsync'), which ensures that the behaviour at synchronization points is deterministic: $\texttt{SP}(d) \Rightarrow |d^{\rightarrow}| = 1$ and $\forall e \in d^{\leftarrow}.|e^{\rightarrow}| = 1$.

This property, i.e., *dsync*, induces a correctness condition when playing the game. It ensures: firstly, that there are no choices to make in synchronization points (as they are not assigned by ∇ to any player); and secondly, that as a synchronization point does not share its preset with any other local position, then local behaviour in the game, which is formally defined later, is truly independent.

Strategies. In a CLG a strategy can be local or global. A *local* strategy $\lambda : \mathcal{D} \rightarrow \mathcal{D}$ is a closure operator partially defined on a board $\mathfrak{D} = (\mathcal{D}, \leq_{\mathcal{D}})$. Being partially defined on $\mathfrak{D}$ means that the properties of closure operators are restricted to those elements where the closure operator is defined. In particular, the relation $a \leq_{\mathcal{D}} a' \Rightarrow \lambda(a) \leq_{\mathcal{D}} \lambda(a')$ holds iff λ is defined in a and a'. Let $\texttt{dfn}(\lambda, d)$ be the predicate that holds iff $\lambda(d)$ is defined or evaluates to false otherwise. The predicate $\texttt{dfn}$ can be defined from a board, for any local strategy, by means of three rules that realise local strategies $\lambda_{\forall}$ for Adam and $\lambda_{\exists}$ for Eve.

Definition 1 (Local Strategies). *Given a board* $\mathfrak{D} = (\mathcal{D}, \leq_{\mathcal{D}})$*, a 'local strategy'* $\lambda_\forall$ *for Adam (resp.* $\lambda_\exists$ *for Eve) is a closure operator defined only in those elements of* $\mathcal{D}$ *given by the following rules:*

1. *The local strategy* $\lambda_\forall$ *(resp.* $\lambda_\exists$*) is defined in the bottom element* $\bot_{\mathcal{D}}$*.*
2. *If* $\lambda_\forall$ *(resp.* $\lambda_\exists$*) is defined in* $d \in \mathcal{D}$*, and either* $\nabla(d) = \exists$ *(resp.* $\nabla(d) = \forall$*) or* $\mathtt{SP}(d)$ *or* $d\not\rightarrow$ *or* $d \rightarrow e \wedge \mathtt{SP}(e)$ *holds, then for all* $d' \in d^{\rightarrow}$ *we have that* $\lambda_\forall$ *(resp.* $\lambda_\exists$*) is defined in* d' *as well.*
3. *If* $\lambda_\forall$ *(resp.* $\lambda_\exists$*) is defined in* $d \in \mathcal{D}$*, and both* $\nabla(d) = \forall$ *(resp.* $\nabla(d) = \exists$*) and* $|d^{\rightarrow}| \geq 1$ *hold, then there exists a* $d' \in d^{\rightarrow}$ *in which* $\lambda_\forall$ *(resp.* $\lambda_\exists$*) is defined.*

Let $\mathtt{dfn}(\lambda_\forall, d)$ *be the predicate that holds iff* $\lambda_\forall(d)$ *is defined, and likewise for* $\lambda_\exists$*. Moreover, the closed elements, i.e., the fixpoints, of* $\lambda_\forall$ *and* $\lambda_\exists$ *are as follows:*

$$\lambda_\forall(d) = d \text{ iff } \nabla(d) = \exists, \text{ or } \mathtt{SP}(d), \text{ or } d\not\rightarrow, \text{ or } d \rightarrow e \wedge \mathtt{SP}(e)$$
$$\lambda_\exists(d) = d \text{ iff } \nabla(d) = \forall, \text{ or } \mathtt{SP}(d), \text{ or } d\not\rightarrow, \text{ or } d \rightarrow e \wedge \mathtt{SP}(e)$$

provided that the predicates $\mathtt{dfn}(\lambda_\forall, d)$ *and* $\mathtt{dfn}(\lambda_\exists, d)$ *hold. Moreover, let* $\lambda_\forall^1$ *and* $\lambda_\exists^1$ *be the 'identity local strategies' of Adam and Eve, respectively, which are defined everywhere in* $\mathfrak{D}$*; thus, formally:* $\lambda_\forall^1(d) = \lambda_\exists^1(d) = d$*, for all* $d \in \mathcal{D}$*.*

Let $\Lambda_{\mathfrak{D}}$ be the set of local strategies on $\mathfrak{D}$, which can be split in two subsets, i.e., $\Lambda_{\mathfrak{D}} = \Lambda_{\mathfrak{D}}^{\exists} \uplus \Lambda_{\mathfrak{D}}^{\forall}$, for Eve and Adam. Informally, Definition 1 says that a local strategy must be able (item 2) to reply to all 'counter-strategies' defined in the same local position, and (item 3) to choose a next local position whenever used. Moreover, item 3 of Definition 1 implies that in order for Eve and Adam to play concurrently, they have to follow a set of local strategies rather than only one.

Definition 1 also characterises the fixpoints of local strategies. Note that a *fixpoint* of a local strategy is a position in the board where a player cannot make a choice, either because it is the other player's turn (e.g., $\nabla(d) = \exists$ for $\lambda_\forall(d)$), or a synchronization must be performed ($\mathtt{SP}(d)$ or $d \rightarrow e \wedge \mathtt{SP}(e)$), or a terminal element is reached ($d\not\rightarrow$), and hence, there are no next local positions to play.

Remark 1. The intuitions as to why a closure operator captures the behaviour in a CLG follow [2]. As boards are *acyclic* ordered structures, there is no reason to move to a previous position and hence strategies should be extensive. They should also be monotonic in order to preserve the *causality* of moves in the game and idempotent to avoid *unnecessary* alternations between sequential steps.

When playing, Eve and Adam will use a set of local strategies $\Lambda_{\daleth}^{\exists} \subseteq \Lambda_{\mathfrak{D}}^{\exists}$ and $\Lambda_{\daleth}^{\forall} \subseteq \Lambda_{\mathfrak{D}}^{\forall}$, whose elements (i.e., local strategies) are indexed by the elements i and j of two sets $\mathbb{K}_\exists = \{1, ..., |\Lambda_{\mathfrak{D}}^{\exists}|\}$ and $\mathbb{K}_\forall = \{1, ..., |\Lambda_{\mathfrak{D}}^{\forall}|\}$; by definition, the identity local strategies are indexed with $i = 1$ and $j = 1$. Moreover, at the beginning of the game Eve and Adam choose (independently and at the same time) two sets of indices $\mathrm{K}_\exists \subseteq \mathbb{K}_\exists$ and $\mathrm{K}_\forall \subseteq \mathbb{K}_\forall$, and consequently the two sets of local strategies $\Lambda_{\daleth}^{\exists} \subseteq \Lambda_{\mathfrak{D}}^{\exists}$ and $\Lambda_{\daleth}^{\forall} \subseteq \Lambda_{\mathfrak{D}}^{\forall}$ they will use to play. This means that both $i \in \mathrm{K}_\exists$ iff $\lambda_\exists^i \in \Lambda_{\daleth}^{\exists}$ and $j \in \mathrm{K}_\forall$ iff $\lambda_\forall^j \in \Lambda_{\daleth}^{\forall}$; by definition, $\lambda_\exists^1$ and $\lambda_\forall^1$ are always included in $\Lambda_{\daleth}^{\exists}$ and $\Lambda_{\daleth}^{\forall}$. Based on this selection of local strategies one can define the sets of global strategies, reachable positions, and moves in the game.

Global strategies are interpreted in a poset of anti-chains since, by definition, a global position is an anti-chain of $\mathfrak{D}$. We will define $\mathbb{A} = (\mathcal{A}, \leq_{\mathcal{A}})$ to be such a suitable poset (a space of anti-chains), and call it the 'arena of global positions' of $\mathfrak{D}$. Then, the concept of arena is formalized in the following way:

Definition 2 (Arena of Global Positions). *Given a board $\mathfrak{D} = (\mathcal{D}, \leq_{\mathcal{D}})$, the poset $\mathbb{A} = (\mathcal{A}, \leq_{\mathcal{A}})$ is its 'arena of global positions', where $\mathcal{A}$ is the set of anti-chains of $\mathfrak{D}$ and $E \leq_{\mathcal{A}} D$ iff $\downarrow E \subseteq \downarrow D$, for all anti-chains of $\mathfrak{D}$.*

The reader acquainted with partial order models of concurrency, in particular with event structures, may have noticed that the poset of anti-chains defined here is similar to the domain of representative elements of a *prime* event structure [15]—and therefore also to the set of states or markings in a *safe* Petri net.

Definition 3 (Global Strategies). *Let $\mathfrak{D} = (\mathcal{D}, \leq_{\mathcal{D}})$ be a board. Given two subsets of indices $\mathrm{K}_\forall$ of $\mathbb{K}_\forall$ and $\mathrm{K}_\exists$ of $\mathbb{K}_\exists$, and hence, two sets of local strategies $\Lambda^{\forall}_{\circlearrowleft}$ and $\Lambda^{\exists}_{\circlearrowleft}$ for Adam and Eve, let the closure operators $\partial_\forall : \mathcal{A} \rightarrow \mathcal{A}$ and $\partial_\exists : \mathcal{A} \rightarrow \mathcal{A}$ on the poset $\mathbb{A} = (\mathcal{A}, \leq_{\mathcal{A}})$, where $\mathbb{A}$ is the arena of global positions associated with $\mathfrak{D}$, be the 'global strategies' for Adam and Eve defined as follows:*

$$\partial_\forall(D) \stackrel{\text{def}}{=} \texttt{max} \bigcup_{d \in D, j \in \mathrm{K}_\forall} \{\lambda^j_\forall(d) \mid \texttt{dfn}(\lambda^j_\forall, d)\}$$
$$\partial_\exists(D) \stackrel{\text{def}}{=} \texttt{max} \bigcup_{d \in D, i \in \mathrm{K}_\exists} \{\lambda^i_\exists(d) \mid \texttt{dfn}(\lambda^i_\exists, d)\}$$

where $D \subseteq \mathcal{D}$ is a global position of $\mathfrak{D}$ (and therefore an element of $\mathcal{A}$) and `max` *is the 'maximal elements' set operation, which is defined as usual.*

The reasons why $\partial_\forall$ and $\partial_\exists$ are closure operators are as follows: extensiveness is given by the identity local strategies $\lambda^1_\forall$ and $\lambda^1_\exists$ and monotonicity and idempotency are inherited from that of local strategies and ensured by `max`. Note that `max` is needed for two reasons: because (1) a global position must be an anti-chain of local ones and (2) only representative elements of $\mathbb{A}$ should be considered.

Now, the dynamics of a game is given by the interaction between the players (together with an external environment Π which is enforced to be deterministic by dsync—the property on boards and synchronization points given before).

Definition 4 (Rounds and Composition of Strategies). *Let a '$(\exists \circ \forall)$-round' be a global step of the game such that if $D \subseteq \mathcal{D}$ is the current global position of the game, $\partial_\exists$ is the strategy of Eve, and $\partial_\forall$ is the strategy of Adam, then the game proceeds first to an intermediate global position $D_{\exists\circ\forall}$ such that:*

$$\begin{aligned} D_{\exists\circ\forall} &= (\partial_\exists \circ \partial_\forall)(D) \\ &= \texttt{max} \bigcup_{d \in D, i \in \mathrm{K}_\exists, j \in \mathrm{K}_\forall} \{(\lambda^i_\exists \circ \lambda^j_\forall)(d) \mid \texttt{dfn}(\lambda^j_\forall, d) \wedge \texttt{dfn}(\lambda^i_\exists, \lambda^j_\forall(d))\} \end{aligned}$$

and then to the next global position $D' = (D_{\exists\circ\forall} \setminus e^{\leftarrow}_{\texttt{SP}}) \bigcup e^{\rightarrow}_{\texttt{SP}}$, given by Π, where:

$$e^{\leftarrow}_{\texttt{SP}} = \bigcup_{e \in D^{\rightarrow}_{\exists\circ\forall}} \{u \in e^{\leftarrow} \mid \texttt{SP}(e) \wedge e^{\leftarrow} \subseteq D_{\exists\circ\forall}\}$$
$$e^{\rightarrow}_{\texttt{SP}} = \bigcup_{e \in D^{\rightarrow}_{\exists\circ\forall}} \{v \in e^{\rightarrow} \mid \texttt{SP}(e) \wedge e^{\leftarrow} \subseteq D_{\exists\circ\forall}\}$$

and call the transition from the global position D to D' a '$\circlearrowleft$-round' of the game.

This definition follows the intuition that in a logic game Eve must respond to any possible move of Adam; moreover, she has to do so in every local position.

Plays. The interaction between the strategies of Eve and Adam define a (possibly infinite) sequence of global positions $\{\bot\}, D_1, ..., D_k, ...$, and hence, a sequence of posets given by the union of the order ideals determined by each D_k. A play is any finite or infinite union of the elements of such posets. Formally, a play $\hbar = (\mathcal{H}, \leq_{\mathcal{D}})$ on a board $\mathfrak{D} = (\mathcal{D}, \leq_{\mathcal{D}})$ is a (possibly infinite) poset such that $\mathcal{H}$ is a downward-closed subset of $\mathcal{D}$. An example can be found in the appendix.

We say that a play can be finite or infinite, and closed or open; more precisely, a play is: *finite* iff all chains of $\hbar$ are finite; *infinite* iff $\hbar$ has at least one infinite chain; *closed* iff at least one of the terminal elements of $\mathfrak{D}$ is in $\mathcal{H}$; *open* iff none of the terminal elements of $\mathfrak{D}$ is in $\mathcal{H}$. This classification of plays is used in a further section to define in a concrete way what the winning sets of a game are. Since for any play $\{\bot\}, D_1, ..., D_k, ...$ the lower subset defined by a global position D_k always includes the lower subsets of all other global positions D_j such that $j < k$, then in a partial order setting any global position D determines a play $\hbar_D = (\mathcal{H}, \leq_{\mathcal{D}})$ on a board $\mathfrak{D} = (\mathcal{D}, \leq_{\mathcal{D}})$ as follows (and let Γ be the set of plays of a game): $\mathcal{H} = \bigcup\{e \in \downarrow d \mid d \in D\} = \bigcup_{d \in D}\{e \in \mathcal{D} \mid e \leq_{\mathcal{D}} d\}$.

Winning Sets and Strategies. The *winning* conditions are the rules that determine when a player has won a play and define the 'winning sets' for each player. Let $\mathcal{W} : \Gamma \to \Upsilon$ be a *partial* function that assigns a winner $\exists, \forall \in \Upsilon$ to a play $\hbar \in \Gamma$, and call it the winning conditions of a game. The winning sets are determined by those plays that contain a terminal element or represent infinite behaviour. On the other hand, *winning* strategies (which are global strategies) are defined as usual, i.e., as for games on graphs. Then, we have:

Definition 5. *$\eth = (\Upsilon, \mathfrak{D}, \Lambda_{\mathfrak{D}}, \nabla, \mathcal{W}, \Gamma)$ is a 'Concurrent Logic Game' (CLG), where $\Upsilon = \{\exists, \forall\}$ is the set of players, the $\bot$-bounded poset $\mathfrak{D} = (\mathcal{D}, \leq_{\mathcal{D}})$ is a board, $\Lambda_{\mathfrak{D}} = \Lambda_{\mathfrak{D}}^{\exists} \uplus \Lambda_{\mathfrak{D}}^{\forall}$ are two disjoint sets of local strategies, $\nabla : \mathcal{D} \to \Upsilon$ is a partial function that assigns players to local positions, and $\mathcal{W} : \Gamma \to \Upsilon$ is a function defined by the winning conditions of $\eth$ over its set of plays Γ.*

A CLG is played as follows: Eve and Adam start by choosing, independently, a set of local strategies. The selection of local strategies is done indirectly by choosing the sets of indices $\mathrm{K}_\forall \subseteq \mathbb{K}_\forall$ and $\mathrm{K}_\exists \subseteq \mathbb{K}_\exists$. The only restriction (which we call '$\forall/\exists$-progress') when choosing the local strategies is that the resulting global strategy ∂, for either player, must preserve joins in the following way:

$$\forall d \in (\uparrow D \cap \downarrow \partial_\forall(D)), \text{ if } \mathtt{BP}(d) \wedge \nabla(d) = \forall \text{ then}$$
$$\forall a, b \in d^{\rightarrow}.\ \mathtt{sync}(a, b) \text{ implies } a, b \in \downarrow \partial_\forall(D).$$

and likewise for Eve, changing $\partial_\forall$ for $\partial_\exists$ and the polarity given by ∇. The predicates BP and sync characterise, respectively, the 'branching points' of a poset and pairs of elements that belong to chains that synchronize; their definitions are: $\mathtt{BP}(d)$ iff $\mid d^{\rightarrow} \mid > 1$ and $\mathtt{sync}(a, b)$ iff $\uparrow a \cap \uparrow b \neq U$, for $U \in \{\emptyset, \uparrow a, \uparrow b\}$. This restriction—which avoids the undesired generation of trivial open plays where nobody wins—is necessary because a synchronization point can be played iff all elements of its preset have been played. Thus, this is a correctness condition.

3 Closure Properties

At least three closure properties are interesting: under dual games, under lower sub-boards, and its order dual, under upper sub-boards. But, before presenting the closures, let us give a simple, though rather useful, technical lemma, which helps ensure that in some sub-boards a number of functionals are preserved.

Lemma 1 (Unique Poset Prefixes). *Let D be a global position of a board $\mathfrak{D}$. There is a unique poset representing all plays containing D up to such position.*

Lemma 1 facilitates reasoning on CLG on posets as it implies that regardless of which strategies the players are using, if a global position D appears in different plays, then the 'poset prefixes' of all such plays, up to D, are isomorphic. Let us now study some of the closure properties the CLG model enjoys. Given a CLG $\eth$ played on a board $\mathfrak{D}$, let $\eth \Downarrow_{\mathfrak{B}}$ be the CLG defined from $\eth$ where $\mathfrak{B}$ is a sub-board of $\mathfrak{D}$ and the other components in $\eth$ are restricted to $\mathfrak{B}$.

Lemma 2 (Closure Under Filters). *Let $\eth$ be a CLG and D a global position of the board $\mathfrak{D}$ of $\eth$. The structure $\eth \Downarrow_{\mathfrak{B}} = (\Upsilon, \bot \oplus \mathfrak{B}, \Lambda_{\mathfrak{B}}, \nabla \Downarrow_{\mathfrak{B}}, \mathcal{W} \Downarrow_{\mathfrak{B}}, \Gamma \Downarrow_{\mathfrak{B}})$ is also a CLG where $\mathfrak{B}$ is the upper sub-board of $\mathfrak{D}$ defined by D.*

Where $\oplus$ is the 'linear sum' operator on posets. The order dual of this closure property is a closure under countable unions of (principal) ideals.

Lemma 3 (Closure Under Ideals). *Let $\eth$ be a CLG and D a global position of the board $\mathfrak{D}$ of $\eth$. The structure $\eth \Downarrow_{\mathfrak{B}} = (\Upsilon, \mathfrak{B}, \Lambda_{\mathfrak{B}}, \nabla \Downarrow_{\mathfrak{B}}, \mathcal{W} \Downarrow_{\mathfrak{B}}, \Gamma \Downarrow_{\mathfrak{B}})$ is also a CLG where $\mathfrak{B}$ is the lower sub-board of $\mathfrak{D}$ defined by D.*

Remark 2. Lemmas 2 and 3 show that the filters of $\mathfrak{D}$ define the 'subgames' of $\eth$; also, that the ideals in $\mathfrak{D}$ can define a subset of the set of plays of Γ. Moreover, notice that games on infinite trees can be reduced to the particular case when D is always a singleton set and where two chains in the board never synchronise.

Since CLG will be used for verification, another useful feature is that of having a game closed under *dual* games, this is, a game used to check the dual of a given property over the same board—i.e., for the same system(s). Formally:

Definition 6 (Dual Games). *Let $\eth = (\Upsilon, \mathfrak{D}, \Lambda_{\mathfrak{D}}, \nabla, \mathcal{W}, \Gamma)$ be a CLG. The dual game $\eth^{op}$ of $\eth$ is $(\Upsilon, \mathfrak{D}, \Lambda_{\mathfrak{D}}, \nabla^{op}, \mathcal{W}^{op}, \Gamma)$, such that for all $d \in \mathcal{D}$ and $\hbar \in \Gamma$:*

- *if $\nabla(d) = \exists$ (resp. $\forall$) then $\nabla^{op}(d) = \forall$ (resp. $\exists$), and*
- *if $\mathcal{W}(\hbar) = \exists$ (resp. $\forall$) then $\mathcal{W}^{op}(\hbar) = \forall$ (resp. $\exists$).*

Moreover, let $\mathfrak{J}$ be a class of CLG where for all $\eth \in \mathfrak{J}$ there is a dual game $\eth^{op} \in \mathfrak{J}$. Then, we say that $\mathfrak{J}$ is closed under dual games.

Lemma 4 (Closure Under Dual Games). *Let $\mathfrak{J}$ be a class of CLG closed under dual games. If Eve (resp. Adam) has a winning strategy in $\eth \in \mathfrak{J}$, then Adam (resp. Eve) has a winning strategy in the dual game $\eth^{op} \in \mathfrak{J}$.*

Lemma 4 does not imply that CLG are determined as the existence of winning strategies has not yet been ensured; let alone the guarantee that finite and open plays in which $D \rightarrow D'$ and $D = D'$ hold are not possible, as this implies that the game is undetermined. Call 'stable' a play where $D \rightarrow D'$ and $D = D'$ hold.

Another condition that is necessary, though not sufficient, for a game to be determined is that all plays that are not finite and open have a winner. This is ensured by requiring $\mathcal{W}$ to be *complete*. We say that $\mathcal{W}$ is complete iff it is a total function on the subset of plays in Γ that are not finite and open.

Lemma 5 (Unique Winner). *Let $\mathfrak{J}$ be a class of CLG closed under dual games for which plays that are stable, finite and open do not occur. If the $\mathcal{W}$ in $\eth \in \mathfrak{J}$ is complete, then every play in $\eth$ and $\eth^{op}$ has a unique winner.*

Remark 3. If a class of games is closed under dual games and has a complete set of winning rules, then a proof of determinacy, which does not rely on Martin's theorem [14], can be given if the game is *sound* (where Adam is a correct falsifier). More importantly, games with these features must also be *complete* (where Eve is a correct verifier), and therefore *determined*. In this way one can reduce reasoning on games by building completeness and determinacy proofs almost for *free*!

Let us finish with a counter-example (Figure 1 in the appendix) that shows that CLG are undetermined. This motivates the definition of a semantic condition that, when satisfied, allows for the construction of a determined CLG.

Proposition 1. *CLG are undetermined in the general case.*

4 Metatheorems for Systems and Property Verification

As a CLG model can be seen as a logic game representation of a verification problem (cf. [5]), then let $\eth_{\mathrm{P}}$ be the CLG associated with a decision problem $\mathrm{V(P)}$, for a given problem P, and $\mathfrak{J}$ the class of CLG representing such a decision problem. We say that $\mathrm{V(P)}$ holds iff such a decision problem has a positive answer, and fails to hold otherwise; then $\mathrm{V(P)}$ is used as a logical predicate.

As usual, $\eth_{\mathrm{P}}$ is correct iff Eve (resp. Adam) has a winning strategy in $\eth_{\mathrm{P}}$ whenever $\mathrm{V(P)}$ holds (resp. fails to hold). Let a 'local configuration' of $\eth_{\mathrm{P}} \in \mathfrak{J}$ be a local position and a 'global configuration' an anti-chain of local positions. Moreover, a 'true/false configuration' is a configuration from which Eve/Adam can win. A global configuration is logically interpreted in a conjunctive way; then, it is true iff it only has true local configurations, and false otherwise.

In order to show the correctness of the family of games $\mathfrak{J}$, in this abstract setting, we need to make sure that the CLG $\eth_{\mathrm{P}}$ associated with a particular verification problem $\mathrm{V(P)}$ has some semantic properties, which are given next.

Definition 7 (Parity/co-Parity Condition). *Let $(A, \leq_A)$ be a poset indexed by a finite subset of $\mathbb{N}$, $\overrightarrow{a}$ be a sequence of elements of A whose order respects $\leq_A$ and downward-closure, and $f^{\omega}_{\mathtt{min}} : A^{\omega} \rightarrow \mathbb{N}$ be a function that characterizes the minimum index that appears infinitely often in $\overrightarrow{a}$. Then, $(\{b \in A \mid b \in \overrightarrow{a}\}, \leq_A)$ is a poset definable by a 'Parity/co-Parity condition' iff $f^{\omega}_{\mathtt{min}}(\overrightarrow{a})$ is even/odd.*

Property 1 (ω-Symmetry: bi-complete ω-regularity) *A family $\mathfrak{J}$ of CLG has Property 1 and is said to be ω-symmetric (bi-complete and ω-regular), iff:*

1. *$\mathfrak{J}$ is closed under dual games;*
2. *for all $\eth_{\mathrm{P}} \in \mathfrak{J}$ we have that $\eth_{\mathrm{P}}$ has a complete set of winning conditions;*
3. *the winning set given by those plays such that $\mathcal{W}(\hbar) = \exists$, i.e., those where Eve wins, is definable by Büchi/Rabin/Parity conditions.*[1]

An immediate consequence of the previous property is the following:

Lemma 6. *If a CLG $\eth_{\mathrm{P}}$ is ω-symmetric, then it also satisfies that the winning set given by those plays such that $\mathcal{W}(\hbar) = \forall$, i.e., those where Adam wins, is definable by co-Büchi/Streett/co-Parity conditions.*

Note that parts 1 and 3 of Property 1 are given by the particular problem to be solved. It is well known that several game characterisations of many verification problems have these two properties. On the other hand, part 2 is a design issue. From a more algorithmic viewpoint, Property 1 and Lemma 6 imply that:

Lemma 7. *The winning sets of Adam are least fixpoint definable; and dually, the winning sets of Eve are greatest fixpoint definable.*

Recall that Büchi and Rabin conditions can be reduced to a Parity one. Moreover, a Parity condition characterises the winning sets (and plays) in the fixpoint modal logic $\mathcal{L}_\mu$ [6] as follows: infinite plays where the smallest index that appears infinitely often is even/odd satisfy greatest/least fixpoints and belong to the winning sets of Eve/Adam. As in our setting plays are posets, the order is the one given by the board. The following semantic condition must hold too:

Property 2 (Local Correctness) *Let $\mathfrak{D}$ be the board of a CLG $\eth_{\mathrm{P}}$. If $d \in \mathcal{D}$ is a false configuration, then either $\nabla(d) = \exists$ and all next configurations are false as well or $\nabla(d) = \forall$. Dually, if $d \in \mathcal{D}$ is a true configuration, then either $\nabla(d) = \forall$ and all next configurations are true as well or $\nabla(d) = \exists$.*

The game interpretation of Property 2 reveals the mathematical property that makes a CLG logically correct. Property 2 implies that not only the local positions that belong to a player must be either true or false local configurations, but also those that belong to II, i.e., the joins of $\mathfrak{D}$. Then, truth and falsity must be transferred to those local positions as well, so that the statements "Eve must preserve falsity" and "Adam must preserve truth" hold. Formally, one needs to ensure that the following restriction (which we call '$\eth$-progress') holds:

$$\bigsqcup D \neq \emptyset \Rightarrow \bigsqcup_{d \in D} \partial_{\forall}(\{d\}) \neq \emptyset \qquad \text{and} \qquad \bigsqcup D \neq \emptyset \Rightarrow \bigsqcup_{d \in D} \partial_{\exists}(\{d\}) \neq \emptyset$$

where $\bigsqcup$ is the 'join operator' on posets; call 'live' a play that is not stable, finite and open, as well as games whose strategies only generate live plays. $\eth$-progress guarantees that only live plays and games—where truth and falsity are preserved—are generated. Moreover, it allow us to show the soundness and completeness of this kind of CLG. A simple technical lemma is still needed: a direct application of Lemma 5 using $\eth$-progress and Property 1 gives Lemma 8:

[1] Büchi and Rabin conditions are defined as expected [12] as well as their duals.

Lemma 8. *Every play of a live, ω-symmetric $\Game_{\mathrm{P}}$ has a unique winner.*

Theorem 1 (Soundness). *If* $\mathrm{V(P)}$ *fails to hold, Adam can always win* $\Game_{\mathrm{P}}$.

And, due to the properties of the game, we get completeness almost for free. Moreover, determinacy with *pure* winning strategies—a property not obvious for concurrent games—follows from the soundness and completeness results.

Theorem 2 (Completeness). *If* $\mathrm{V(P)}$ *holds, Eve can always win* $\Game_{\mathrm{P}}$.

Corollary 1 (Determinacy). *Eve has a winning strategy in* $\Game_{\mathrm{P}}$ *iff Adam does not have it, and vice versa.*

5 Algorithmic Applications and Further Work

Solving a CLG $\Game_{\mathrm{P}}$ using the approach we presented here requires the construction of a winning game $\Game_{\mathrm{P}} \Downarrow_{\mathfrak{B}}$ (and with it a winning strategy) for either player, according to Theorems 1 and 2. This is in general an undecidable problem since the board $\mathfrak{D}$ can be infinitely large. However, in many practical cases $\mathfrak{D}$ can be given a finite representation where all information needed to solve the verification problem is contained. Let us finish this section with the following result:

Theorem 3 (Decidability). *The winner of any CLG* $\Game_{\mathrm{P}}$ *can be decided in finite time if the board* $\mathfrak{D}$ *in* $\Game_{\mathrm{P}}$ *has finite size.*

Although Theorem 3 is not a surprising result, what is interesting is that several partial order models can be given a finite poset representation which, in a number of cases, can be *smaller* than their interleaving counterparts. Therefore, the decidability result may have important practical applications whenever the posets can be kept small. This opens up the possibility of defining new *concurrent* decision procedures for different verification problems.

Remark 4. The size of a board $\mathfrak{D}$ in a CLG model can be smaller than the unfolding $\mathfrak{U}$ of the interleaving structure representing the same problem. The exact difference, which can be exponential, depends on the degree of independence or concurrency in $\mathfrak{D}$, i.e., on the number of elements of those chains that are independent and therefore must be interleaved in order to get $\mathfrak{U}$ from $\mathfrak{D}$. Only experiments can tell if the CLG model can have an important practical impact.

Applications to Bisimulation and Model-Checking. The game framework described in this manuscript can be applied to solve, in a uniform way, the bisimulation and modal μ-calculus [6] model-checking problems. In particular, the induced decision procedures generalize those defined by Stirling in [16]. A detailed description of the two reductions can be found in [12] (Chapter 5).

Future Work. The work presented here can be extended in various ways. In particular, related to the author's previous work, we intend to study the expressivity and applicability of the CLG model with respect to logics and equivalences for so-called 'true concurrency', where interleaving approaches, either concurrent or sequential, are not semantically powerful enough. Results on this direction would allow us to give a concurrent alternative to the higher-order logic games for bisimulation and (local) model-checking previously presented in [11, 13].

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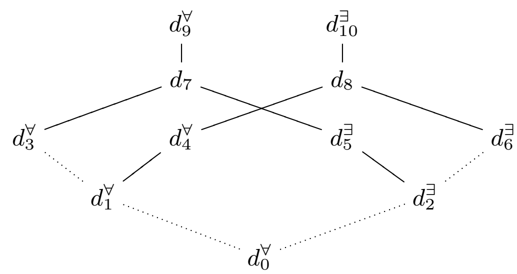
Appendix

All proofs and some examples can be found in Chapter 5 of the author's PhD thesis [12].[2] Here we present only some selected ones which closely relate with the main technical results in the paper. We also include the proof of the closure under dual games since this lemma is the key property that makes the proof of completeness of the game model rather short, even though we are working on a partial order setting. A simple example that illustrates how a CLG is played between Eve and Adam is also given in the proof of the following proposition.

Proposition 1. *CLG are undetermined in the general case.*

[2] This research was supported by an Overseas Research Studentship (ORS) award, a School of Informatics PhD scholarship of the University of Edinburgh, and the EPSRC Research Grant EP/G012962/1 'Solving Parity Games and Mu-Calculi'.



$\Game$-rounds: $\{d_0^\forall\} \to \{d_3^\forall, d_6^\exists\} \to \{d_3^\forall, d_6^\exists\} \to ...$

Fig. 1. Local positions are labelled with their polarities and the dotted lines are the player's moves. Here $\mathcal{W}(\downarrow D) = \exists$ iff $d_{10}^\exists \in D$ and $\mathcal{W}(\downarrow D) = \forall$ otherwise, for all global positions D containing a terminal element. The play is stable, finite and open.

Proof. Neither player can have a winning strategy in the game presented in Figure 1 since Eve and Adam can enforce plays for which $\mathcal{W}$ is not defined (a stable, finite and open play). Notice that $\forall/\exists$-progress is not violated. $\square$

Notice that the play presented in Figure 1 is the best that both players can do, since any other strategy they choose to play will lose against the strategy their opponent is currently playing in the example.

Lemma 4 (Closure Under Dual Games). *Let $\mathfrak{J}$ be a class of CLG closed under dual games. If Eve (resp. Adam) has a winning strategy in $\Game \in \mathfrak{J}$, then Adam (resp. Eve) has a winning strategy in the dual game $\Game^{op} \in \mathfrak{J}$.*

Proof. Suppose that Eve has a winning strategy ∂_W in $\Game$. Since for all global positions in the game $\Game$ one has that the next global position is initially defined by $\partial_W \circ \partial_\forall$, then whenever Adam has to make a move in $\Game^{op}$ he can use the winning strategy ∂_W of Eve because for all $d \in D$, if $\nabla(d) = \exists$ then we have that $\nabla(d)^{op} = \forall$. However, notice that in each local position of the game board Adam must always "play first" both in $\Game$ and in $\Game^{op}$ because the global evolution of the game, which is determined by the rounds being played, is always defined by pairs of local strategies $\lambda_\exists^i$ and $\lambda_\forall^j$ such that $\lambda_\exists^i \circ \lambda_\forall^j(d)$, for any local position d, regardless of whether we are playing $\Game$ or $\Game^{op}$.

So, there are actually two cases: firstly, consider those $d \in D$, for any global position D, such that $\nabla(d) = \exists$. In this case $\nabla^{op}(d) = \forall$ and then in $\Game^{op}$ Adam can simply play Eve's strategy in $\Game$ at position d. The second case is that of those $d \in D$ such that $\nabla(d) = \forall$. In this case, $\nabla^{op}(d) = \exists$ and hence Adam can play d itself, and let Eve decide on the new local position d', for which, by hypothesis, Eve has a winning strategy in $\Game$ and the two previous cases apply again, though in a new round of the game; moreover, the behaviour at synchronization points, which are played deterministically by the environment Π, remains as in $\Game$. In this way, Adam can enforce in $\Game^{op}$ all plays that Eve can enforce in $\Game$.

Finally, since for all such plays in Γ it was, by hypothesis, Eve who was the winner, then Adam is the winner in all plays in ∂^{op} as now for all $\hbar \in \Gamma$, one has that $\mathcal{W}^{op}(\hbar) = \forall$. The case when Adam has a winning strategy in ∂ is dual. □

Theorem 1 (Soundness). *If* $\mathrm{V(P)}$ *fails to hold, Adam can always win* ∂_{P}*.*

Proof. We show that Adam can win all plays of ∂_{P} if $\mathrm{V(P)}$ fails to hold by providing a winning strategy for him. The proof has two parts: first, we provide a board where Adam can always win and show how to construct a game on that board, in particular, the local strategies in the game—and hence, a strategy for Adam; then, we show that in such a game Adam can always win by checking that his strategy is indeed a *pure* winning strategy.

Let $\partial_{\mathrm{P}} \Downarrow_{\mathfrak{B}}$ be a CLG on a poset $\mathfrak{B} = (\mathcal{B}, \leq_{\mathcal{D}})$, which is a subset of $\mathfrak{D} = (\mathcal{D}, \leq_{\mathcal{D}})$, the initial board of the game. Let the set $\mathcal{B}$ be a downward-closed subset of $\mathcal{D}$ with respect to $\leq_{\mathcal{D}}$; the bottom element $\bot_{\mathcal{B}} = \bot_{\mathcal{D}}$ (where every play of the game starts) is, by hypothesis, a false configuration.

The construction of the board is as follows: $\mathfrak{B}$ contains only the winning choices for Adam (i.e., those that preserve falsity) as defined by the local correctness semantic property 2. After those elements of the poset have been selected, adjoin to them all possible responses or moves available to Eve that appear in $\mathfrak{D}$. Do this, starting from $\bot$, either infinitely often for infinite chains or until a terminal element is reached in finite chains. This construction clearly ensures that $\mathcal{B}$ is a downward-closed set with respect to $\leq_{\mathcal{D}}$. As in the proofs of Lemmas 2 and 3 (see [12] for further details), the polarity function ∇ for $\mathfrak{B}$ is as in $\mathfrak{D}$.

Using the constructions given in the proof of Lemma 2, one can define all other elements of $\partial_{\mathrm{P}} \Downarrow_{\mathfrak{B}}$. In particular, the local strategies for Eve and Adam will be 'stable' closure operators;[3] based on Definition 1 such stable closure operators are completely defined once one has determined what the 'output' functions will be (the local positions d' in item 3) since the fixpoints are completely determined already in Definition 1. Then, each local strategy $\lambda^j_\forall$ for Adam and $\lambda^i_\exists$ for Eve—where $j \in \mathrm{K}_\forall \subseteq \mathbb{K}_\forall$ and $i \in \mathrm{K}_\exists \subseteq \mathbb{K}_\exists$, respectively—is defined as follows:[4]

$$\lambda^i_\exists(d) = d \vee f^i_\exists(d)$$
$$\lambda^j_\forall(d) = d \vee g^j_\forall(d)$$

where:

$$\begin{array}{ll} \lambda^i_\exists(d) = d & \text{, if } \mathtt{fix}_\exists(\lambda^i_\exists, d) \\ \lambda^i_\exists(d) = f^i_\exists(d) & \text{, otherwise} \\ \lambda^j_\forall(d) = d & \text{, if } \mathtt{fix}_\forall(\lambda^j_\forall, d) \\ \lambda^j_\forall(d) = g^j_\forall(d) & \text{, otherwise} \end{array}$$

where each 'output' function $g^j_\forall$ necessarily preserves falsity and each output function $f^i_\exists$ must preserve truth (because $\mathfrak{B}$ was constructed taking into account Property 2). Moreover $\mathtt{fix}_\exists$ and $\mathtt{fix}_\forall$ are predicates that characterise the fixpoints of the local strategies for Eve and Adam in the following way:

[3] Good references for stable maps on posets are [1, 2] as well as some references therein.
[4] Recall that $i, j > 1$ as $\lambda^1_\forall$ and $\lambda^1_\exists$ are the identity local strategies of Adam and Eve.

$$\texttt{fix}_\exists(\lambda_\exists, d) \stackrel{\text{def}}{=} \texttt{dfn}(\lambda_\exists, d) \text{ and } (\ \nabla(d) = \forall, \text{ or } \texttt{SP}(d), \text{ or } d\not\rightarrow, \text{ or } d \rightarrow e \wedge \texttt{SP}(e)\ )$$
$$\texttt{fix}_\forall(\lambda_\forall, d) \stackrel{\text{def}}{=} \texttt{dfn}(\lambda_\forall, d) \text{ and } (\ \nabla(d) = \exists, \text{ or } \texttt{SP}(d), \text{ or } d\not\rightarrow, \text{ or } d \rightarrow e \wedge \texttt{SP}(e)\ )$$

where $d \in \mathcal{D}$, $\lambda_\forall \in \Lambda^\forall_{\mathfrak{D}}$, and $\lambda_\exists \in \Lambda^\exists_{\mathfrak{D}}$ in a game board $\mathfrak{D} = (\mathcal{D}, \leq_\mathcal{D})$.

However, since in $\mathfrak{B}$ all choices available to Eve were preserved, then the set of local strategies for Eve (i.e., $\Lambda^\exists_{\mathfrak{B}}$) can be safely chosen to be simply the same set of local strategies in $\mathfrak{D}$ (i.e., $\Lambda^\exists_{\mathfrak{D}}$); therefore, $\Lambda^\exists_{\mathfrak{B}} = \Lambda^\exists_{\mathfrak{D}}$ (because, formally, they play in $\mathfrak{D}$, even though Adam can prevent the positions in $\mathfrak{D} \setminus \mathfrak{B}$ to be reached) and $\Lambda^\forall_{\mathfrak{B}} \subseteq \Lambda^\forall_{\mathfrak{D}}$; moreover, the definition of global strategies immediately follows from this specification of local strategies as given by Definition 3 – of course, subject the restriction that any such global strategy must preserve the existence of joins in $\mathfrak{B}$ (progress restrictions). Finally, the sets of plays and winning conditions are defined from $\mathfrak{B}$ and the new sets of strategies as done in the proof of Lemma 2.

For the second part of this proof, let us show that the game $\eth_\mathrm{P} \Downarrow_{\mathfrak{B}}$ is winning for Adam, i.e., that he has a winning strategy. Then, let us analyse the outcome of plays to certify that he indeed wins all plays in such a game. First consider finite plays, which must be closed because all valid strategies must preserve the existence of joins. All such plays have a global position D_f which contains at least one local position that is a terminal element of $\mathfrak{B}$. Due to Property 1 (part 2), all those plays are effectively recognised as winning for one of the players, in this case for Adam: since $\perp$ is a false configuration, Eve must preserve falsity, and Adam is only playing strategies that also preserve falsity, then D_f contains at least one local position d_f which also is a false configuration, and therefore D_f is a false configuration as well since it is interpreted conjunctively. As a consequence all finite plays are winning for Adam. The same argument also applies for infinite, closed plays. The final case is that of open, infinite plays.

The correctness of this case is shown by a transfinite induction on a well-founded poset of sub-boards of $\mathfrak{B}$; this technique generalizes the analysis of approximants of fixpoints on interleaving structures (i.e., on total orders) to a partial ordered setting. So, let $(\mathcal{O}, \leq_\mathcal{O})$ be the following partial order on sub-boards (i.e., posets):

$$\mathcal{O} = \{\ \eth \Downarrow_{\uparrow D} \mid D \text{ is a global position of } \mathfrak{B}\}$$
$$\eth \Downarrow_{\uparrow D} \leq_\mathcal{O} \eth \Downarrow_{\uparrow D'} \text{ iff } \uparrow D' \subseteq \uparrow D$$

The relation $\leq_\mathcal{O}$ is clearly well-founded because all finite and infinite chains in the poset $(\mathcal{O}, \leq_\mathcal{O})$ have $\perp_\mathcal{O} = \eth \Downarrow_{\uparrow \perp_\mathcal{B}} = \eth \Downarrow_{\mathfrak{B}}$ as their bottom element. Since any particular play in the game corresponds to a chain of $(\mathcal{O}, \leq_\mathcal{O})$, then let us also define a valuation $[\![\cdot]\!] : \mathcal{O} \rightarrow \{\textbf{true},\textbf{false}\}$ and a total order on the sub-boards (i.e., posets), and therefore subgames, associated with $\mathfrak{B}$. Let $\hbar$ be any open, infinite play (an infinite chain of $(\mathcal{O}, \leq_\mathcal{O})$) and let $\alpha, \varpi \in \mathbb{O}\mathrm{rd}$ be two ordinals, where ϖ is a limit ordinal. Then:

$$\begin{aligned}
[\![\hbar^0]\!] &= [\![\perp_\mathcal{O}]\!] && \text{(the base case)}\\
[\![\hbar^{\alpha+1}]\!] &= [\![\rightarrow_\mathcal{O} (\hbar^\alpha)]\!] && \text{(the induction step)}\\
[\![\hbar^\varpi]\!] &= [\![\textstyle\bigcup_{\alpha<\varpi}(\hbar^\alpha)]\!] && \text{(because } \varpi \text{ is a limit ordinal)}
\end{aligned}$$

where $\rightarrow_\mathcal{O}$ is the accessibility relation of $\leq_\mathcal{O}$ restricted to the elements of the chain $\hbar$. Then, for Adam, we have the following:

$$
\begin{array}{lll}
[\![\bot_{\mathcal{O}}]\!] & = \textbf{false} & \text{(by hypothesis, } \bot_{\mathcal{O}} \text{ is a false configuration)} \\
[\![\rightarrow_{\mathcal{O}} (\hbar^{\alpha})]\!] & = [\![\hbar^{\alpha}]\!] & \text{(due to Property 2, } \rightarrow_{\mathcal{O}} \text{ preserves falsity)} \\
[\![\bigcup_{\alpha<\varpi}(\hbar^{\alpha})]\!] & = \bigvee_{\alpha<\varpi} [\![\hbar^{\alpha}]\!] & \text{(because due to Lemma 7,} \\
 & & \text{Adam's winning sets are least fixpoint definable)}
\end{array}
$$

Due to the principle of (transfinite) fixpoint induction, the result holds for all ordinals, and therefore for all global positions of any open, infinite play. Note that we can actually repeat this analysis for all ordinals $\beta < \alpha$ (and thus for all global positions), due to Property 1 (part 3), since winning configurations, and hence winning sets, are fixpoint definable. But, since ordinals are well-founded such a process of checking subgames and open, infinite plays always terminates regardless of which α one chooses. Hence, there can be neither a D nor a game $\Game_{\mathrm{P}} \Downarrow_{\bot_{\mathcal{B}} \oplus \uparrow D}$ where Eve wins.

As she cannot win any finite or infinite play in $\Game_{\mathrm{P}} \Downarrow_{\mathfrak{B}}$, and due to Lemma 8 all plays have a unique winner, Adam's strategy is indeed a winning strategy in $\Game_{\mathrm{P}}$; more precisely, it clearly is a *pure* winning strategy. Then, one can ensure that If V(P) fails to hold then Adam can win all plays of $\Game_{\mathrm{P}}$. □

A similar proof can be given to show the completeness of the game. Nevertheless, due to the properties of the game (notably, the closure under dual games), we can get the proof of completeness almost for free!

Theorem 2 (Completeness). *If* V(P) *holds, Eve can always win* $\Game_{\mathrm{P}}$.

Proof. Due to Property 1 (part 1) there exists a dual CLG $\Game_{\mathrm{P}}^{op}$ for the dual verification problem V(P^{op}) of V(P) such that V(P^{op}) does not hold. And, due to Theorem 1 Adam has a winning strategy in the game $\Game_{\mathrm{P}}^{op}$ for the dual problem P^{op}. Therefore, due to Lemma 4 and Lemma 8, Eve can use the local strategies of Adam in $\Game_{\mathrm{P}}^{op}$ to be the unique winner of all plays $\hbar \in \Gamma$ of $\Game_{\mathrm{P}}$, and hence the existence of a winning strategy for Eve in $\Game_{\mathrm{P}}$ follows. □

Theorem 3 (Decidability). *The winner of any CLG* $\Game_{\mathrm{P}}$ *can be decided in finite time if the board* $\mathfrak{D}$ *in* $\Game_{\mathrm{P}}$ *has finite size.*

Proof. Since $\mathfrak{D}$, by hypothesis, has finite size, then there are finitely many sub-boards $\mathfrak{B}$, and consequently, finitely many subgames $\Game_{\mathrm{P}} \Downarrow_{\mathfrak{B}}$ that must be checked before constructing a winning one for either player. Moreover, constructing a particular game $\Game_{\mathrm{P}} \Downarrow_{\mathfrak{B}}$ either for Eve or Adam as described in the proofs of Theorems 1 and 2 can be effectively done also because $\mathfrak{D}$ is finite, as follows.

Firstly, since $\mathfrak{B}$ is finite there are finitely many different strategies for Eve and Adam. Moreover, since those strategies are closure operators in a finite structure, then their sets of closed elements eventually stabilize. As a consequence, there are only finitely many possible different plays (and game configurations), whose winner can always be checked—because the game is determined and its set of winning conditions is complete. Therefore, a winning strategy can be chosen from the set of strategies of the game by exhaustively searching such a set, simply by comparing it against all possible strategies of the other player. As we assume that Properties 1 and 2 hold, they need not be verified. □