DPLL is Abstract Interpretation

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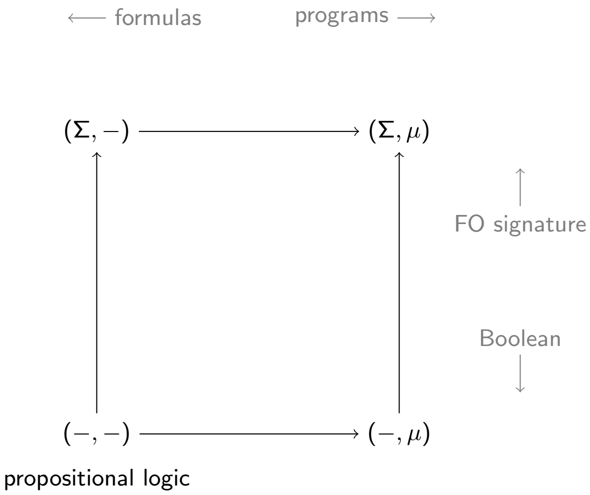
Analysing Logic and Programs



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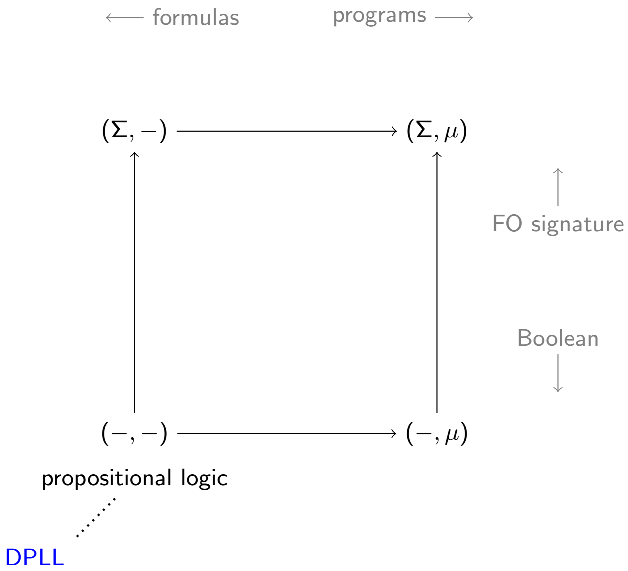

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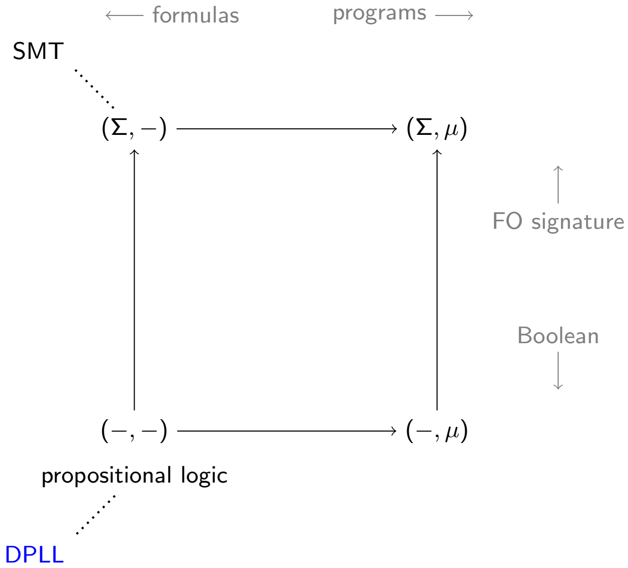

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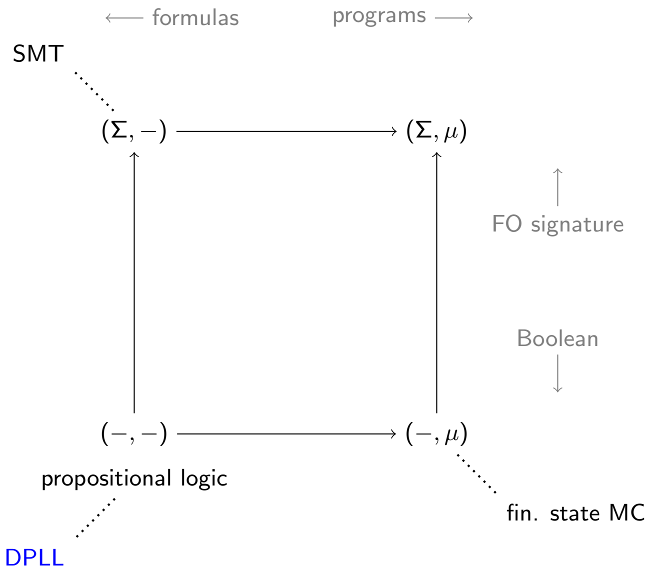

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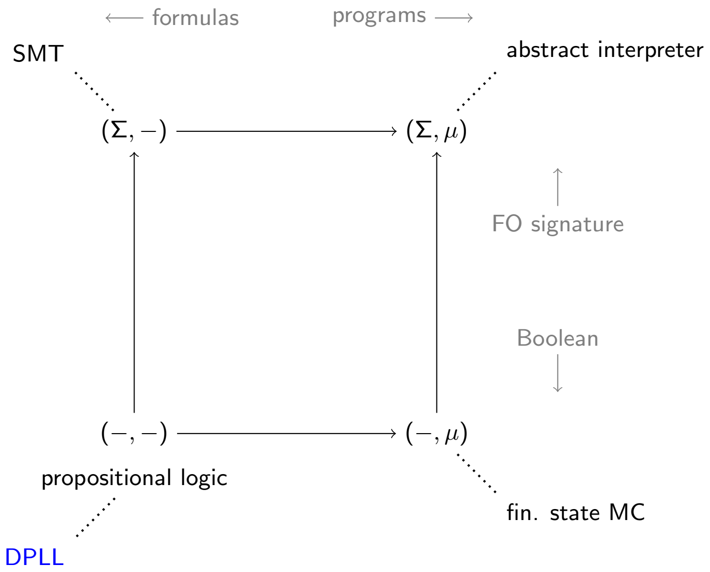
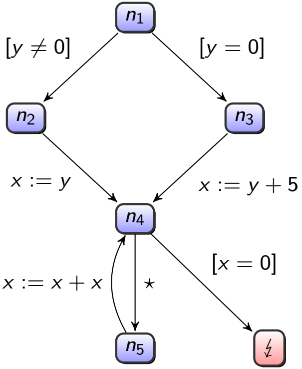


SAT Solvers are Efficient

(Malik and Zhang 2009)

Prove the following program safe using the domain of intervals:



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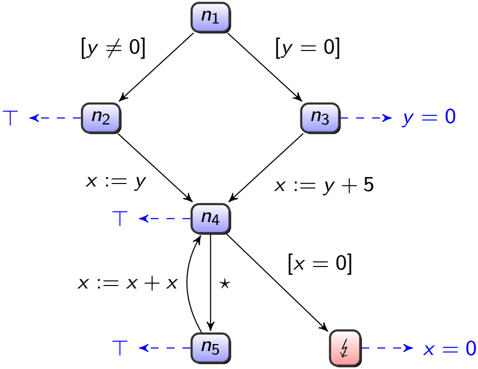
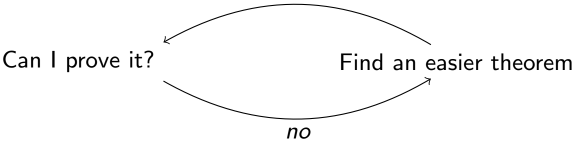


Flowchart for Proving a Theorem

Prove theorem T



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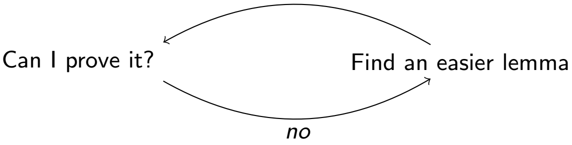

Flowchart for Proving a Theorem

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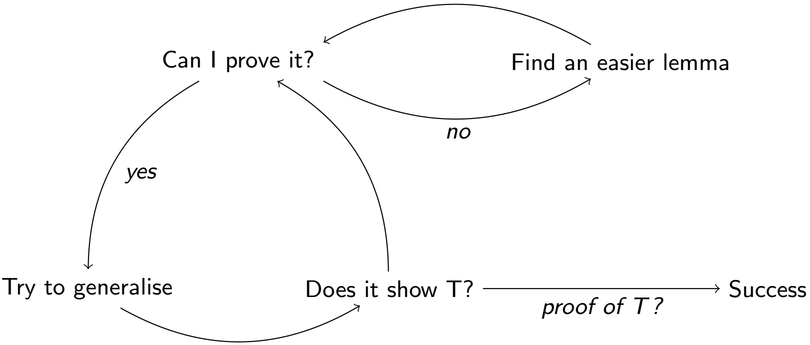
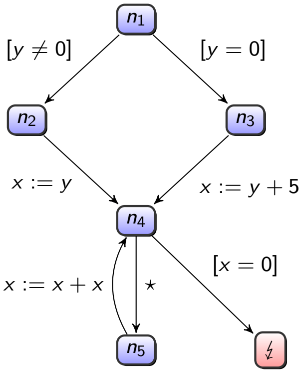


Refining the analysis

Let's try this on the program:



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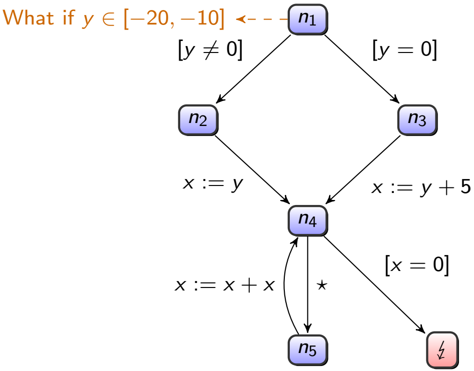

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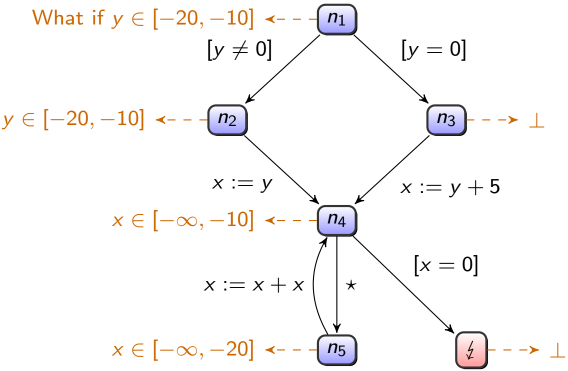

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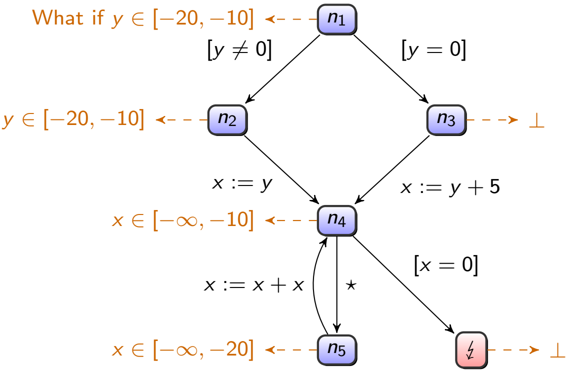

Analyse proof

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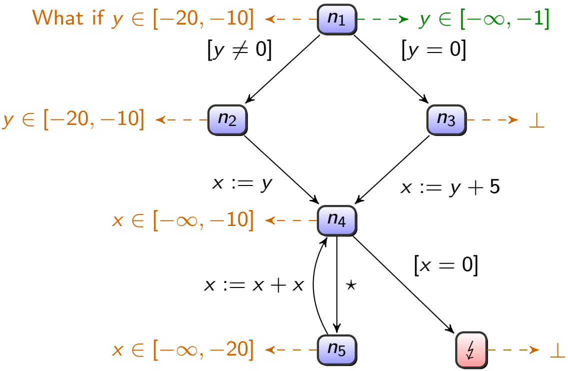

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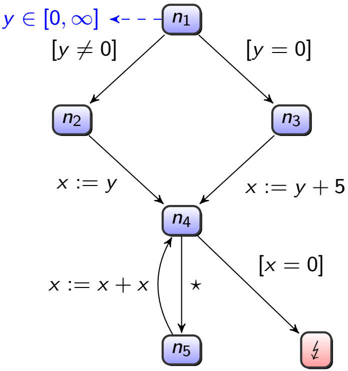

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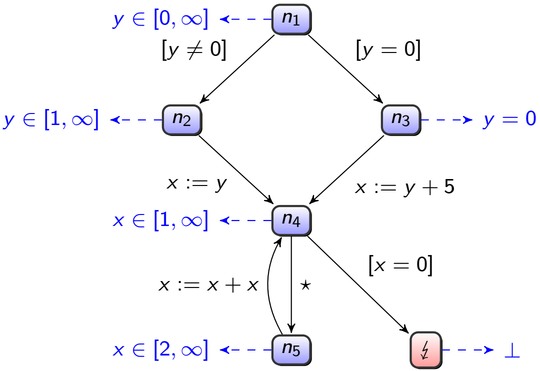
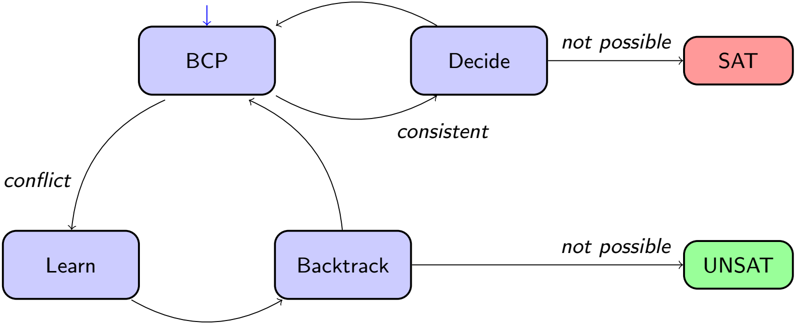


The Modern DPLL algorithm

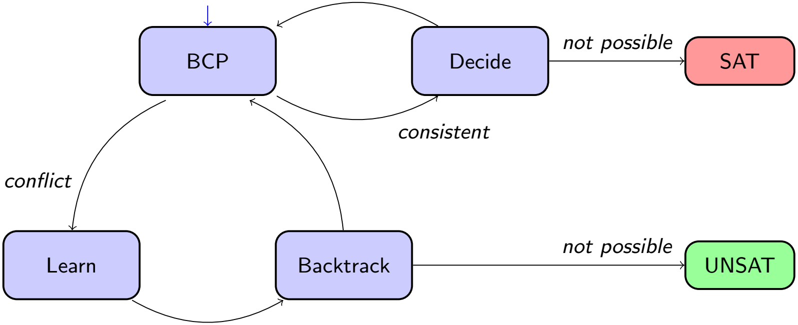
Decide satisfiability of propositional formula ϕ in conjunctive normal form (CNF)



Preliminaries

- Set of variables Var
- $v, \neg v$ are literals for $v \in$ Var
- Disjunction of literals is a clause $v_1 \vee \neg v_2 \vee v_3$
- CNF formula is a conjunction of clauses

The Modern DPLL algorithm



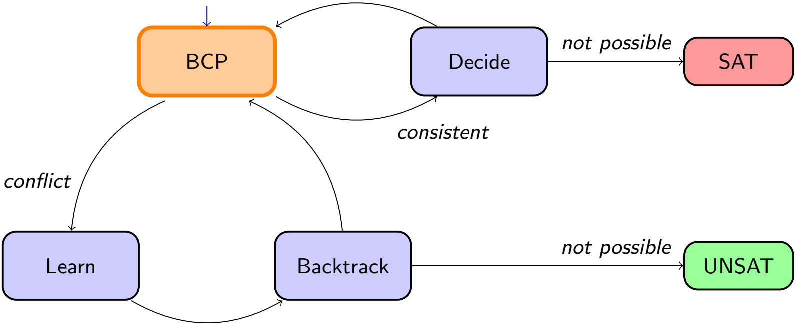
Partial assignments

SAT solver explores the space of partial functions:

$$\mathsf{Var} \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

Uses deduction and search to find a satisfying assignment or exhaustively search space.

The Modern DPLL algorithm

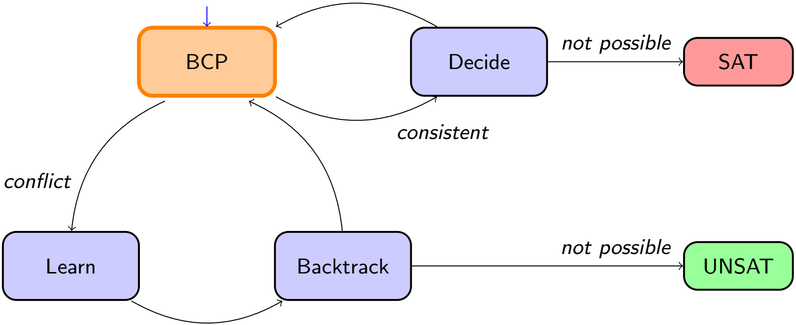


$$\phi = {\color{red}x} \wedge (\neg x \vee \neg y) \wedge (y \vee z \vee \neg w) \wedge (y \vee z \vee w)$$

BCP example

BCP: $\emptyset \rightarrow$

The Modern DPLL algorithm

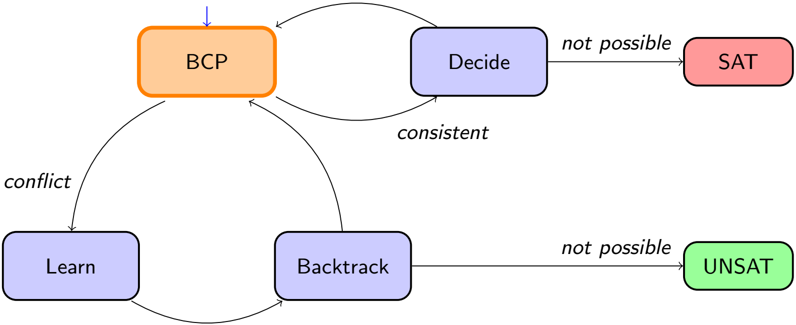


$$\phi = x \wedge (\neg x \vee \neg y) \wedge (y \vee z \vee \neg w) \wedge (y \vee z \vee w)$$

BCP example

BCP: $\emptyset \rightarrow \{x \mapsto \mathrm{t}\} \rightarrow$

The Modern DPLL algorithm



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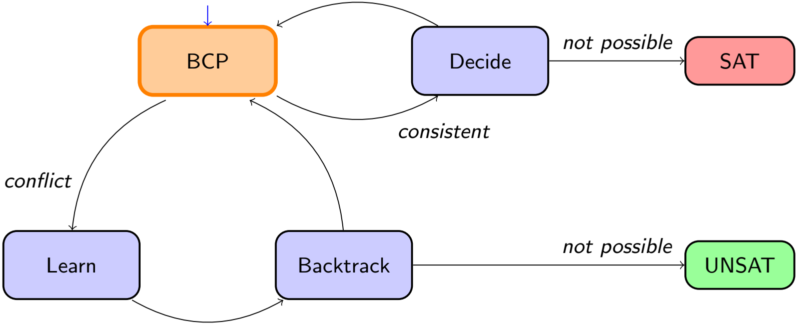
BCP example

BCP: $\emptyset \rightarrow \{x \mapsto \mathrm{t}\} \rightarrow \{x \mapsto \mathrm{t}, y \mapsto \mathrm{f}\}$

The Modern DPLL algorithm

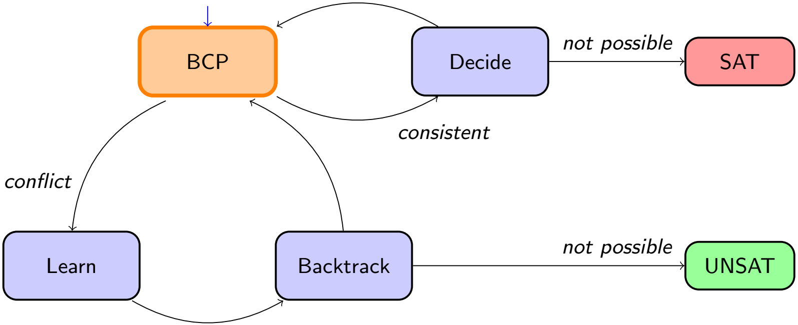


Unit rule

Deduction uses the unit rule:

$$\text{unit}(\rho, l_1 \vee \ldots l_i \ldots \vee l_k) = \begin{cases} \text{conflict} & \text{all literals are contradicted by } \rho \\ \rho[\mathrm{l_i} \mapsto \mathrm{t}] & \text{all literals but } l_i \text{ contradicted } \rho \\ \rho & \text{otherwise} \end{cases}$$

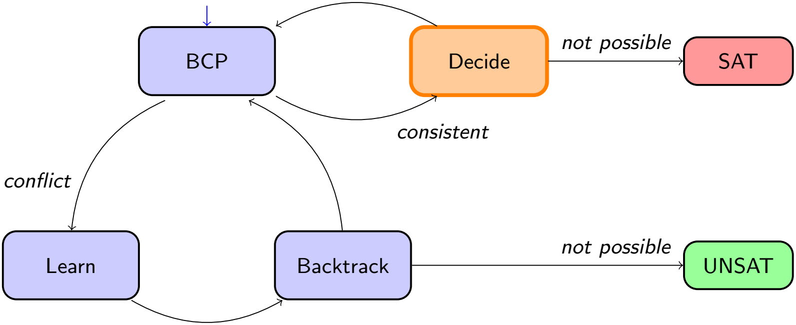
The Modern DPLL algorithm



Boolean Constraint Propagation

BCP(ϕ, ρ) {
    repeat
        $\rho' \leftarrow \rho$;
        for *Clause* $c \in \phi$ **do** $\rho \leftarrow \text{unit}(c, \rho)$;
    until $\rho = \rho'$;
}

The Modern DPLL algorithm



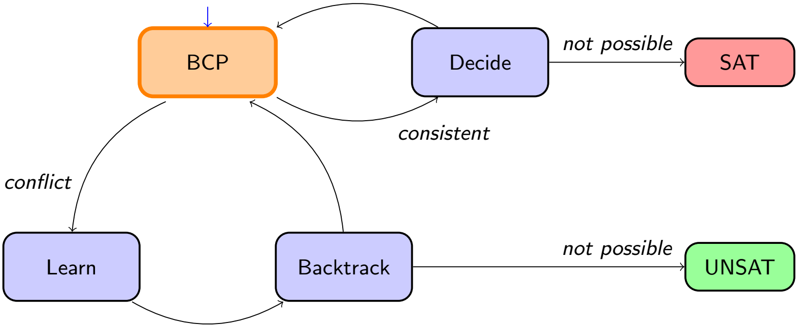
$$\phi = x \wedge (\neg x \vee \neg y) \wedge (y \vee z \vee \neg w) \wedge (y \vee z \vee w)$$

Decision

BCP: $\emptyset \rightarrow \{x \mapsto \mathsf{t}\} \rightarrow \{x \mapsto \mathsf{t}, y \mapsto \mathsf{f}\} \rightarrow$

Add assumption to partial assignment:
Decision: $\{x \mapsto \mathsf{t}, y \mapsto \mathsf{f}, z \mapsto \mathsf{f}\}$

The Modern DPLL algorithm



$$\phi = x \wedge (\neg x \vee \neg y) \wedge (y \vee z \vee \neg w) \wedge (y \vee z \vee w)$$

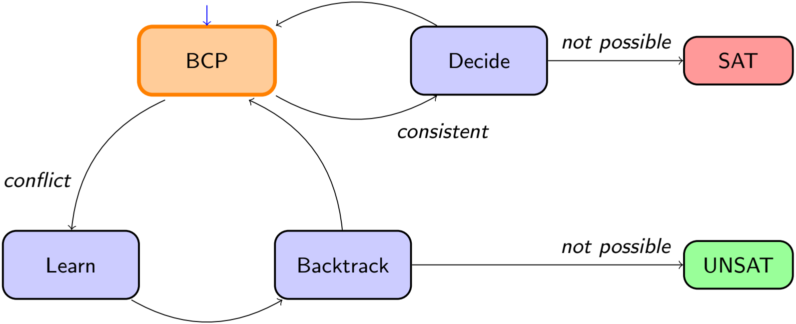
BCP

BCP: $\emptyset \rightarrow \{x \mapsto t\} \rightarrow \{x \mapsto t, y \mapsto f\} \rightarrow$

Decision: $\{x \mapsto t, y \mapsto f, z \mapsto f\} \rightarrow$

BCP:

The Modern DPLL algorithm

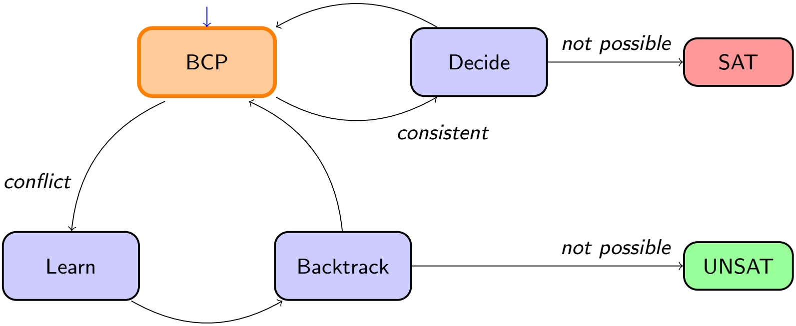


$$\phi = x \wedge (\neg x \vee \neg y) \wedge (y \vee z \vee \neg w) \wedge \color{red}{(y \vee z \vee w)}$$

BCP

BCP:	$\emptyset \to \{x \mapsto \mathrm{t}\} \to \{x \mapsto \mathrm{t}, y \mapsto \mathrm{f}\} \to$
Decision:	$\{x \mapsto \mathrm{t}, y \mapsto \mathrm{f}, \color{blue}{z \mapsto \mathrm{f}}\} \to$
BCP:	$\{x \mapsto \mathrm{t}, y \mapsto \mathrm{f}, z \mapsto \mathrm{f}, w \mapsto \mathrm{f}\}$

The Modern DPLL algorithm



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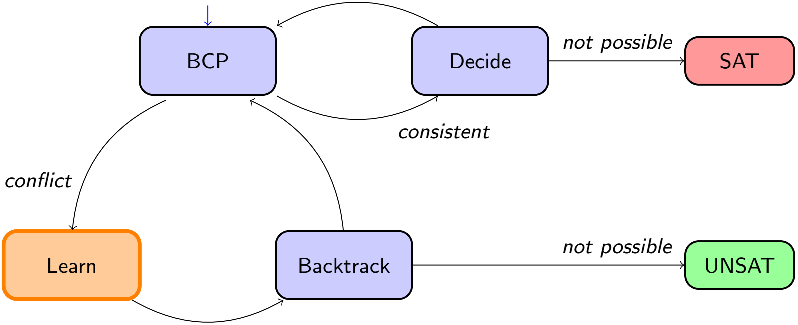
BCP

BCP:	$\emptyset \rightarrow \{x \mapsto \mathrm{t}\} \rightarrow \{x \mapsto \mathrm{t}, y \mapsto \mathrm{f}\} \rightarrow$
Decision:	$\{x \mapsto \mathrm{t}, y \mapsto \mathrm{f}, z \mapsto \mathrm{f}\} \rightarrow$
BCP:	$\{x \mapsto \mathrm{t}, y \mapsto \mathrm{f}, z \mapsto \mathrm{f}, w \mapsto \mathrm{f}\} \rightarrow$ conflict

The Modern DPLL algorithm



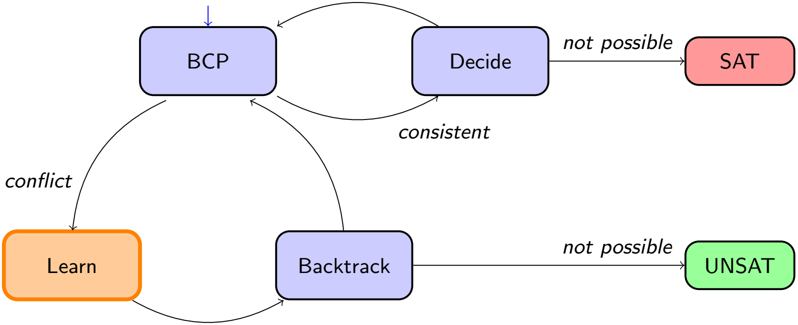
$$\phi = x \wedge (\neg x \vee \neg y) \wedge (y \vee z \vee \neg w) \wedge (y \vee z \vee w)$$

Learn

BCP: $\{x \mapsto \text{t}, y \mapsto \text{f}, z \mapsto \text{f}, w \mapsto \text{f}\} \rightarrow$ conflict

Find reason for the conflict,

The Modern DPLL algorithm



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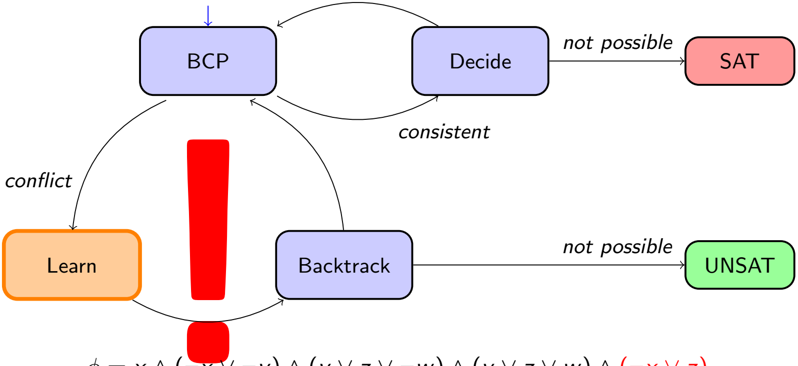
Learn

BCP: $\{x \mapsto \mathrm{t}, y \mapsto \mathrm{f}, z \mapsto \mathrm{f}, w \mapsto \mathrm{f}\} \rightarrow$ conflict

Find reason for the conflict, e.g., $x \wedge \neg z$ can never be true:

The Modern DPLL algorithm



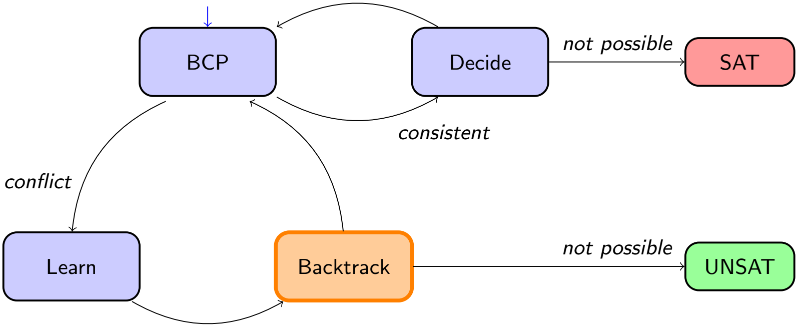
$$\phi = x \wedge (\neg x \vee \neg y) \wedge (y \vee z \vee \neg w) \wedge (y \vee z \vee w) \wedge (\neg x \vee z)$$

Learn

BCP: $\{x \mapsto \text{t}, y \mapsto \text{f}, z \mapsto \text{f}, w \mapsto \text{f}\} \rightarrow$ conflict

Find reason for the conflict, e.g., $x \wedge \neg z$ can never be true:
Learn $\neg(x \wedge \neg z) = \neg x \vee z$.

The Modern DPLL algorithm

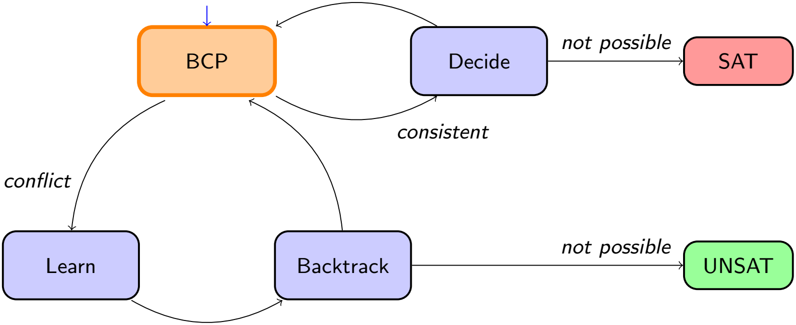


$$\phi = x \wedge (\neg x \vee \neg y) \wedge (y \vee z \vee \neg w) \wedge (y \vee z \vee w) \wedge (\neg x \vee z)$$

Backtrack

Undo assumptions that contradict learned clause:
Backtrack: $\{x \mapsto \mathrm{t}, y \mapsto \mathrm{t}\}$

The Modern DPLL algorithm



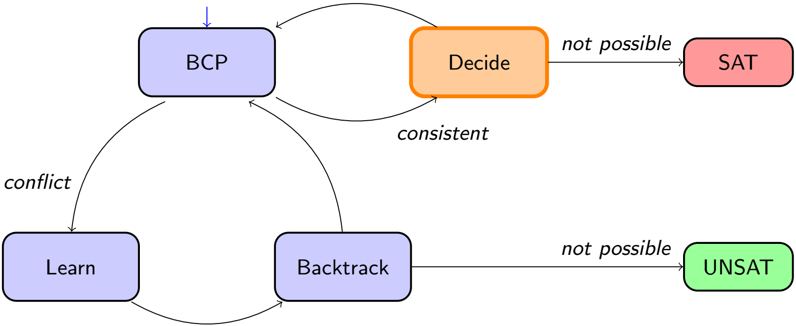
$$\phi = x \land (\neg x \lor \neg y) \land (y \lor z \lor \neg w) \land (y \lor z \lor w) \land (\neg x \lor z)$$

BCP

Backtrack: $\{x \mapsto \mathrm{t}, y \mapsto \mathrm{t}\} \rightarrow$

BCP automatically takes us to a new part of the search space:
BCP: $\{x \mapsto \mathrm{t}, y \mapsto \mathrm{t}, z \mapsto \mathrm{t}\}$

The Modern DPLL algorithm



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Decide

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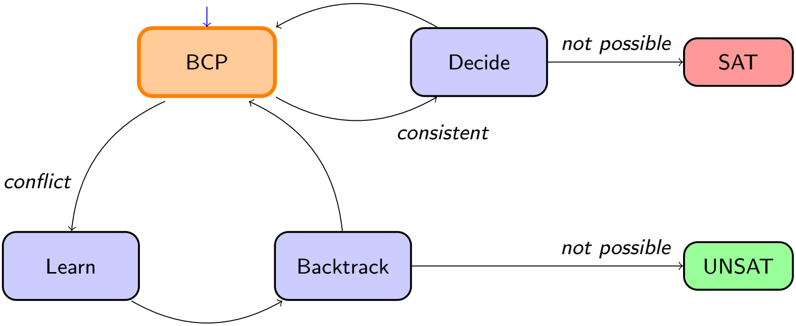
BCP:	$\{x \mapsto \mathrm{t}, y \mapsto \mathrm{t}, z \mapsto \mathrm{t}\} \rightarrow$
Decide:	$\{x \mapsto \mathrm{t}, y \mapsto \mathrm{t}, z \mapsto \mathrm{t}, w \mapsto \mathrm{f}\}$

The Modern DPLL algorithm



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Decide

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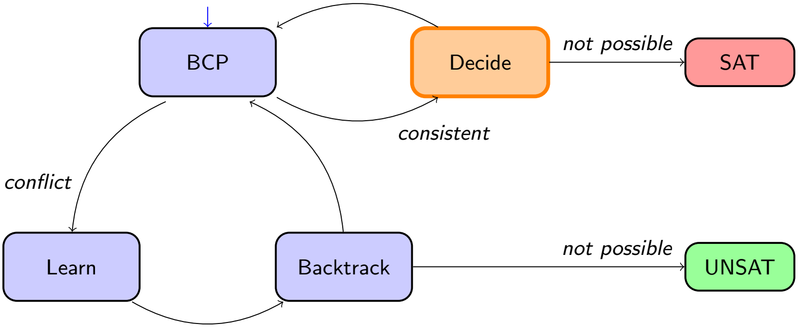

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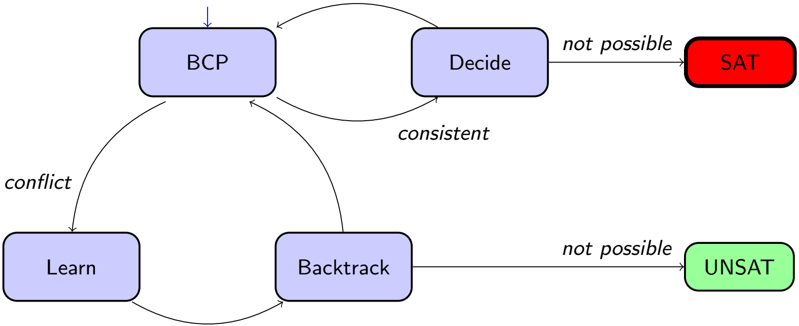

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Decide

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BCP:	$\{x \mapsto \mathsf{t}, y \mapsto \mathsf{t}, z \mapsto \mathsf{t}\} \rightarrow$
Decide:	$\{x \mapsto \mathsf{t}, y \mapsto \mathsf{t}, z \mapsto \mathsf{t}, w \mapsto \mathsf{f}\}$

```
bool v_1, ..., v_k;
if(φ)
  assert(0);
// program safe iff φ UNSAT
```

Logic for Static Analysis People

$$C = \langle \wp(\mathrm{Var} \to \{\mathrm{t}, \mathrm{f}\}), \subseteq, \cap, \cup \rangle$$
$$post_{\phi}^{C}(X) = \{\varepsilon \in X \mid \phi \text{ is true under } \varepsilon\}$$

Logic for Static Analysis People

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Examples

$$post^C_{a \wedge b}(\top) = \{\langle a \mapsto \mathsf{t}, b \mapsto \mathsf{t}\rangle\}$$
$$post^C_{a \vee \neg b}(\top) = \{\langle a \mapsto \mathsf{t}, b \mapsto \mathsf{t}\rangle, \langle a \mapsto \mathsf{t}, b \mapsto \mathsf{f}\rangle, \langle a \mapsto \mathsf{f}, b \mapsto \mathsf{f}\rangle\}$$

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ϕ is satisfiable exactly if $post^C_\phi(\top) \neq \emptyset$.

Abstraction and DPLL

DPLL datastructure

Partial assignment $\text{Var} \mapsto \{t, f\}$ and additional conflict states.
Each variable is:

unknown

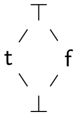
true false

conflicting

Abstraction and DPLL

DPLL operates over an abstract domain

$$\mathsf{Var} \to \wp(\{\mathsf{t}, \mathsf{f}\}) \quad = \quad \mathsf{Var} \longrightarrow$$



$$A = \langle \mathsf{Var} \to \wp(\{\mathsf{t}, \mathsf{f}\}), \sqsubseteq, \sqcap, \sqcup \rangle$$

$$C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$$

DPLL datastructure

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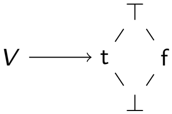
Best Abstract Transformers over the Cartesian Abstraction



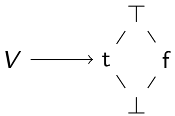
$$post^A_\phi(\top) = \bot \quad \Longrightarrow \quad \phi \text{ is unsatisfiable}$$

Best abstract transformer for a clause $a \vee b$

We synthesize the transformer $post^A_{a\vee b}$ for the argument $(a \mapsto \mathsf{f}, b \mapsto \top)$:

$$post^A_{a\vee b}(\langle a \mapsto \mathsf{f}, b \mapsto \top\rangle) =$$

Best Abstract Transformers over the Cartesian Abstraction



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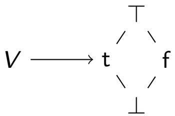
$$post^A_{a\vee b}(\langle a \mapsto \mathsf{f}, b \mapsto \top\rangle) = \alpha \circ post^C_{a\vee b} \circ \gamma(\langle a \mapsto \mathsf{f}, b \mapsto \top\rangle)$$

Best Abstract Transformers over the Cartesian Abstraction



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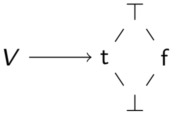
$$\begin{aligned} post^A_{a\vee b}(\langle a \mapsto \mathrm{f}, b \mapsto \top\rangle) &= \alpha \circ post^C_{a\vee b} \circ \gamma(\langle a \mapsto \mathrm{f}, b \mapsto \top\rangle) \\ &= \alpha \circ post^C_{a\vee b}(\{\langle a : \mathrm{f}, b : \mathrm{f}\rangle, \langle a : \mathrm{f}, b : \mathrm{t}\rangle\}) \end{aligned}$$

Best Abstract Transformers over the Cartesian Abstraction



$$post^A_\phi(\top) = \bot \quad \Longrightarrow \quad \phi \text{ is unsatisfiable}$$

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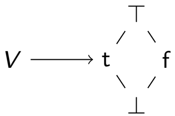
Best Abstract Transformers over the Cartesian Abstraction

DPLL operates over an abstract domain.
The unit rule is the best abstract transformer.



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Abstract Transformer Semantics for CNF

Literals:

$$post^A_v(a) = a \sqcap \langle v \mapsto \mathrm{t} \rangle$$
$$post^A_{\neg v}(a) = a \sqcap \langle v \mapsto \mathrm{f} \rangle$$

Disjunction and Conjunction:

$$post^A_{\phi \vee \psi}(a) = post^A_\phi(a) \sqcup post^A_\psi(a)$$
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Imprecision example:

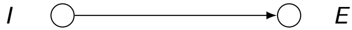
$$\phi = a \wedge (\neg a \vee b)$$

$$\begin{aligned} post^A_\phi(\top) &= post^A_a(\top) \sqcap post^A_{\neg a \vee b}(\top) \\ &= \langle a \mapsto \mathrm{t} \rangle \sqcap (\langle a \mapsto \mathrm{f} \rangle \sqcup \langle b \mapsto \mathrm{t} \rangle) \\ &= \langle a \mapsto \mathrm{t} \rangle \sqcap \top = \langle a \mapsto \mathrm{t} \rangle \end{aligned}$$

→analysis too imprecise

Refined Abstract Analyses through gfp Iteration

Can I reach E?



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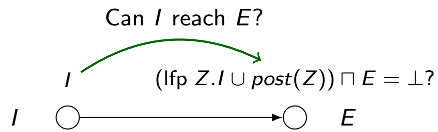

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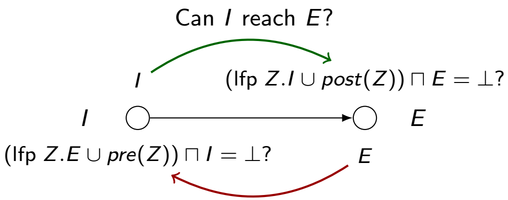

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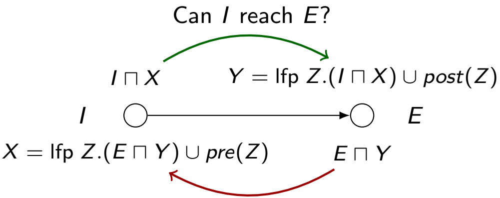

(P. Cousot 1978)

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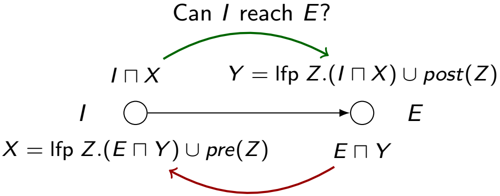

(P. Cousot 1978)

Iterating abstractions for increased precision

Iterate forward and backward analysis

$$\text{gfp}\langle X, Y\rangle. \quad \langle\ \text{lfp } Z.(E \sqcap Y) \sqcup \mathit{pre}(Z),\ \ \text{lfp } Z.(I \sqcap X) \sqcup \mathit{post}(Z)\ \rangle$$

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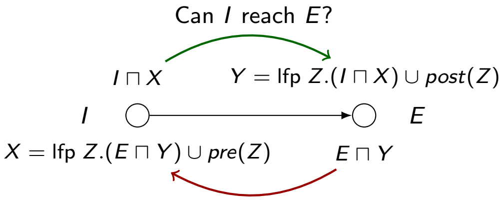

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In propositional logic, it is the case that $pre = post$

$$\text{gfp}\langle X, Y\rangle. \quad \langle \ post^A_\phi(\top \sqcap Y), \ \ post^A_\phi(\top \sqcap X) \ \rangle$$

Fixed point semantics: gfp $post^A_\phi$

Fixed Point Semantics

Fixed point semantics example: $\phi = a \wedge (\neg a \vee b)$

$$
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post^A_\phi(\top) &= \langle a \mapsto \mathrm{t} \rangle \sqcap \top & &= \langle a \mapsto \mathrm{t} \rangle \\
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Boolean Constraint Propagation

```
BCP(φ, ρ) {
    repeat
        ρ′ ← ρ;
        for Clause c ∈ φ do  ρ ← unit(c, ρ);
    until ρ = ρ′ ;
}
```

Fixed Point Semantics

DPLL operates over an abstract domain.
The unit rule is the best abstract transformer.
BCP is the gfp semantics of the abstract transformer

Fixed point semantics example: $\phi = a \wedge (\neg a \vee b)$

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BCP(ϕ, ρ) {
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 until $\rho = \rho'$;
}

Partitioning of gfp Semantics

Example

$$\phi = x \wedge (\neg x \vee \neg y) \wedge (y \vee z \vee \neg w) \wedge (y \vee z \vee w)$$

$$\text{gfp } post_{\phi}^{A} = \langle x \mapsto \mathrm{t}, y \mapsto \mathrm{f} \rangle \rightarrow \text{too imprecise}$$

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Trace partitioning (Mauborgne and Rival 2005)

Partition gfp $post_\phi^A(X)$ using transformers F and F'

1. $$\text{gfp } X.(post_\phi^A(X) \sqcap F(X)) \qquad \rightarrow F(X) = post_{\neg z}^A(X)$$
2. $$\text{gfp } X.(post_\phi^A(X) \sqcap F'(X)) \qquad \rightarrow F'(X) = post_{\neg x \vee z}^A(X)$$

Partitioning of gfp Semantics

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Decisions and learning are dynamic construction of trace partitionings

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DPLL is Abstract Interpretation

1. DPLL operates over an abstract domain
2. The unit rule is the best abstract transformer.
3. BCP is the gfp semantics of the abstract transformer
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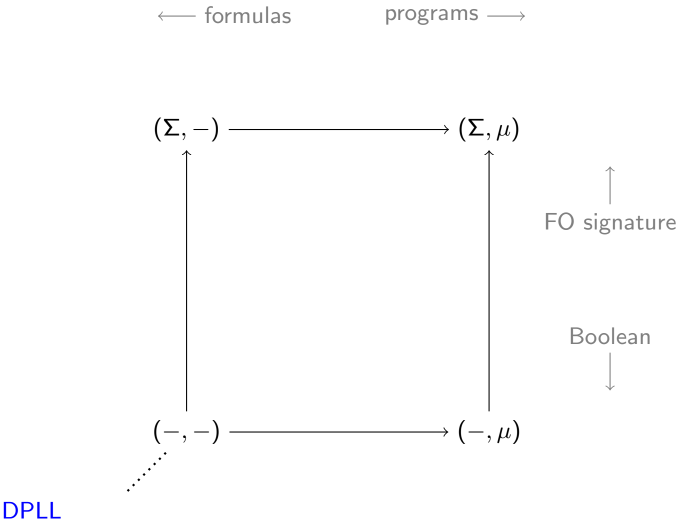
DPLL = AI + gfp semantics + dynamic trace partioning

- DPLL operates over an abstraction to compute a precise concrete result
- DPLL iteratively computes a minimal "error-preserving" transformer

World Domination through Abstract Interpretation



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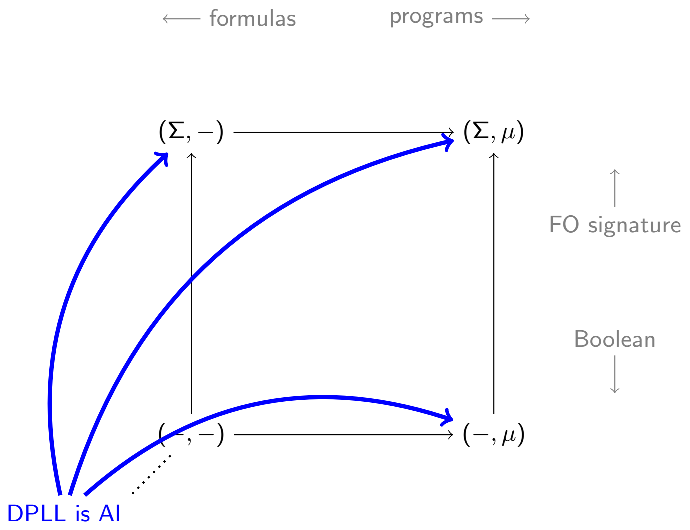
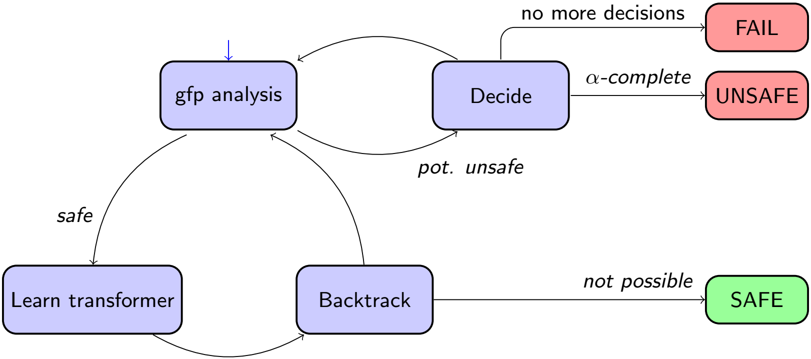


Generalising DPLL



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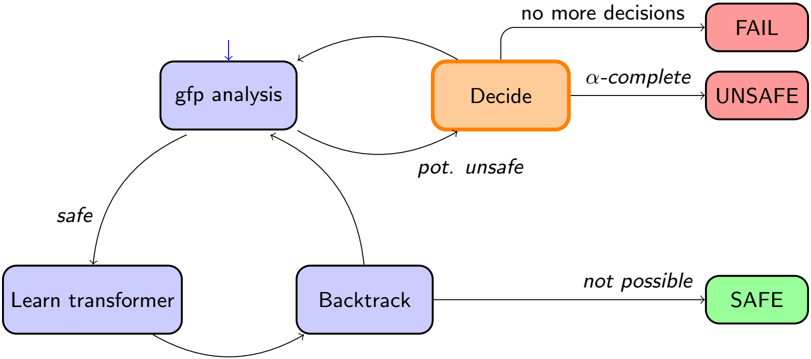
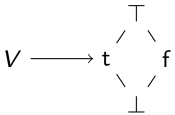


Generalising Decisions



Complementable decompositions

Take $a = \langle x \mapsto \mathrm{t}, y \mapsto \mathrm{f} \rangle \in \mathrm{Var} \to \wp(\{\mathrm{t}, \mathrm{f}\})$.

- Complement: $\neg a = \langle x \mapsto \mathrm{f} \rangle \vee \langle y \mapsto \mathrm{t} \rangle$
  Not precisely expressible in $\mathrm{Var} \to \wp(\{\mathrm{t}, \mathrm{f}\})$.
- Decomposition: $a = \langle x \mapsto \mathrm{t} \rangle \sqcap \langle y \mapsto \mathrm{f} \rangle$
  Each element of the decomposition has a precise complement.

Decisions are made over complementable meet-irreducible elements.

Decisions in Numeric Domains

Intervals

$a = \langle x \in [0, 10], y \in [-\infty, 0] \rangle$ has no precise complements

$$a = \underbrace{\langle x \geq 0 \rangle}_{\neg\langle x<0 \rangle} \sqcap \underbrace{\langle x \leq 10 \rangle}_{\neg\langle x>10 \rangle} \sqcap \underbrace{\langle y \leq 0 \rangle}_{\neg\langle y>0 \rangle}$$

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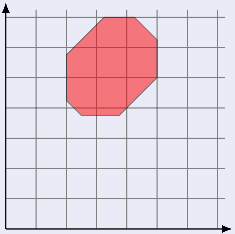
Octagons



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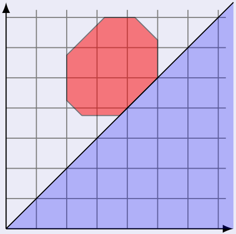

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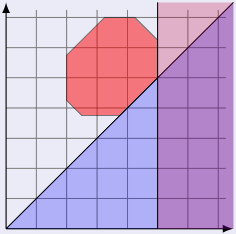

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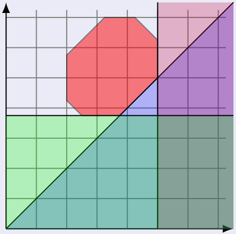


Decisions in Trace Abstractions

Control-flow abstraction

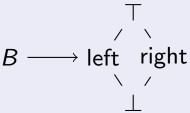
Set of control-flow branches B

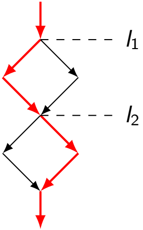
$$B \longrightarrow \begin{array}{ccc} & \top & \\ / & & \backslash \\ \text{left} & & \text{right} \\ \backslash & & / \\ & \bot & \end{array}$$

Decisions in Trace Abstractions

Control-flow abstraction

Set of control-flow branches B





Decomposition

$$
\begin{aligned}
a &= \langle l_1 \mapsto \text{left}, l_2 \mapsto \text{right} \rangle \\
&= \underbrace{\langle l_1 \mapsto \text{left} \rangle}_{\neg \langle l_1 \mapsto \text{right} \rangle} \sqcap \underbrace{\langle l_2 \mapsto \text{right} \rangle}_{\neg \langle l_2 \mapsto \text{left} \rangle}
\end{aligned}
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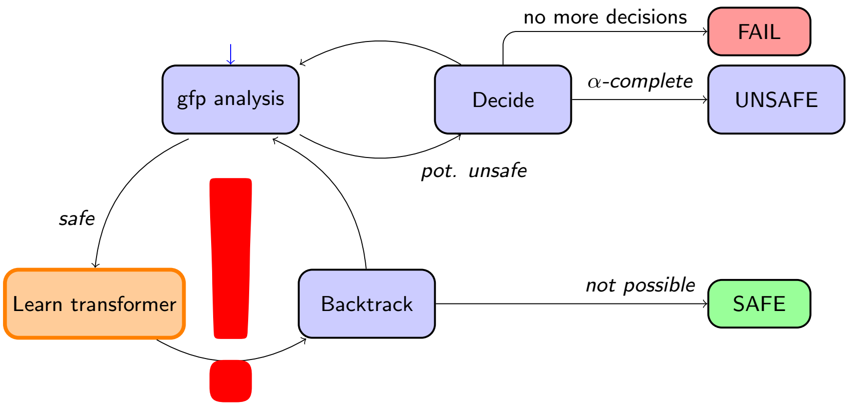
Generalising DPLL



DPLL Learning Example

Learn deeper reason for a conflict using an implication graph

$$\neg 1 \wedge (1 \vee \neg 2 \vee \neg 3) \wedge (\neg 4 \vee 5) \wedge (\neg 6 \vee 7) \wedge (\neg 6 \vee \neg 8) \wedge (\neg 7 \vee 8 \vee \neg 9) \wedge (3 \vee 9 \vee 1)$$

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DL0

$\overline{1}$

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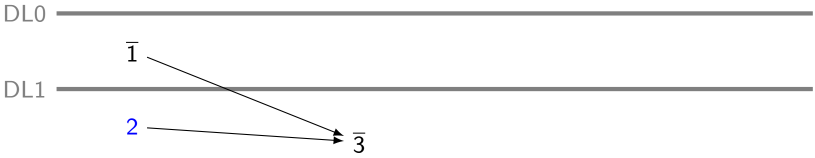

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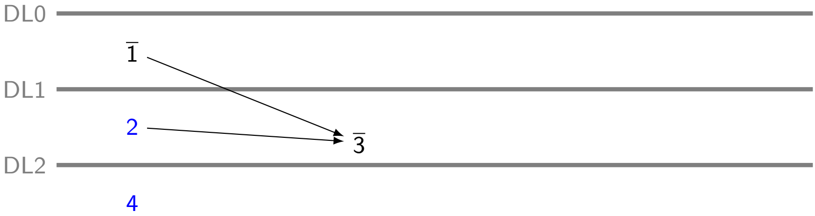

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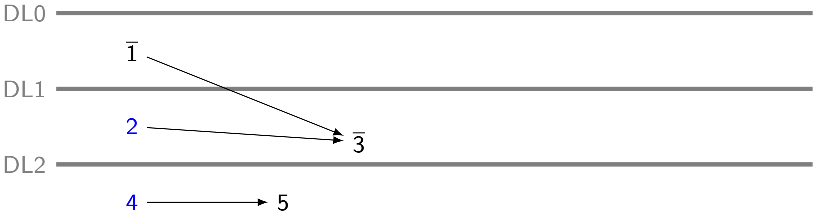

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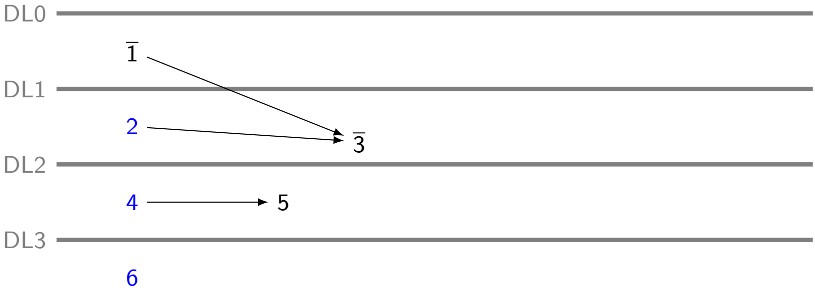

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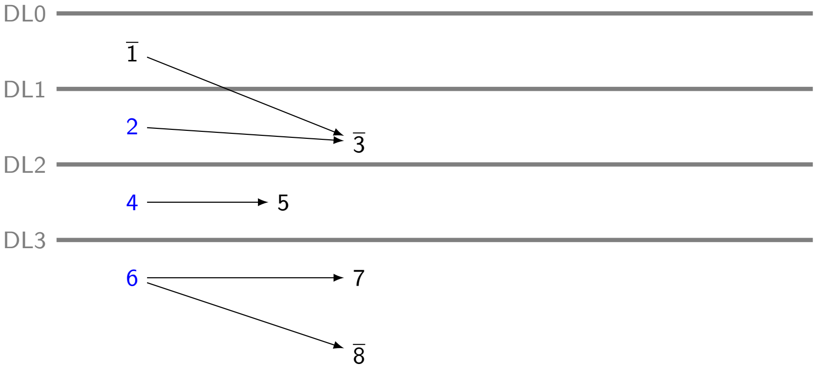

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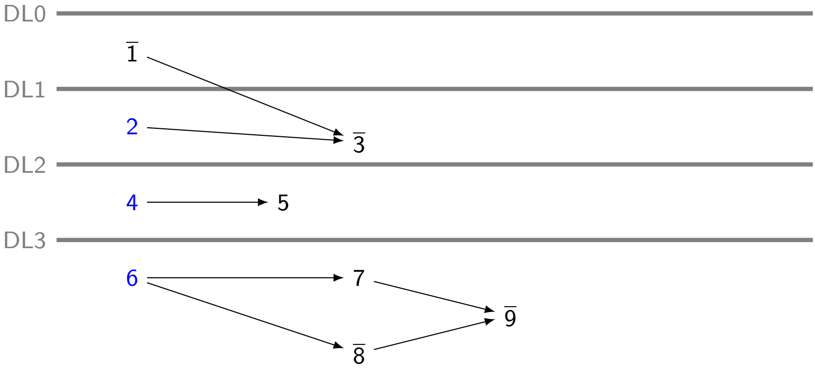

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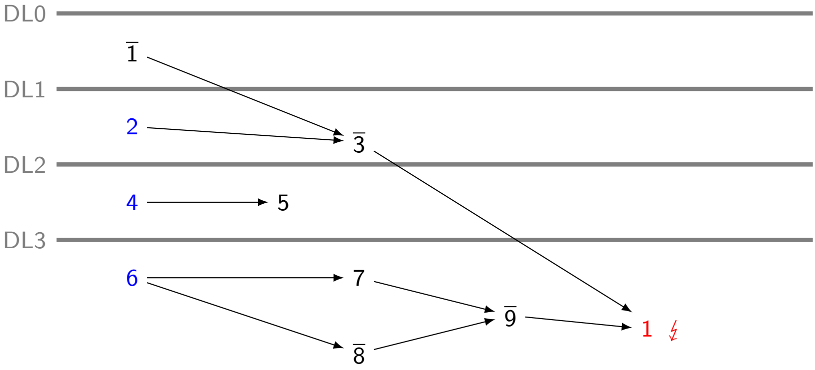

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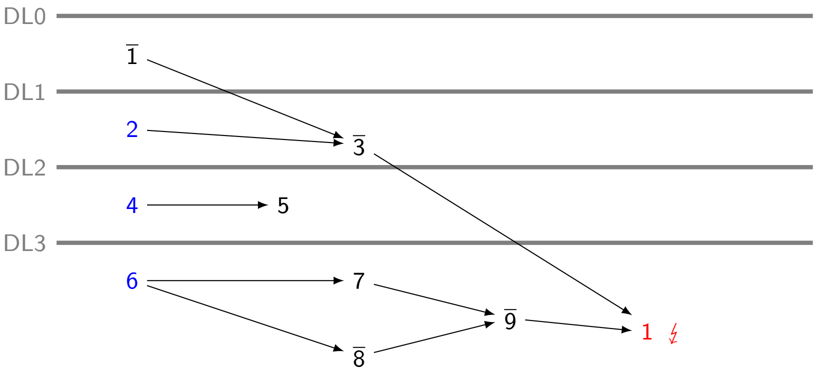

Every cut that disconnects the roots from the error is a reason

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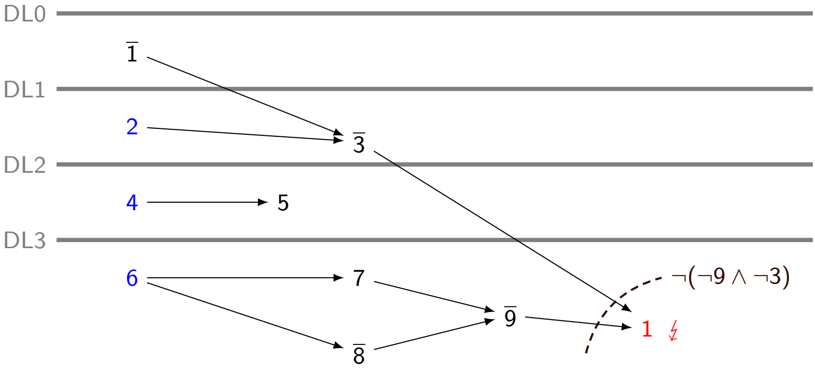
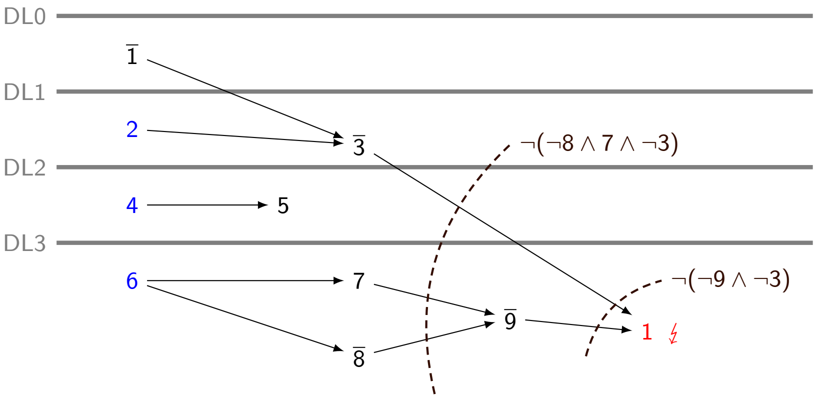


Every cut that disconnects the roots from the error is a reason

DPLL Learning Example

Learn deeper reason for a conflict using an implication graph

$\neg 1 \wedge (1 \vee \neg 2 \vee \neg 3) \wedge (\neg 4 \vee 5) \wedge (\neg 6 \vee 7) \wedge (\neg 6 \vee \neg 8) \wedge (\neg 7 \vee 8 \vee \neg 9) \wedge (3 \vee 9 \vee 1)$



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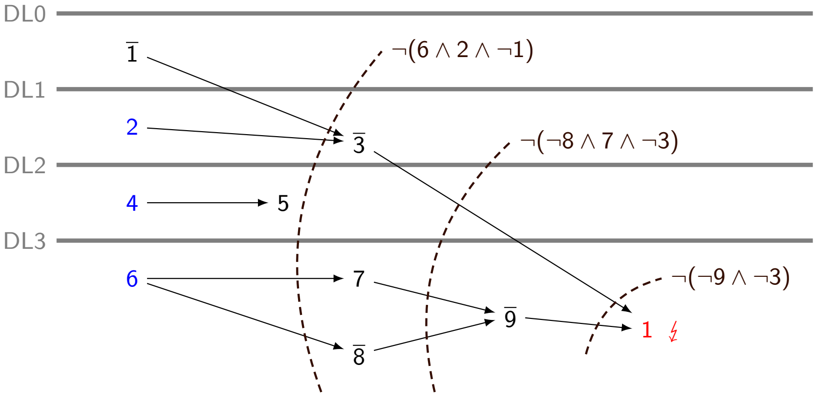

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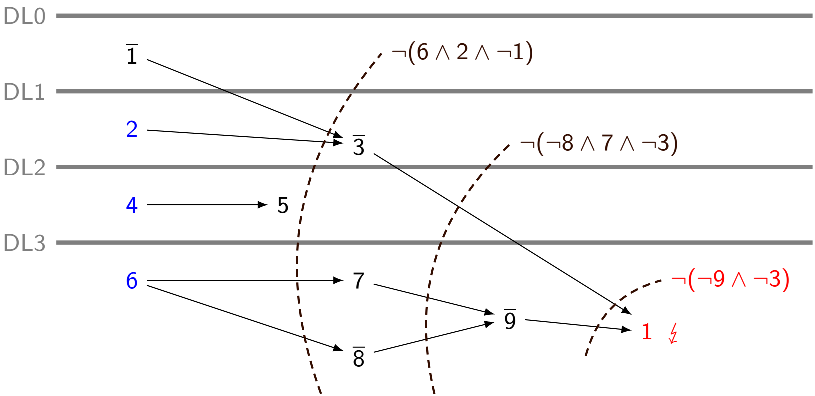

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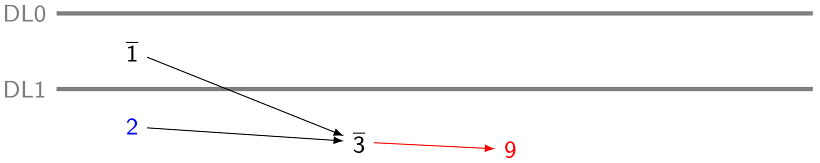
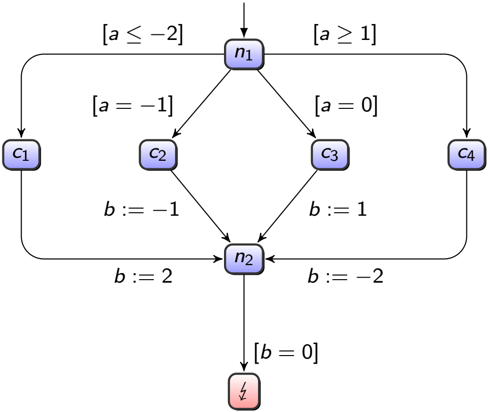
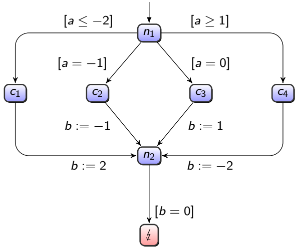


Putting it All Together: Simple Program Example

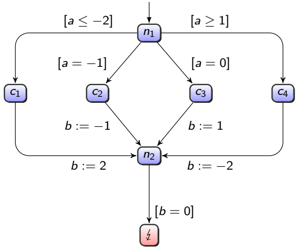


Abstract Implication Graph



Abstract Implication Graph

DL0



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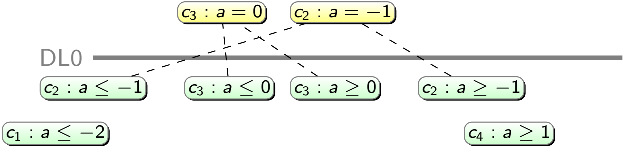

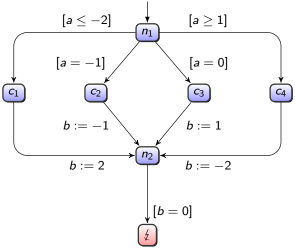

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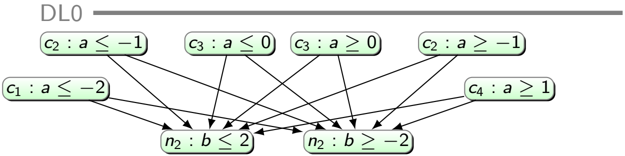

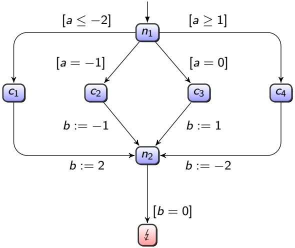

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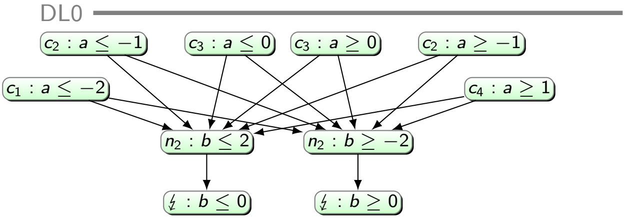

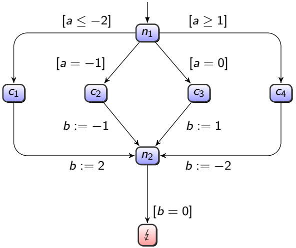

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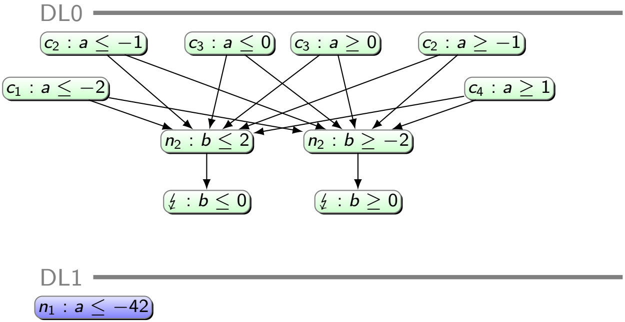

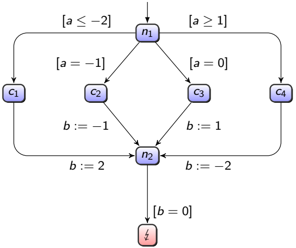

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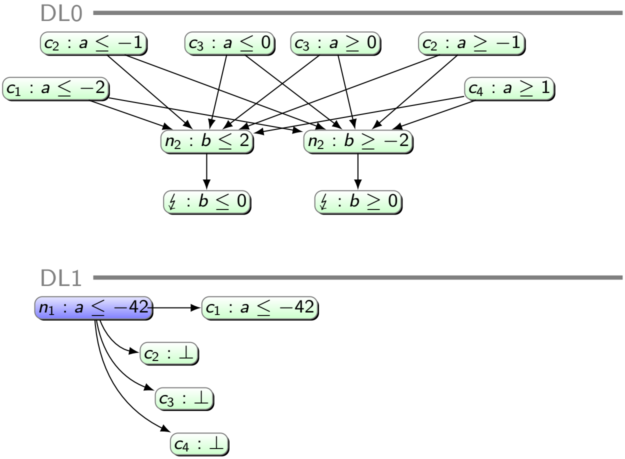

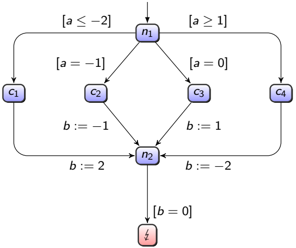

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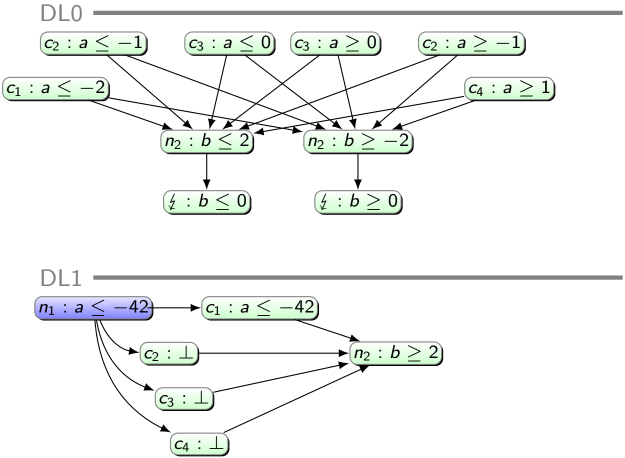

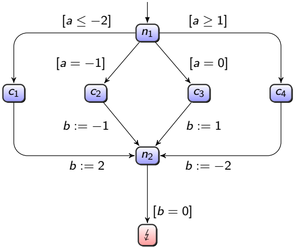

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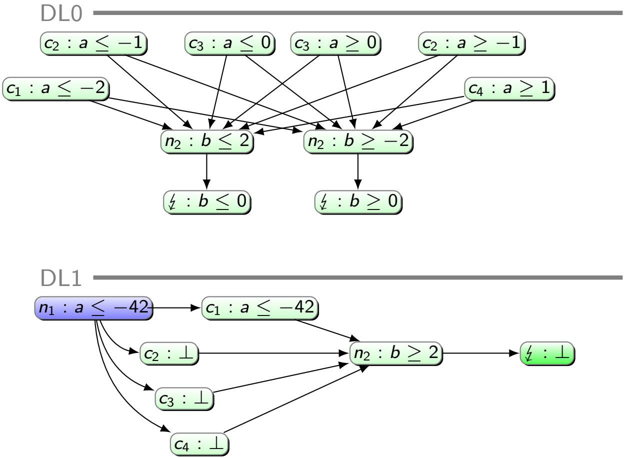
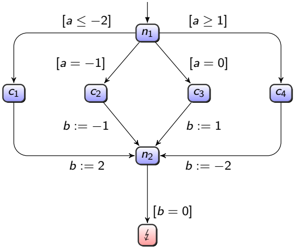
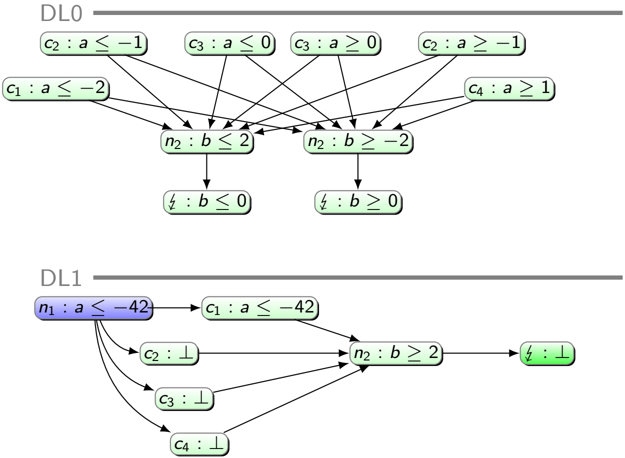
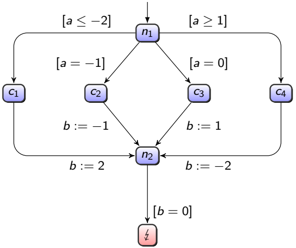




SAFE

Abstract Implication Graph





SAFE $\rightarrow$ Generalise!

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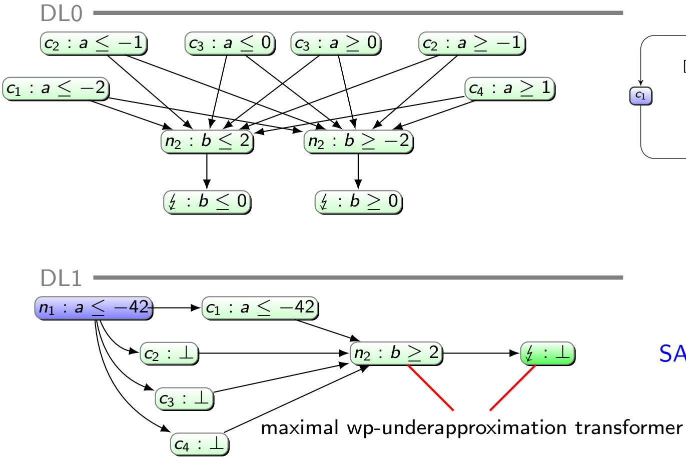

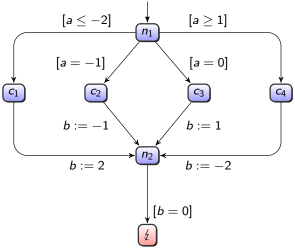

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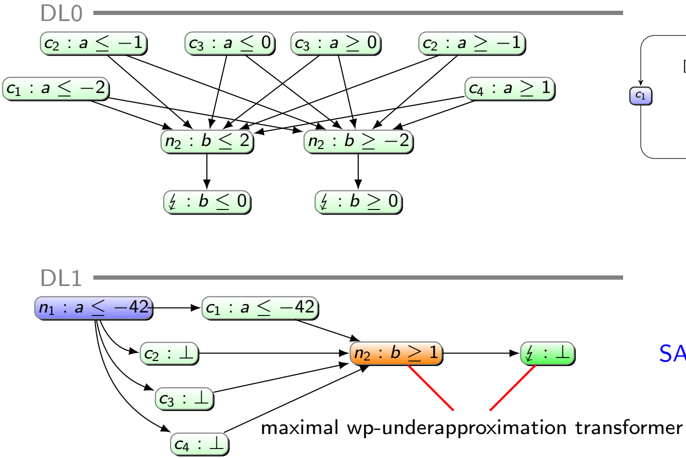

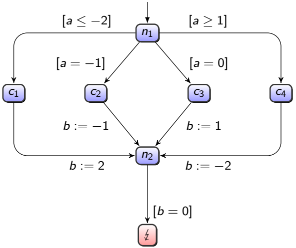

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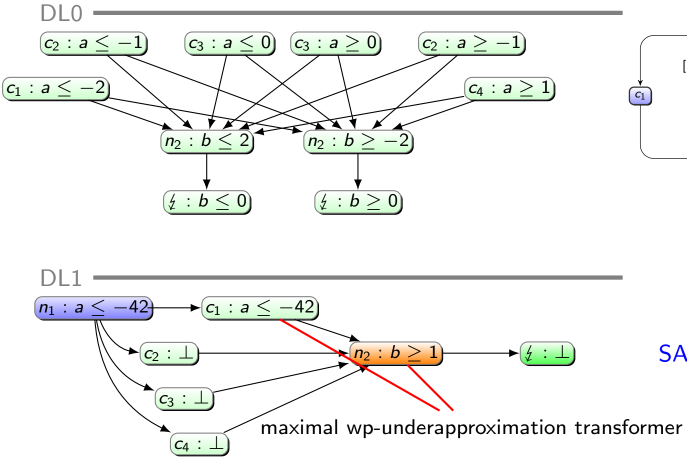

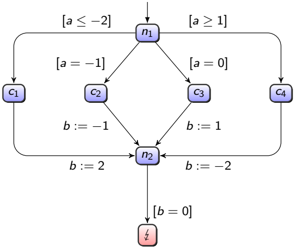

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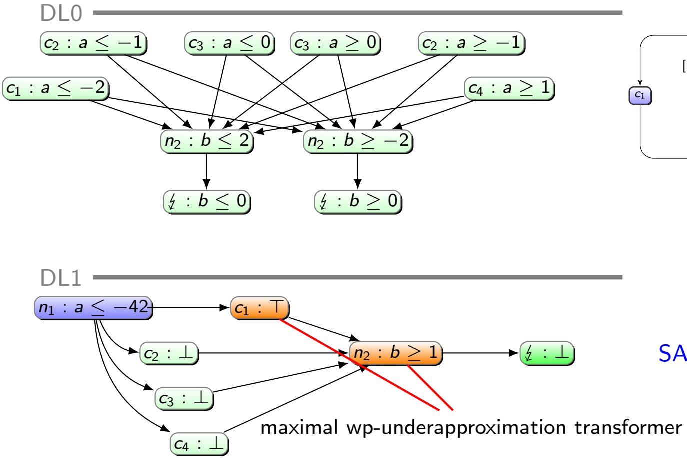

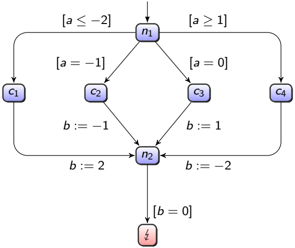

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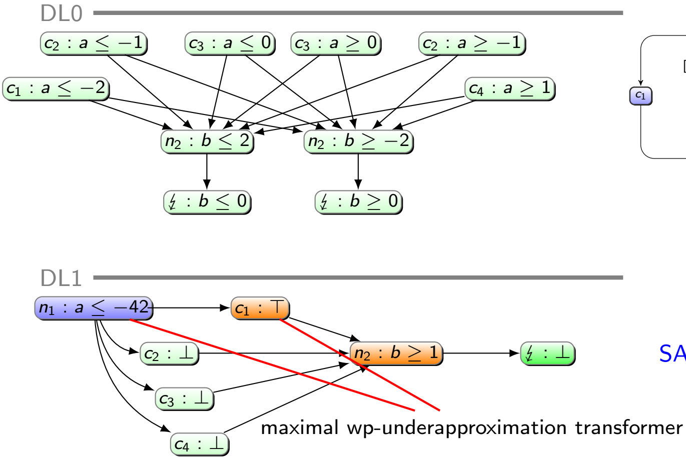

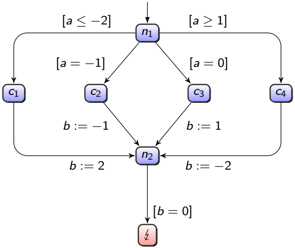

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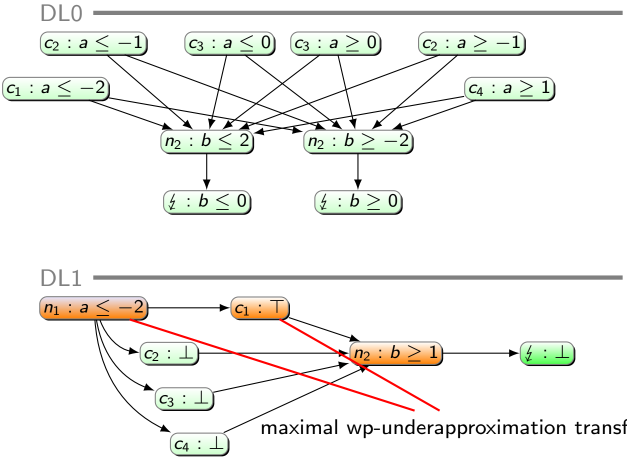

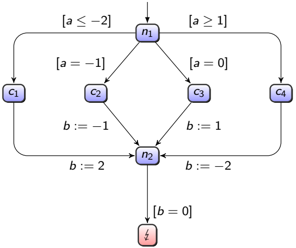

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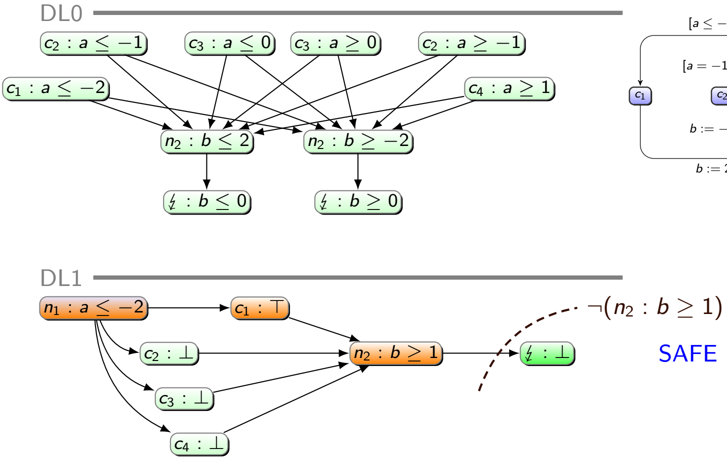

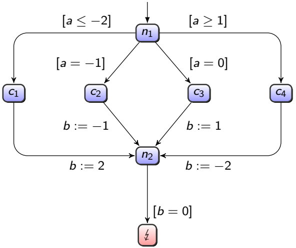

SAFE $\rightarrow$ find cut

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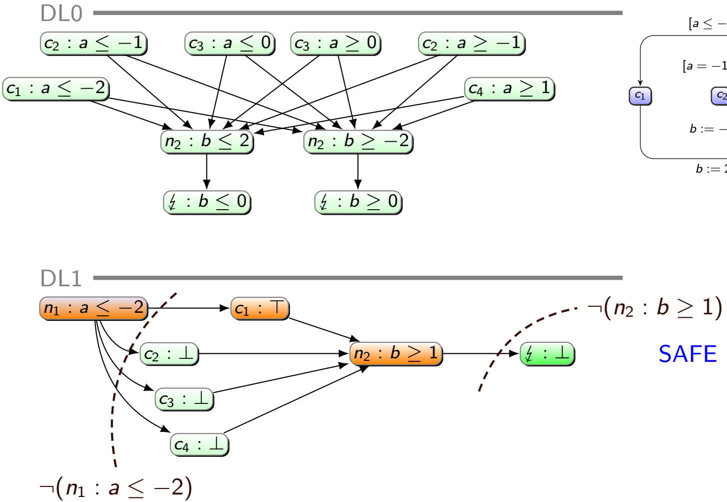

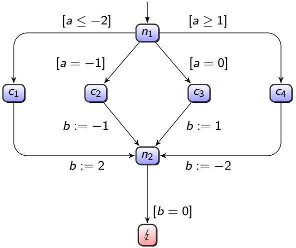

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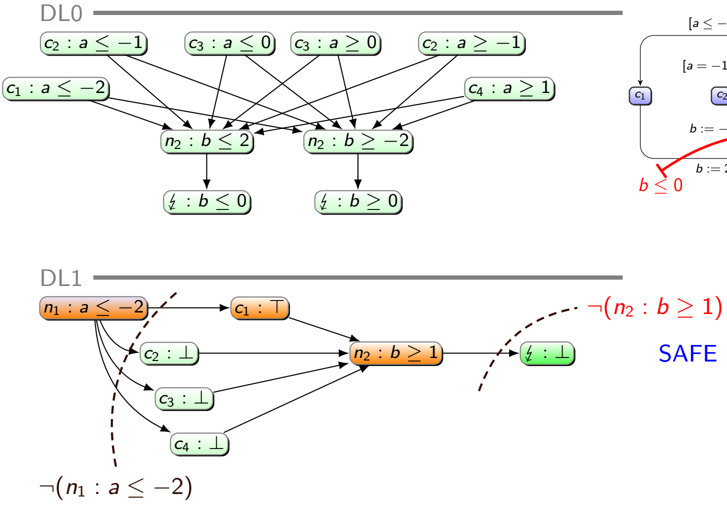

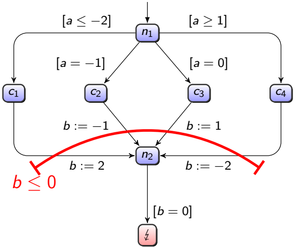

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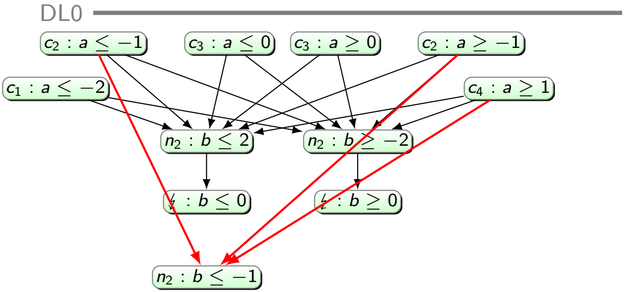

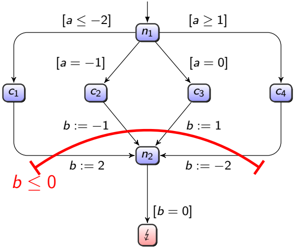

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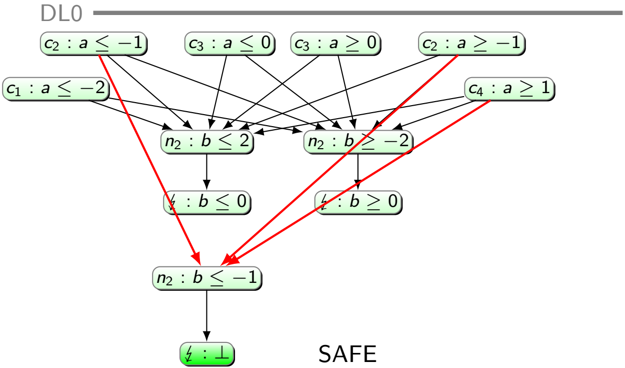

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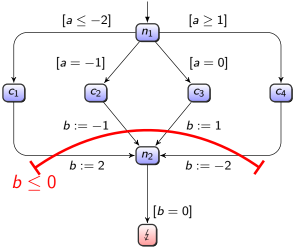
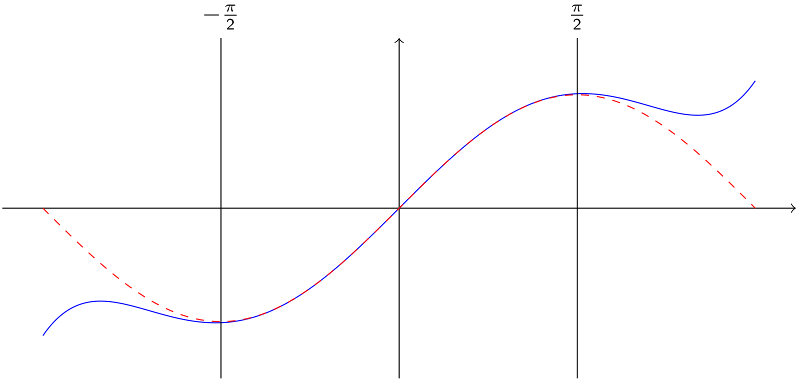


Demo

Property-dependent Trace Partitioning



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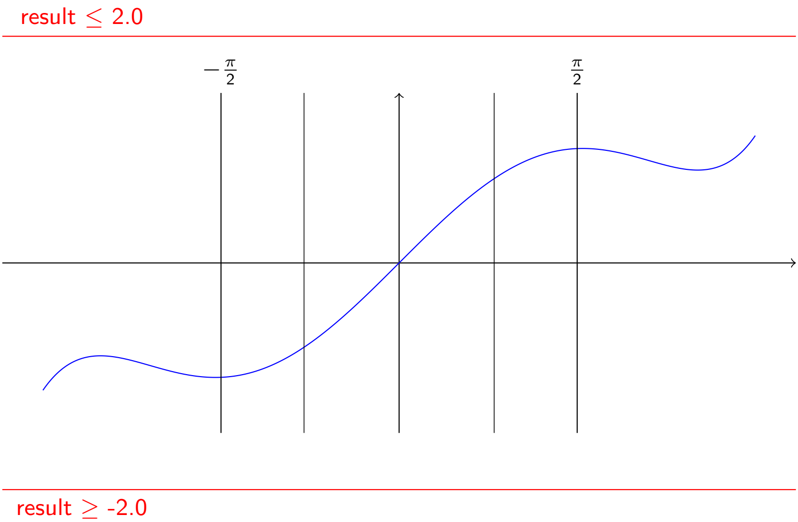

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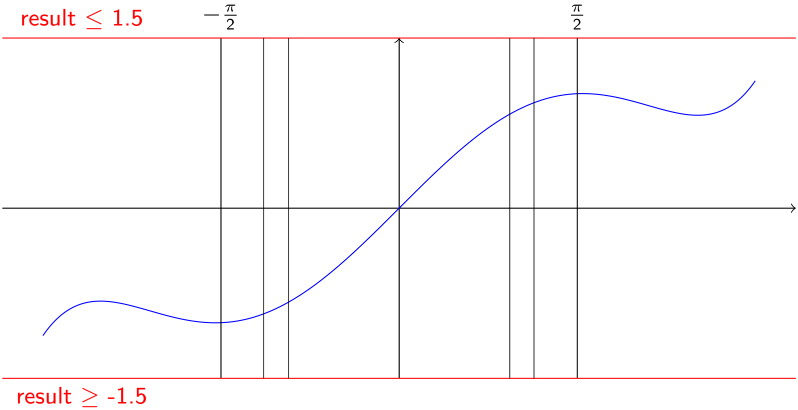

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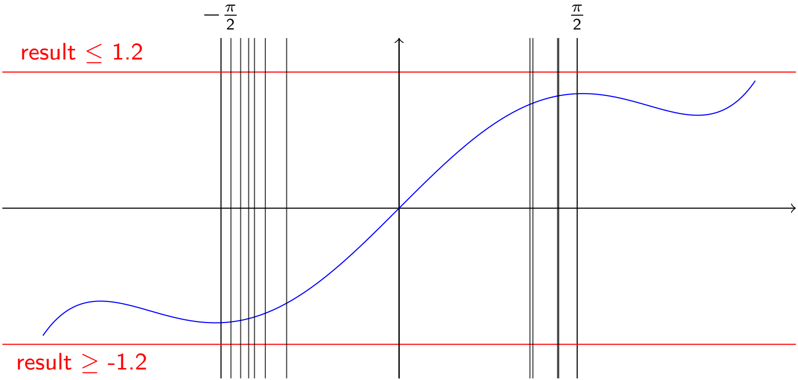

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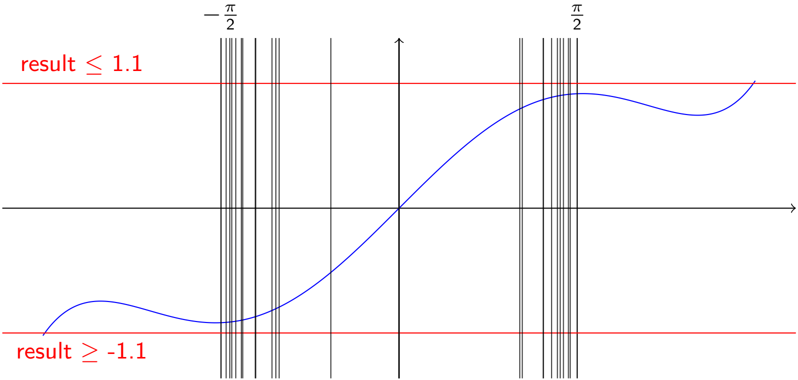

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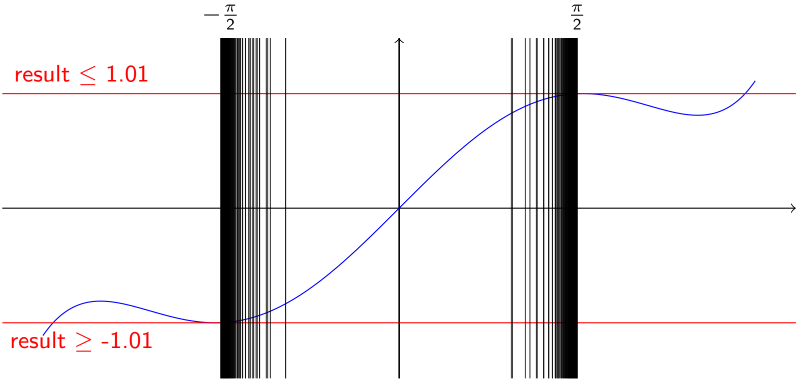

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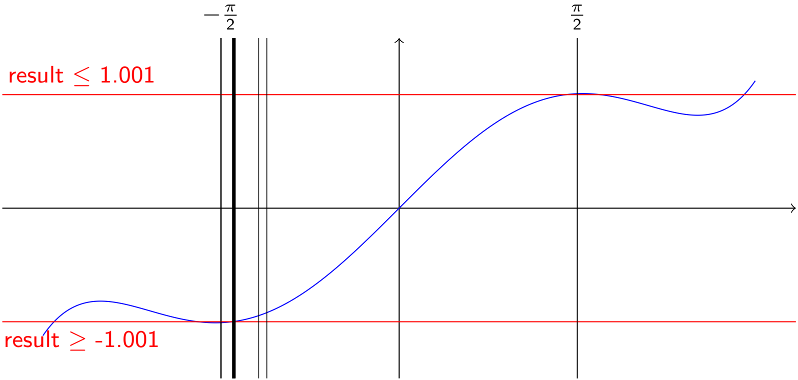

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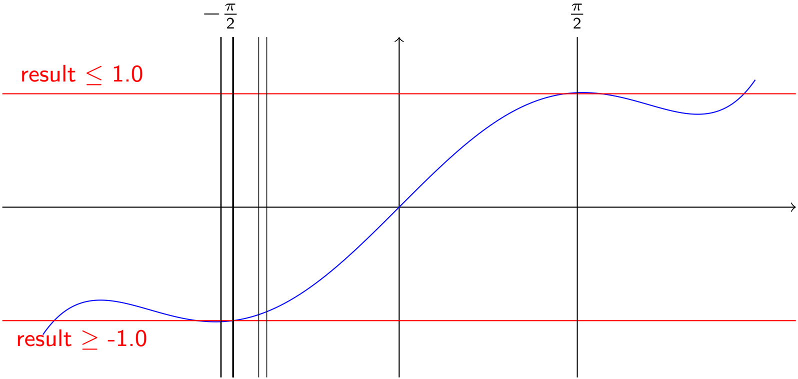

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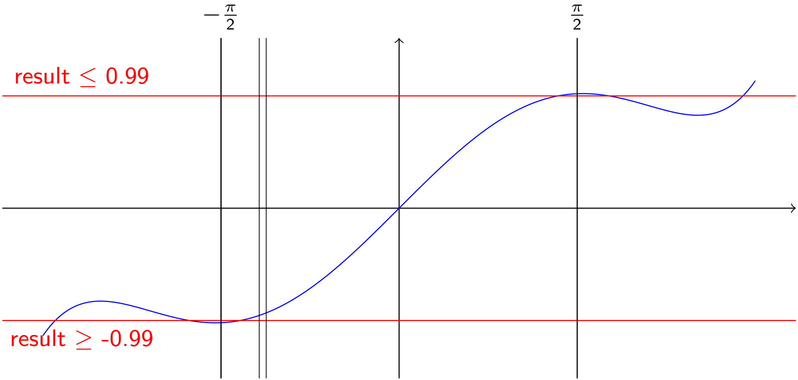
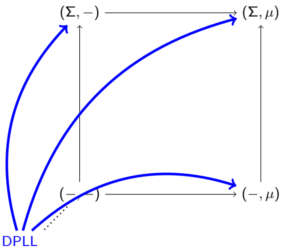


Conclusion



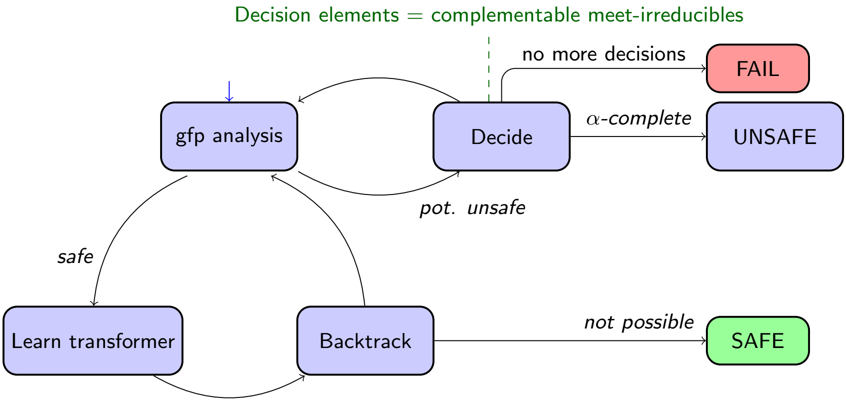
- A DPLL solver . . .
  - is an abstract interpreter over the Cartesian Boolean abstraction
  - AI + gfp-iteration + dynamic trace partitioning
  - iteratively computes a minimal property-preserving abstract transformer

- We can simply change the domain to lift DPLL to richer logics and programs

Thank you for your attention

Appendix

Generalising DPLL



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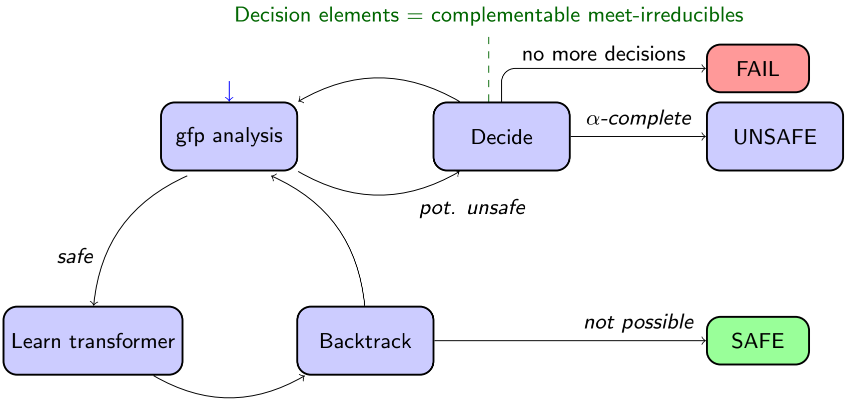

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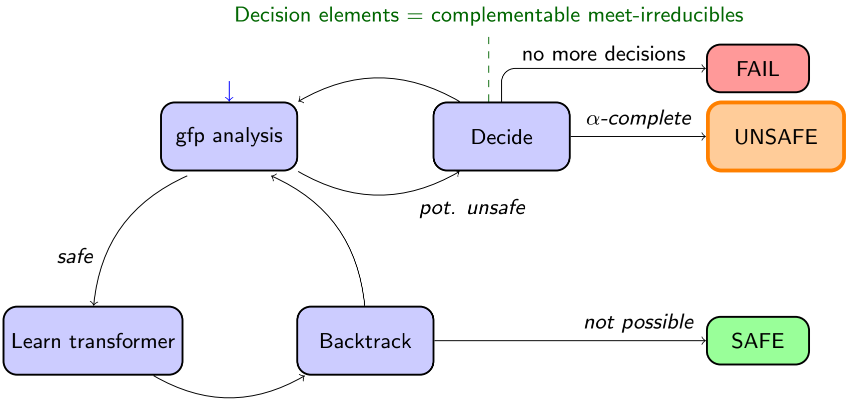

Satisfying Assignments and Proofs

Satisfying assignments

DPLL constructs restricted transformers. When can we stop?

$$\phi = (a \vee b) \wedge (\neg b \vee c)$$

$$r = \langle a \mapsto \mathrm{t}, c \mapsto \mathrm{t} \rangle$$

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Construct restricted transformer

$$\widehat{F}_r(X) = post^A_\phi(X \sqcap r)$$

α-completeness: $\widehat{F}_r \circ \alpha = \alpha \circ F_r$

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$\widehat{F}_r$ is a sufficient to show SAT $\iff$ $\widehat{F}_r$ is α-complete and $\widehat{F}_r(\top) \sqsupset \bot$

$\widehat{F}_r$ is witness to SAT assignments $\iff$ $\widehat{F}_r$ is γ-complete and $\widehat{F}_r(\top) \sqsupset \bot$

Satisfying Assignments and Proofs

Interval counter-example

When can we stop refining in a program?

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int x, y; x = y; assert(x != 0);
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Witness for counter-example:

$$\widehat{F}_{\langle y=0 \rangle}(X) = \widehat{F}(X \sqcap \langle y = 0 \rangle)$$

$$\widehat{F}_{\langle y=0 \rangle} \text{ is } \gamma\text{-complete and } \widehat{F}_{\langle y=0 \rangle}(\top) \sqsupset \bot$$