Strong Spatial Mixing with Fewer Colours for Lattice Graphs*

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Abstract

Recursively-constructed couplings have been used in the past for mixing on trees. We show how to extend this technique to non-tree-like graphs such as lattices. Using this method, we obtain the following general result. Suppose that G is a triangle-free graph and that for some $\Delta \geq 3$, the maximum degree of G is at most Δ. We show that the spin system consisting of q-colourings of G has strong spatial mixing, provided $q > \alpha\Delta - \gamma$, where $\alpha \approx 1.76322$ is the solution to $\alpha^\alpha = e$, and $\gamma = \frac{4\alpha^3 - 6\alpha^2 - 3\alpha + 4}{2(\alpha^2 - 1)} \approx 0.47031$. Note that we have no additional lower bound on q or Δ. This is important for us because our main objective is to have results which are applicable to the lattices studied in statistical physics such as the integer lattice $\mathbb{Z}^d$ and the triangular lattice. For these graphs (in fact, for any graph in which the distance-k neighbourhood of a vertex grows sub-exponentially in k), strong spatial mixing implies that there is a unique infinite-volume Gibbs measure. That is, there is one macroscopic equilibrium rather than many. Our general result gives, for example, a "hand proof" of strong spatial mixing for 7-colourings of triangle-free 4-regular graphs. (Computer-assisted proofs of this result were provided by Salas and Sokal (for the rectangular lattice) and by Bubley, Dyer, Greenhill and Jerrum.) It also gives a hand proof of strong spatial mixing for 5-colourings of triangle-free 3-regular graphs. (A computer-assisted proof for the special case of the hexagonal lattice was provided earlier by Salas and Sokal.) Towards the end of the paper we show how to improve our general technique by considering the geometry of the lattice. The idea is to construct the recursive coupling from a system of recurrences rather than from a single recurrence. We use the geometry of the lattice to derive the system of recurrences. This gives us an analysis with a horizon of more than one level of induction, which leads to improved results. We illustrate this idea by proving strong spatial mixing for $q = 10$ on the lattice $\mathbb{Z}^3$. Finally, we apply the idea to the triangular lattice, adding computational assistance. This gives us a (machine-assisted) proof of strong spatial mixing for 10-colourings of the triangular lattice. (Such a proof for 11 colours was given by Salas and Sokal.) For completeness, we also show that our strong spatial mixing proof implies rapid mixing of Glauber dynamics for sampling proper colourings of neighbourhood-amenable graphs. (It is known that strong spatial mixing often implies rapid mixing, but existing proofs seem to be written for $\mathbb{Z}^d$.) Thus our strong spatial mixing results give rapid mixing corollaries for neighbourhood-amenable graphs such as lattices.

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1 Introduction

This paper is concerned with (proper) colourings of an infinite graph G such as the integer lattice $\mathbb{Z}^d$. A colouring is an assignment of colours from the set $\{1, \ldots, q\}$ to the vertices. It is *proper* if adjacent vertices receive different colours. Proper colourings correspond to configurations in the *zero-temperature antiferromagnetic Potts model.* Two important closely-related questions which have received a lot of recent attention are

- Do boundary effects decay exponentially? This notion is known as "strong spatial mixing", and
- Is there a unique infinite-volume Gibbs measure? (The converse situation is often called a "phase transition".)

See Weitz's PhD thesis [31] and Martinelli's lecture notes [22] for an exposition of this material and the papers [1, 3, 13, 20, 23, 27] for some recent (and not so recent) results. For graphs like regular lattices (in fact, for any graph in which the distance-k neighbourhood of a vertex grows sub-exponentially in k), strong spatial mixing implies that there is a unique infinite-volume Gibbs measure. See [31] and [22] for details.[1] For these graphs, the two questions above are also known to be closely related to a third question:

- Is Glauber dynamics rapidly mixing on finite pieces of the graph?

For graphs such as lattice graphs, strong spatial mixing implies rapid mixing. More details are given in Section 7. A number of papers have given bounds on the number of colours that are necessary for rapid mixing, both for general graphs [9, 10, 14, 16, 17, 19, 25, 29] and for specific graphs and lattices [1, 15, 21, 24].

1.1 Definitions and background

In order to define "strong spatial mixing" and "infinite-volume Gibbs measure", we need notation for describing colourings of finite regions of the infinite graph G. A *region* R of G is a (not necessarily connected) subset of the vertices. A *colouring* of R is a function from R to the set of *colours* $Q = \{1, \ldots, q\}$. If R is non-empty and finite then ∂R denotes the vertex boundary around R. That is, ∂R is the set of vertices that are not in R, but are adjacent to R. A colouring of ∂R is a function from ∂R to the set $\{0\} \cup Q$. The colour "0" corresponds to an unconstrained boundary vertex. Given a colouring $\mathcal{B}$ of ∂R, a colouring C of R is said to be *proper* if adjacent vertices in R receive different colours, and vertices in R receive colours different from adjacent boundary vertices. $S(\mathcal{B})$ denotes the set of proper colourings of R and $\pi_{\mathcal{B}}$ denotes the uniform distribution on $S(\mathcal{B})$. For any $\Lambda \subseteq R$, $\pi_{\mathcal{B},\Lambda}$ denotes the distribution on colourings of Λ induced by $\pi_{\mathcal{B}}$.

A measure μ on the set of proper colourings of G is an infinite-volume *Gibbs measure* (with respect to the uniform specification) if, for any finite region R, the conditional probability distribution $\mu(\cdot \mid \sigma_{\overline{R}})$ (conditioned on the colouring $\sigma_{\overline{R}}$ of all vertices other than those in R) is the uniform distribution on proper colourings of R. Infinite-volume Gibbs measures exist for any G. The problem of determining whether there is more than one infinite-volume Gibbs

[1] The formal definition of "strong spatial mixing" that we use [13, 22] requires that there be exponential decay in the effect of a single discrepancy at the boundary of a region. If the graph has sub-exponential growth (e.g., the distance-k neighbourhood of a vertex grows sub-exponentially in k) then this implies uniqueness.

measure for a given "specification" is known as the DLR problem (Dobrushin, Lanford and Ruelle) in statistical physics (see [3]).

An important notion in statistical physics is whether the system (as specified by the finite-volume Gibbs measures) satisfies *strong spatial mixing* [22]. Informally, this means that for any finite set of vertices R, if you consider two different colourings $\mathcal{B}$ and $\mathcal{B}'$ of the boundary of R which differ at a single vertex y then the effect that this difference has on a subset $\Lambda \subseteq R$ decays exponentially with the distance from Λ to y. For the formal definition (which we take from [13]), recall that the *total variation distance* between distributions θ_1 and θ_2 on Ω is

$$d_{\mathrm{tv}}(\theta_1, \theta_2) = \frac{1}{2} \sum_{i \in \Omega} |\theta_1(i) - \theta_2(i)| = \max_{A \subseteq \Omega} |\theta_1(A) - \theta_2(A)|.$$

We can now define strong spatial mixing.

Definition 1 *The spin system specified by uniform finite-volume Gibbs measures on proper q-colourings of G has strong spatial mixing if there are constants β and $\beta' > 0$ such that for any non-empty finite region R, any $\Lambda \subseteq R$, any vertex $y \in \partial R$, and any pair of colourings $(\mathcal{B}, \mathcal{B}')$ of ∂R which differ only at y,*

$$d_{\mathrm{tv}}(\pi_{\mathcal{B},\Lambda}, \pi_{\mathcal{B}',\Lambda}) \leq \beta |\Lambda| \exp(-\beta' d(y, \Lambda)),$$

where $d(y, \Lambda)$ is the distance within R from the vertex y to the region Λ.

For a wide family of graphs, this notion of strong spatial mixing implies that there is a unique infinite-volume Gibbs measure with exponentially decaying correlations. For further details on this connection see [22, 30, 31].

In order to demonstrate that there is strong spatial mixing for the systems studied in this paper, we will consider an arbitrary finite set of vertices R and two different colourings $\mathcal{B}$ and $\mathcal{B}'$ of the boundary of R which differ at a single vertex y. We will show inductively that there is a coupling of the two conditional distributions in which, for every vertex $v \in R$, the probability of disagreement at v is exponentially small (as a function of its distance to the boundary discrepancy).

Another issue that we address is the mixing time of Glauber dynamics for *sampling* proper graph colourings. Let R be a finite region and let $\mathcal{B}$ be a colouring of ∂R. The (heat-bath) Glauber dynamics is a Markov chain that can be used to sample from $S(\mathcal{B})$, the set of proper colourings that are consistent with the colouring $\mathcal{B}$ of ∂R. The transition from a colouring $\sigma \in S(\mathcal{B})$ is made by choosing a vertex v uniformly at random from R and then recolouring v from the conditional distribution induced by the colours of the neighbours of v.

A sufficient condition for the Glauber dynamics Markov chain to be *connected* (i.e. any proper colouring can be obtained from another proper colouring by a series of the transitions described) is to have $q \geq \Delta + 2$, where Δ is the maximum degree of the graph. The stationary distribution of this Markov chain is $\pi_{\mathcal{B}}$, the uniform distribution on $S(\mathcal{B})$. In this setting, the question of interest is to determine the *mixing time*, $\tau(\delta)$, of the Glauber dynamics chain, defined as

$$\tau(\delta) = \min\{t : d_{\mathrm{tv}}(P^{(t')}(\sigma, \cdot), \pi_{\mathcal{B}}) \leq \delta \quad \forall t' \geq t\}.$$

Here $P^{(t)}(\sigma, \nu)$ is the probability of moving from σ to ν in exactly t steps of the Markov chain.

Heat-bath dynamics on larger regions is defined similarly except that a "block" of K vertices is updated during each transition. See [13] for one example of heat-bath dynamics on the lattice $\mathbb{Z}^2$. We discuss a general version of heat-bath dynamics later when we examine the connections between strong spatial mixing and rapid mixing more closely.

For some graphs with sub-exponential growth (that is, for graphs in which the volume of increasing balls around any vertex increases sub-exponentially with the radius), it is well known that strong spatial mixing implies rapid mixing of Glauber dynamics. For example, [13] provides a purely combinatorial proof that when G is the d-dimensional integer lattice $\mathbb{Z}^d$, if the system has strong spatial mixing then there exists a finite integer K for which the heat-bath dynamics on a "cube" of side length K mixes in $O(n \log n)$ time, where $n = |R|$. This result holds for the "permissive" case, which corresponds to the restriction $q > \Delta + 1$ in our setting. As [13] observes, it is also known for $\mathbb{Z}^d$ that strong spatial mixing implies $O(n \log n)$ mixing for Glauber dynamics (see [6, 22]) though no purely combinatorial proof of this fact is known. Also, the proofs as written may need to be modified to apply to the "zero-temperature" (proper colouring) case. Even without using these results in the zero-temperature case, we can deduce that Glauber dynamics mixes in polynomial time (in fact, in $O(n^2)$ time) for a general family of graphs. This can be shown by using the comparison method of Diaconis and Saloff-Coste [7] to turn a rapid mixing result for heat-bath dynamics (for a fixed K) into a rapid mixing result for Glauber dynamics. For an example on the integer lattice $\mathbb{Z}^2$, refer to Theorem 2 of [1] which shows rapid mixing for Glauber dynamics on 6-colourings of square pieces of $\mathbb{Z}^2$. (For convenience, the authors have bounded the mixing time as $O(n^2 \log n)$ but if one wanted to tighten the bound to $O(n^2)$ by tuning the parameters in the comparison, this is possible. See, for example, [11, Example 9].)

The theorems in [13] are explicitly stated for the integer lattice $\mathbb{Z}^d$ but the authors state that similar techniques apply to any lattice with sub-exponential growth. This is mentioned as a footnote in [13] and is discussed more fully in [31]. We provide a proof that strong spatial mixing implies rapid mixing of Glauber dynamics for a class of graphs that we call neighbourhood-amenable, whose definition is given below.

First, for any vertex $v \in G$ and a non-negative integer d, let $Ball_d(v)$ denote the set of vertices that are at most distance d from v. Thus we have $Ball_0(v) = \{v\}$.

Definition 2 *For a non-negative integer d, let $T_d = \sup_{v \in G} \frac{|\partial Ball_d(v)|}{|Ball_d(v)|}$. G is said to be* neighbourhood-amenable *if $\inf_d T_d = 0$.*

Neighbourhood-amenability is a related, yet different, notion to amenability in graphs.[2] From the definition we can see that for a neighbourhood-amenable graph, given any real number $c > 0$ we can find $d \geq 0$ such that $\frac{|\partial Ball_d(v)|}{|Ball_d(v)|} \leq c$, uniformly in v, meaning that the "surface-area-to-volume" ratio of balls can be made arbitrarily small with a suitable choice of radius d. Most natural lattices, such as the triangular lattice and $\mathbb{Z}^k$, are neighbourhood-amenable.

Conditions under which we can prove rapid mixing of Glauber dynamics on neighbourhood-amenable graphs are given in Theorem 8 in Section 1.3.

[2] An infinite graph G is *amenable* if

$$\inf \left\{ \frac{|\partial S|}{|S|} : S \text{ is a finite and non-empty subset of } V(G) \right\} = 0.$$

1.2 The framework

Our results rely on considering proper colourings of a finite region for a pair of boundary colourings of that region that differ on the colour of a single vertex. First, we outline the general framework in which we operate.

Let G denote an infinite graph with maximum degree Δ. Let R be a *finite* subgraph of G, and as before define ∂R to be the boundary of R, i.e. those vertices in G that are not in R but are joined by an edge to at least vertex of R. We first give the following definition.

Definition 3 *A* vertex-boundary *pair X consists of*

- *a non-empty finite region R_X of the graph G,*
- *a distinguished vertex v_X in ∂R_X, and*
- *a pair $(\mathcal{B}_X^1, \mathcal{B}_X^2)$ of colourings of ∂R_X (using the colours $Q \cup \{0\}$) which differ only on the vertex v_X. We require that the two colours $\mathcal{B}_X^1(v_X)$ and $\mathcal{B}_X^2(v_X)$ are both in the set Q. That is, the two boundary colourings differ on the colour of v_X, but this vertex is not an unconstrained vertex (with colour 0) in either boundary colouring.*

We are interested in the effect that the difference in colour at v_X has on the other vertices. Let $S(\mathcal{B}_X^1)$ be the set of proper q-colourings of R_X that are consistent with the boundary colouring $\mathcal{B}_X^1$, and similarly define $S(\mathcal{B}_X^2)$. We use $\pi_{\mathcal{B}_X^1}$ (resp. $\pi_{\mathcal{B}_X^2}$) to denote the uniform distribution on $S(\mathcal{B}_X^1)$ (resp. $S(\mathcal{B}_X^2)$).

We want to construct a *coupling* Ψ_X of the distributions $\pi_{\mathcal{B}_X^1}$ and $\pi_{\mathcal{B}_X^2}$, i.e., a joint distribution on $S(\mathcal{B}_X^1) \times S(\mathcal{B}_X^2)$ that has $\pi_{\mathcal{B}_X^1}$ and $\pi_{\mathcal{B}_X^2}$ as its marginal distributions. For such a coupling Ψ_X and for each vertex $f \in R_X$, we define the indicator random variable $1_{\Psi_X,f}$ for the event that, when a pair of colourings is drawn according to Ψ_X, the colour of f differs in these two colourings. We would like to show that $\sum_{f \in R_X} \mathbb{E}[1_{\Psi_X,f}]$ is small. If this quantity is small enough for all vertex-boundary pairs X, we can use that conclusion to infer strong spatial mixing. We can also show rapid mixing of Glauber dynamics for a general class of graphs. One way to show the sum is small is to show that $\mathbb{E}[1_{\Psi_X,f}]$ decreases rapidly as the distance between v_X and f grows. We give a method to construct a coupling using couplings of subgraphs which may overlap. In the course of the proof we use what we call an ε-coupling cover for G, whose definition follows.

Definition 4 *Let G denote an infinite graph with maximum degree Δ. Fix $\varepsilon > 0$. We say that G has an* ε-coupling cover *if for all vertex-boundary pairs X, there is a coupling Ψ_X of $\pi_{\mathcal{B}_X^1}$ and $\pi_{\mathcal{B}_X^2}$ such that*

$$\sum_{f \in R_X} \mathbb{E}[1_{\Psi_X,f}] \leq \frac{\Delta}{\varepsilon}.$$

Thus, if G has an ε-coupling cover, then the sum $\sum_{f \in R_X} \mathbb{E}[1_{\Psi_X,f}]$ is small. The precise manner in which this property is used to prove strong spatial mixing is described in Section 4 after we lay the groundwork in Sections 2 and 3.

1.3 Our results

We define two constants. Let α be the solution to $\alpha^\alpha = e$ (so $\alpha \approx 1.76322$) and $\gamma = \frac{4\alpha^3-6\alpha^2-3\alpha+4}{2(\alpha^2-1)} \approx 0.47031$.

Our first main result is

Theorem 5 *Let G denote an infinite triangle-free graph and suppose that for some $\Delta \geq 3$ the maximum degree of G is at most Δ. The spin system specified by uniform finite-volume Gibbs measures on proper q-colourings of G has strong spatial mixing if $q > \alpha\Delta - \gamma$.*

With some additional consideration of the structure of the graph we can prove two additional results about strong spatial mixing. Theorem 6, proven in Section 5, uses a system of recurrence relations to show strong spatial mixing. The geometry of $\mathbb{Z}^3$ play an important role in deriving this system of recurrences. This special case is not covered by our general result above since $\alpha \cdot 6 - \gamma \approx 10.10901$.

Theorem 6 *The spin system specified by uniform finite-volume Gibbs measures on proper 10-colourings of $\mathbb{Z}^3$ has strong spatial mixing.*

With computational assistance, we also show another special case that is not covered by our general theorem (this graph has lots of triangles in it!). In this case we again derive a system of recurrence relations to show strong spatial mixing. See Section 6 for more details.

Theorem 7 *The spin system specified by uniform finite-volume Gibbs measures on proper 10-colourings of the triangular lattice has strong spatial mixing.*

In addition to the results on strong spatial mixing, we prove a general result on rapid mixing of Glauber dynamics for sampling proper colourings. Provided there exists an ε-coupling cover, Glauber dynamics is rapidly mixing for neighbourhood-amenable graphs.

Theorem 8 *Let G denote an infinite neighbourhood-amenable graph with maximum degree Δ. Let R be a finite subgraph of G with $|R| = n$ and $\mathcal{B}(R)$ denote a colouring of $\partial(R)$ using the colours $Q \cup \{0\}$. (We assume that $q \geq \Delta + 2$.)*

Suppose there exists $\varepsilon > 0$ such that G has an ε-coupling cover. Then the Glauber dynamics Markov chain on $S(\mathcal{B}(R))$ is rapidly mixing and $\tau(\delta) \in O(n(n + \log \frac{1}{\delta}))$.

Path coupling is used to prove this theorem. To show Theorem 8 we first examine a heat-bath Markov chain on "windows" in the graph (more specifically, small regions of the form $Ball_d(v) \cap R$ for a suitable d) and prove this chain mixes in time $O(n \log n)$. Then using the standard technique of comparing Markov chains, we are able to conclude that the simpler Glauber dynamics (single-vertex) chain is rapidly mixing in time $O(n^2)$. It is for this reason that we require an ε-coupling cover for G, since in our analysis we examine proper colourings of R that differ at a single vertex and we need to determine what happens in a single step of the heat-bath chain.

A brief review of path coupling, the comparison method, and all the details of the proof of Theorem 8 can be found in Section 7.

In the proof of Theorem 5 we construct an ε-coupling cover (see Lemma 15 and Lemma 20). Taking these and Theorem 8 together, we obtain the following corollary regarding rapid mixing of Glauber dynamics. See Section 7.5 for some remarks on the "neighbourhood-amenable" condition in the hypothesis of Theorem 8 and in the corollary.

Corollary 9 *Let G denote an infinite triangle-free, neighbourhood-amenable graph and suppose that for some $\Delta \geq 3$ the maximum degree of G is at most Δ. Suppose $q > \alpha\Delta - \gamma$. Let R be a finite subgraph of G with $|R| = n$ and $\mathcal{B}(R)$ denote a colouring of $\partial(R)$ using the colours $Q \cup \{0\}$. (We assume that $q \geq \Delta + 2$.)*

The Glauber dynamics chain on proper colourings of R compatible with $\mathcal{B}(R)$ is rapidly mixing with $\tau(\delta) \in O(n(n + \log \frac{1}{\delta}))$.

Since $\mathbb{Z}^3$ and the triangular lattice are neighbourhood-amenable graphs, we also obtain the following corollaries. We note that in the course of showing strong spatial mixing for 10-colourings of $\mathbb{Z}^3$ and the triangular lattice, we prove the existence of an ε-coupling cover in each case. These results, together with Theorem 8, give us the two corollaries below. See Sections 5 and 6, respectively, for more details on the existence of an ε-coupling cover in each case.

Corollary 10 *Let R denote a finite subgraph of $\mathbb{Z}^3$ with $|R| = n$. Let $\mathcal{B}(R)$ denote a colouring of ∂R with the colours $\{1, \ldots, 10\} \cup \{0\}$. Glauber dynamics is rapidly mixing on the set of 10-colourings compatible with $\mathcal{B}(R)$ and $\tau(\delta) \in O(n(n + \log \frac{1}{\delta}))$.*

Corollary 11 *Let R denote a finite subgraph of the triangular lattice with $|R| = n$. Let $\mathcal{B}(R)$ denote a colouring of ∂R with the colours $\{1, \ldots, 10\} \cup \{0\}$. Glauber dynamics is rapidly mixing on the set of 10-colourings compatible with $\mathcal{B}(R)$ and $\tau(\delta) \in O(n(n + \log \frac{1}{\delta}))$.*

1.4 Related work

Previous papers [20, 23, 24] have used recursively-constructed couplings to show rapid mixing and exponential decay of correlations on trees. To apply this approach more generally (i.e., to graphs other than trees) we need a mechanism for constructing a coupling from couplings of subgraphs (even though these subgraphs may overlap). Our approach (see Lemma 12) gives an upper bound on the effect of the discrepancy at a site by summing over discrepancies at adjacent edges (using the triangle inequality as in path coupling). The subgraphs corresponding to these edges overlap but the triangle inequality is used a second time to bound the quality of the resulting coupling.

The closest directly applicable result similar to ours is that of Salas and Sokal [27]. They showed that strong spatial mixing occurs whenever $q > 2\Delta$. Given that strong spatial mixing and rapid mixing are sometimes interchangeable (as we noted above), it is perhaps more appropriate to compare our result with recent (stronger) results about rapid mixing for colourings. There are lots of these results. Since our goal is to have results which apply to lattices such as those studied in statistical physics (i.e., small Δ and small q) the most relevant result is the new theorem of Dyer, Frieze, Hayes and Vigoda [10]. They show that if the girth of the graph is at least 5 (i.e., there are no 4-cycles or triangles) then Glauber dynamics is rapidly mixing provided $q > \max(\alpha\Delta, C)$ where C is an absolute constant (it depends upon $q - \alpha\Delta$ but not upon the number of vertices) which is at least 200. Our result (Theorem 5) can be viewed as a companion to that one. Both results apply when $q > \alpha\Delta$. Ours gives strong spatial mixing when the girth is at least 4 (and the maximum degree $\Delta \geq 3$). The result in [10] gives rapid mixing when the girth is at least 5 and $q \geq C$. The two results are interesting for different, but overlapping, classes of graphs. Ours is interesting (since it implies uniqueness of Gibbs measure and rapid mixing) even for graphs with very small degree (all the way down to $\Delta = 3$ and $q = 6$) but the applications to uniqueness and rapid mixing only apply if the graph is neighbourhood-amenable (or some similar condition). (All natural lattices satisfy this.) The result of [10] is interesting even for graphs with other neighbourhood growth properties, but it only applies if $q \geq C$.

Better results for rapid mixing are known when the degree, or the girth, is guaranteed to be large. (For graphs with large degree, the distribution is concentrated, so strong results are possible.) These results include rapid mixing for $q > \alpha\Delta$ assuming $\Delta = \Omega(\log n)$ and girth at least 4 (Hayes and Vigoda [18]), rapid mixing for $q > \alpha\Delta$ assuming $\Delta = \Omega(\log n)$ and "local sparsity" (Frieze and Vera [14]), rapid mixing for $q > (1+\varepsilon)\Delta$ assuming $\Delta = \Omega(\log n)$ and girth at least 9 (Hayes and Vigoda [17]), and rapid mixing for graphs with girth at least 6 when $q > \max(\beta\Delta, C')$ for some constant C' and $\beta \approx 1.49$ (Dyer et al. [10]).

Theorem 5 provides the first hand proof of strong spatial mixing for 7-colourings of triangle-free graphs with degree at most 4. A machine-assisted proof for the rectangular lattice was provided by Salas and Sokal [27] and a machine-assisted proof of rapid mixing for triangle-free 4-regular graphs was provided by Bubley, Dyer, Greenhill and Jerrum [5]. Our result also shows strong spatial mixing for $q = 5$ and $\Delta = 3$. This is the first hand proof of strong spatial mixing for 5-colourings of triangle-free graphs with degree at most 3. A machine-assisted proof for the special case of the hexagonal lattice was proved by Salas and Sokal [27]. They also give a machine-assisted proof for 4-colourings of this lattice.

In Section 5 we show how to improve our general technique by considering the geometry of the lattice. The idea is to construct the recursive coupling from a system of recurrences rather than from a single recurrence. We use the geometry of the lattice to derive the system of recurrences. This gives us an analysis with a horizon of more than one level of induction, which leads to improved results. We illustrate this idea by proving strong spatial mixing for $q = 10$ on the lattice $\mathbb{Z}^3$.

In Section 6 we further extend our results using computational assistance. An idea that gets used to reduce the amount of computation is the notion of a "relevant" boundary pair. In order to reduce the search space, we want to look just at "relevant" boundary pairs, and not at all of them. Boundary pairs induced by vertex boundaries are "relevant", and we can show by induction that our method recurses from relevant boundary pairs to relevant boundary pairs of sub-problems. The proof of this fact again relies on the geometry of the lattice. See Section 6 for details. Using the approach we obtain a (machine-assisted) proof of strong spatial mixing (and therefore, uniqueness of the infinite-volume Gibbs measure) for $q = 10$ on the triangular lattice. This improves an earlier result of Salas and Sokal [27] which used machine-assisted proof to show strong spatial mixing for $q = 11$. Our approach can also be used to show (with computational assistance) strong spatial mixing for $q = 6$ on the rectangular lattice. This gives an alternative proof of the result of Achlioptas, Molloy, Moore and Van Bussel [1] (which was also proved with machine assistance).

As we have previously mentioned, using standard techniques, our results can be used to show rapid mixing for Glauber dynamics for a wide class of graphs. See Section 7 for full details.

2 Exponential decay and edge discrepancies

Let R be a non-empty finite region of the graph. For most of the technical part of the paper it will be convenient to consider the edge-boundary of R rather than the boundary ∂R of vertices surrounding R. Here is the notation that we will use. The *boundary* of the region R

is the collection of all edges that have exactly one endpoint in R. A *colouring* of the boundary is a function from the set of edges in the boundary to the set $\{0\} \cup Q$.

Let R be a finite region and let B be a colouring of its boundary. A colouring C of R is said to be *proper* if

- adjacent vertices in R receive different colours, and
- vertices in R receive colours different from adjacent boundary edges.

Let $S(B)$ denote the set of proper colourings of R and let π_B be the uniform distribution on $S(B)$. We will be interested in studying how much $S(B)$ varies when we change the boundary colouring B by recolouring a single edge. This small change to the boundary is formalised in the following notation.

A *boundary pair* X consists of

- a non-empty finite region R_X of the graph,
- a distinguished edge s_X on the boundary of R_X, and
- a pair (B_X, B'_X) of colourings of the boundary of R_X which differ only on the edge s_X. We require that the two colours $B_X(s_X)$ and $B'_X(s_X)$ are both in Q. That is, the two boundary colourings differ on the colouring of edge s_X, but this edge is not an unconstrained edge (with colour 0) in either boundary colouring.

For any boundary pair X, we define f_X to be the endpoint of s_X that is in R_X and w_X to be the other endpoint of s_X. Let E_X be the set of edges which connect f_X to another vertex in R_X. A *coupling* Ψ of π_{B_X} and $\pi_{B'_X}$ is a distribution on $S(B_X) \times S(B'_X)$ which has marginal distributions π_{B_X} and $\pi_{B'_X}$. For such a coupling Ψ, we define $1_{\Psi,f}$ to be the indicator random variable for the event that, when a pair of colourings is drawn from Ψ, the colour of f differs in these two colourings.

For any boundary pair X we define Ψ_X to be some coupling of π_{B_X} and $\pi_{B'_X}$ minimising $\mathbb{E}[1_{\Psi,f_X}]$. For every pair of colours c and c', let $p_X(c,c')$ be the probability that, when a pair of colourings (C, C') is drawn from Ψ_X, f_X is coloured with colour c in C and with colour c' in C'.

Suppose that X is a boundary pair and that f is a vertex in R_X. Let $d(f, s_X)$ denote the distance within R_X from f to s_X. Thus, $d(f_X, s_X) = 1$ and if vertex f in R_X adjoins f_X then $d(f, s_X) = 2$ and so on.

A main objective is to prove that the effect of the discrepancy at the boundary edge s_X decays exponentially with the distance from s_X (see Lemma 19). In order to do this, we use a recursive coupling (Lemma 12). The technique in Lemma 12 does not require that the graph be triangle-free — the general technique should also be applicable to models other than colourings.

To aid our analysis, we define a labelled tree T_X associated with each boundary pair X. The tree T_X is constructed as follows. Start with a vertex r which will be the root of T_X. For every pair of colours $c \in Q$ and $c' \in Q$, add an edge labelled $(p_X(c,c'), f_X)$ from r to a new node $r_{c,c'}$. If E_X is empty, $r_{c,c'}$ is a leaf. Otherwise, let $e_1, \ldots, e_k$ be the edges in E_X. For each $i \in \{1, \ldots, k\}$, let $X_i(c,c')$ be the boundary pair consisting of

- the region $R_X - f_X$;

- the distinguished edge e_i;
- the colouring B of the boundary of $R_X - f_X$ that
  - agrees with B_X on common edges,
  - colours $e_1, \ldots, e_{i-1}$ with colour c', and
  - colours $e_i, \ldots, e_k$ with colour c; and
- the colouring B' that agrees with B except that it colours e_i with colour c'.

Recursively construct $T_{X_i(c,c')}$, the tree corresponding to boundary pair $X_i(c,c')$. Add an edge with label $(1,\cdot)$ from $r_{c,c'}$ to the root of $T_{X_i(c,c')}$. That completes the construction of T_X.

We say that an edge e of T_X is *degenerate* if the second component of its label is "$\cdot$". For edges e and e' of T_X, we write $e \to e'$ to denote the fact that e is an ancestor of e'. That is, either $e = e'$, or e is a proper ancestor of e'. Define the *level* of edge e to be the number of non-degenerate edges on the path from the root down to, and including, e. Suppose that e is an edge of T_X with label (p, f). We say that the *weight* $w(e)$ of edge e is p. Also the *name* $n(e)$ of edge e is f. The *likelihood* $\ell(e)$ of e is $\prod_{e':e'\to e} w(e)$. The *cost* $\gamma(f, T_X)$ of a vertex f in T_X is $\sum_{e:n(e)=f} \ell(e)$.

Lemma 12 *For every boundary pair X there exists a coupling Ψ of π_{B_X} and $\pi_{B'_X}$ such that, for all $f \in R_X$, $\mathbb{E}[1_{\Psi,f}] \le \gamma(f, T_X)$.*

Proof. The coupling Ψ is constructed recursively in the same manner as the tree T_X, where at each stage, the discrepancy at a given vertex is broken to discrepancies at single edges, so at every stage of the recursion we only need to consider a pair of colourings with a discrepancy at a single edge (i.e., a boundary pair).

Let (C, C') denote the random variable corresponding to a pair of colourings from Ψ. If $|R_X| = 1$ then $\Psi = \Psi_X$. Otherwise, let $e_1, \ldots, e_k$ be the edges in E_X, i.e., those that are adjacent both to f_X and to another vertex in R_X. We will use Ψ_X to couple the colouring of vertex f_X and we will recursively construct a different coupled colouring of the other vertices in R_X. We will assign $C(f_X) = c$ and $C'(f_X) = c'$ with probability $p_X(c, c')$.

Let $X(c, c')$ be an "extended boundary pair" consisting of

- the region $R_X - f_X$,
- the colouring B_c of the boundary of $R_X - f_X$ that agrees with B_X on common edges and colours edges in E_X with colour c, and
- the colouring $B_{c'}$ of the boundary of $R_X - f_X$ that agrees with B_X on common edges and colours edges in E_X with colour c'.

We can complete the coupling Ψ by constructing (for each c and c') a coupling $\Psi_{c,c'}$ of π_{B_c} and $\pi_{B_{c'}}$. (The reader may verify that this ensures that the marginal distributions of Ψ are correct.) The particular choice that we make for $\Psi_{c,c'}$ is either the perfect coupling if $c = c'$ or, if $c \ne c'$, the composition[3] of $\Psi_1(c, c'), \ldots, \Psi_k(c, c')$ where $\Psi_i(c, c')$ is a recursively-constructed coupling for boundary pair $X_i(c, c')$.

[3]The composition that we have in mind is the natural one — to choose a pair (σ_0, σ_k) from $\Psi_{c,c'}$ first choose (σ_0, σ_1) from $\Psi_1(c, c')$. Say $\sigma_0 = x_0$ and $\sigma_1 = x_1$. Then choose (σ_1, σ_2) from the conditional distribution of $\Psi_2(c, c')$, conditioned on $\sigma_1 = x_1$. Say that $\sigma_2 = x_2$. Now choose (σ_2, σ_3) from the conditional distribution of $\Psi_3(c, c')$ conditioned on $\sigma_2 = x_2$, and so on. This is the same as the composition that occurs in path coupling [4].

We will now show that, for all $f \in R_X$, $\mathbb{E}[1_{\Psi,f}] \leq \gamma(f, T_X)$. The proof is by induction on $|R_X|$. If $f = f_X$ then $\mathbb{E}[1_{\Psi,f}] = \gamma(f, T_X)$ by the construction of Ψ and T_X. This handles the base case, $|R_X| = 1$. Suppose $f \neq f_X$ and $|R_X| > 1$. Then

$$\begin{aligned}\mathbb{E}[1_{\Psi,f}] = \sum_{c,c'} p_X(c,c')\mathbb{E}[1_{\Psi_{c,c'},f}] &\leq \sum_{c,c'} p_X(c,c') \sum_{i=1}^{k} \mathbb{E}[1_{\Psi_i(c,c'),f}] \\ &\leq \sum_{c,c'} p_X(c,c') \sum_{i=1}^{k} \gamma(f, T_{X_i(c,c')}) = \gamma(f, T_X),\end{aligned}$$

where the second inequality uses the inductive hypothesis. □

Suppose that X is a boundary pair. Lemma 12 ensures that there is a coupling of π_{B_X} and $\pi_{B'_X}$ with substantial agreement as long as, for most vertices $f \in R_X$, $\gamma(f, T_X)$ is small. A key ingredient from the construction of T_X which affects $\gamma(f, T_X)$ is the quantity $\mathbb{E}[1_{\Psi_X, f_X}]$, which we denote $\nu(X)$. (Thus, $\nu(X) = \min_\Psi \mathbb{E}[1_{\Psi, f_X}]$, where the minimum is over all couplings Ψ of π_{B_X} and $\pi_{B'_X}$.) An important part of our method is to determine good upper bounds on $\nu(X)$.

Let $d = B_X(s_X)$ and $d' = B'_X(s_X)$. Let $Q' = Q - \{d, d'\}$. For $c \neq d$, let $n_c(X)$ denote the number of colourings in $S(B_X)$ which colour f_X with colour c. Let $n_d(X)$ denote the number of colourings in $S(B'_X)$ which colour f_X with colour d. Let $N(X) = \sum_{c \in Q'} n_c(X)$. Define $\mu(X)$ to be the maximum of the probabilities $\Pr_{\pi_{B_X}}(f_X = d')$ and $\Pr_{\pi_{B'_X}}(f_X = d)$. That is,

$$\mu(X) = \frac{\max(n_d(X), n_{d'}(X))}{N(X) + \max(n_d(X), n_{d'}(X))}.$$

We use the following straightforward lemma to derive upper bounds on $\nu(X)$. Figure 1 is an illustration of part (ii) of this lemma. The basic idea is to pick a subregion R' that contains the vertex f_X. Compute the maximum value of μ for that subregion, where we maximize over colourings of the boundary of R' that agree with $B(R)$ on the common overlap of these boundaries. This maximum value is an upper bound for $\nu(X)$.

Lemma 13 *Suppose that X is a boundary pair. Let R' be any subset of R_X which includes f_X. Let χ be the set of boundary pairs $X' = (R_{X'}, s_{X'}, B_{X'}, B'_{X'})$ such that $R_{X'} = R'$, $s_{X'} = s_X$, $B_{X'}$ agrees with B_X on common edges, and $B'_{X'}$ agrees with B'_X on common edges. Then $\nu(X) \leq \max_{X' \in \chi} \mu(X')$.*

Proof. We will show
(i) $\nu(X) \leq \mu(X)$, and
(ii) $\mu(X) \leq \max_{X' \in \chi} \mu(X')$.

Let X be a boundary pair. To shorten the notation we will use n_c to denote $n_c(X)$ and N to denote $N(X)$. Let $d = B_X(s_X)$ and $d' = B'_X(s_X)$. Let $Q' = Q - \{d, d'\}$.

For (i), we can construct a coupling Ψ of π_{B_x} and $\pi_{B'_X}$ which matches N colourings, each of which occurs with probability at least

$$\frac{1}{\max(|S(B_X)|, |S(B'_X)|)} = \frac{1}{N + \max(n_d, n_{d'})}.$$

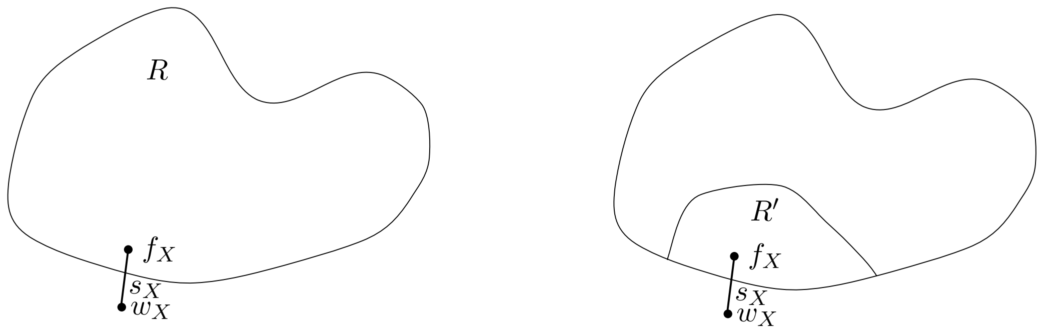

Figure 1: Fix a subregion R' containing f_X. Then maximize $\mu(X')$ over boundary colourings of R' that agree with $B(R)$ on the overlapping part of the boundary.

Thus,

$$\nu(X) \leq 1 - \frac{N}{N + \max(n_d, n_{d'})} = \mu(X).$$

(In fact, $\nu(X) = \mu(X)$, but we will not need this fact.)

Part (ii) will follow from the fact that $\Pr_{\pi_{B_X}}(f_X = d')$ is a convex combination of $\Pr_{\pi_{B_{X'}}}(f_X = d')$ for $X' \in \chi$ and $\Pr_{\pi_{B'_X}}(f_X = d)$ can be decomposed similarly. Let $W = R_X - R_{X'}$. Let H be the set of colourings of W. For $c \neq d$ and $\rho \in H$, let $n_{c,\rho}$ denote the number of colourings in $S(B_X)$ which colour f_X with colour c and W with colouring ρ. For $\rho \in H$, let $n_{d,\rho}$ denote the number of colourings in $S(B'_X)$ which colour f_X with colour d and W with colouring ρ. Let $N_\rho = \sum_{c \in Q'} n_{c,\rho}$. Then

$$\begin{aligned}
\mu(X) &= 1 - \frac{N}{N + \max(n_d, n_{d'})} \\
&= 1 - \frac{\sum_{\rho \in H} N_\rho}{\sum_{\rho \in H} N_\rho + \max(\sum_{\rho \in H} n_{d,\rho}, \sum_{\rho \in H} n_{d',\rho})} \\
&\leq 1 - \frac{\sum_{\rho \in H} N_\rho}{\sum_{\rho \in H} N_\rho + \sum_{\rho \in H} \max(n_{d,\rho}, n_{d',\rho})} \\
&= \frac{\sum_{\rho \in H} \max(n_{d,\rho}, n_{d',\rho})}{\sum_{\rho \in H}(N_\rho + \max(n_{d,\rho}, n_{d',\rho}))} \\
&\leq \max_{\rho \in H} \frac{\max(n_{d,\rho}, n_{d',\rho})}{N_\rho + \max(n_{d,\rho}, n_{d',\rho})} \\
&= \max_{\rho \in H} \mu(X'),
\end{aligned}$$

where X' is the boundary pair that is induced by ρ. □

3 Bounding $\mu(X)$

In this section we show how to bound $\mu(X)$ for triangle-free graphs with sufficiently many colours. So that we can separate the task of bounding $\mu(X)$ from the task of showing strong spatial mixing, we define the notion of "ε-good". Informally, the number of colours q will be

ε-good for a graph G (for some $\varepsilon \in (0,1)$) whenever we can show strong spatial mixing for q-colourings of G.

Definition 14 *Suppose $\varepsilon \in (0,1)$. We will say that the number of colours, q, is ε-good for the graph G if*

$$\mu(X) \leq \frac{1}{\max(1,r)}\left(\frac{1}{1+\varepsilon}\right)$$

for every boundary pair X in which R_X consists of a node f_X plus $r \geq 0$ neighbours $y_1, \ldots, y_r$ of f_X.

The purpose of this section is to prove Lemma 15 given below, which enables us to establish strong spatial mixing whenever $q > \alpha\Delta - \gamma$. The basic idea of the lemma is this. Consider a triangle-free region R_X and boundary condition $\mathcal{B}_X$. Suppose that the region contains sufficiently many neighbours of a vertex f_X which is adjacent to the boundary. Then we derive an upper bound on the probability, in the equilibrium distribution, that f_X is assigned a particular colour (in particular, the colour d' from the definition of $\mu(X)$).

Lemma 15 *Let α be the solution to $\alpha^\alpha = e$ (so $\alpha \approx 1.76322$), and $\gamma = \frac{4\alpha^3-6\alpha^2-3\alpha+4}{2(\alpha^2-1)} \approx 0.47031$. Suppose that the graph G is triangle-free and that for some $\Delta \geq 3$ the maximum degree of G is at most Δ. If $q \geq \alpha\Delta - \gamma + \alpha\varepsilon(\Delta - 1)$ then q is ε-good for G.*

We prove Lemma 15 by reducing to the case in which R_X contains only f_X and its neighbours. We then use the fact that the graph has no triangles to count the number of colourings as a product. This leaves us with an optimization problem, the solution of which gives the result. Before we can prove Lemma 15, it helps to prove a preliminary lemma.

Lemma 16 *Let r, q, Δ be integers satisfying $q > \Delta > r \geq 2$. Define $p = q - \Delta + 1 \geq 2$. Consider a set of $\{0,1\}$-variables $\{\delta_{c,j} :\ 1 \leq c \leq q;\ 1 \leq j \leq r\}$ and an integer q', subject to the bounds:*

$$s_j = \sum_{c=1}^{q} \delta_{c,j} \geq p \text{ for } 1 \leq j \leq r; \quad q \geq q' \geq p + r.$$

Define

$$n_c = \prod_{j=1}^{r}(s_j - \delta_{c,j}) \text{ for } 1 \leq c \leq q', \quad \textit{and} \quad Z = \sum_{c=3}^{q'} n_c/n_1.$$

Then a minimal value of Z is attained by taking $q' = p + r$ and some choice of δ's such that $s_j = p$ for all j and $\delta_{c,j} = 0$ if $c \notin \{3, \ldots, q'\}$.

Proof. The choice $q' = p + r$ is clearly optimal. Fix some j and consider the dependence of Z on $\delta_{c,j}$ for all c, $1 \leq c \leq q'$. For some positive $a_1, \ldots, a_{q'}$, we can write

$$Z = \sum_{c=3}^{q'} \frac{a_c(s_j - \delta_{c,j})}{a_1(s_j - \delta_{1,j})}.$$

We now suppose that Z is minimal and derive the properties claimed. First, we can ensure that $\delta_{c,j} = 0$ for $c \notin \{3, \ldots, q'\}$. If any of these values $\delta_{c,j}$ is positive, we can set it to zero without increasing Z. If s_j had been at its lower bound of p then we can restore this value by increasing $\delta_{c,j}$ from 0 to 1 for some $c \in \{3,\ldots,q'\}$. Note that such a c exists since $q'-2 \geq p$.

Now $n_1 = a_1 s_j$ (since $\delta_{1,j} = 0$), and

$$Z = \sum_{c=3}^{q'} \frac{a_c}{a_1}(1 - \frac{\delta_{c,j}}{s_j}) = A - \frac{1}{a_1}\frac{\sum_{c=3}^{q'} a_c \delta_{c,j}}{\sum_{c=3}^{q'} \delta_{c,j}}, \text{ where } A = \sum_{c=3}^{q'} \frac{a_c}{a_1}.$$

Since $\sum_{c=3}^{q'} a_c \delta_{c,j} / \sum_{c=3}^{q'} \delta_{c,j}$, is the average of s_j of the a's, a maximal value for this quotient can be obtained by taking $s_j = p$, i.e., as small as possible, and selecting a set of p largest a's with the corresponding δ's. □

Here is the proof of Lemma 15.

Proof. We will show that q is ε-good for G, assuming that $q \geq \alpha\Delta - \gamma + \varepsilon\alpha(\Delta - 1)$. Again, let $p = q - \Delta + 1$.

Suppose that X is a boundary pair in which R_X consists of a node f_X plus $r \leq 1$ neighbours. By Part (ii) of Lemma 13, $\mu(X) \leq \max_{X'} \mu(X')$ where X' is a boundary pair containing f_X only. The numerator of $\mu(X')$ is at most 1. The denominator is at least $q - \Delta$ so $\mu(X) \leq 1/(q - \Delta)$. Now we have $q - \Delta \geq (\alpha - 1)\Delta - \gamma + \varepsilon\alpha(\Delta - 1) > 1 + \varepsilon$ for $\Delta \geq 3$, so $\mu(X) < 1/(1 + \varepsilon)$.

For a boundary pair X we will use the notation $\mu_1(X)$ to denote $n_1(X)/(N(X)+n_1(X))$. Define $\mu_2(X)$ similarly. We will show that, for every boundary pair X in which R_X consists of a node f_X plus $r > 1$ neighbours $y_1, \ldots, y_r$ of f_X, we have $\mu_1(X) \leq \frac{1}{r}\left(\frac{1}{1+\varepsilon}\right)$. By symmetry, every such pair has the same upper bound on $\mu_2(X)$ and therefore on $\mu(X)$.

Suppose without loss of generality that $B_X(s_X) = 1$ and $B'_X(s_X) = 2$. Let K be the set of all colours which B_X assigns to neighbours of f_X other than s_X. We can assume without loss of generality that colour 1 is not in K. Otherwise, $\mu_1(X) = 0$. Let $\delta_{c,j}$ be the Boolean indicator variable which is 0 if colour c is used at a neighbour of y_j in the boundary of R_X. Let $Q' = Q - K$.

Now for every $c \in Q - Q'$ we have $n_c(X) = 0$. Since the graph is triangle-free, every $c \in Q'$ satisfies $n_c(X) = \prod_{j=1}^{r}(s_j - \delta_{c,j})$, where $s_j = \sum_{c=1}^{q} \delta_{c,j}$. Thus,

$$\frac{1}{\mu_1(X)} - 1 = \frac{\sum_{c=3}^{q} n_c(X)}{n_1(X)} = \frac{\sum_{c \in Q' - \{1,2\}} n_c(X)}{n_1(X)}.$$

Without loss of generality, we can assume that the colours in $Q' - \{1,2\}$ are colours $3, \ldots, q'$, for some q' so we have

$$\frac{1}{\mu_1(X)} - 1 = \frac{\sum_{c=3}^{q'} n_c(X)}{n_1(X)}. \tag{1}$$

We are now in the framework of Lemma 16. We have the constraints $s_j \geq p$ since the degree of y_j is at most Δ, and $q' \geq p + r$ since $q' - 2 = |Q' - \{1,2\}| \geq q - 2 - (\Delta - (r+1))$. Let Z be the right-hand-side of (1). Z is minimized by taking $q' = p + r$ and some choice of δ's such that $s_j = p$ for all j and $\delta_{c,j} = 0$ if $c \notin \{3, \ldots, q'\}$. Writing $m_c = \sum_{j=1}^{r} \delta_{c,j}$ and plugging in $n_c(X) = \prod_{j=1}^{r}(s_j - \delta_{c,j})$, we have

$$\frac{1}{\mu_1(X)} - 1 \;=\; Z \;\geq\; \sum_{c=3}^{q'} \prod_{j=1}^{r} \left(1 - \frac{\delta_{c,j}}{p}\right)$$

$$= \sum_{c=3}^{q'} \left(1 - \frac{1}{p}\right)^{m_c}, \tag{2}$$

where $\sum_{c=3}^{q'} m_c = \sum_{j=1}^{r} s_j = rp$, and $q' - 2 = p + r - 2 \geq p$.

Our goal is to derive a lower bound for $(Z+1)/r$ with respect to r and the m_c's. Let S denote the expression (2) for an arbitrary choice of m_c's, and consider the effect of increasing r by 1. The sum of the m_c's is increased by p and the number of them, q', is increased by 1. One possibility is to leave the existing m_c's unchanged and add the extra term $(1 - 1/p)^p$, corresponding to the new m_c. To show that the lower bound given by minimizing $(S+1)/r$ with respect to the m_c's is decreasing in r, it is sufficient to verify that

$$\frac{S+1}{r} \geq \frac{S + 1 + \left(1 - \frac{1}{p}\right)^p}{r+1}, \quad \text{i.e.,} \quad S + 1 \geq r\left(1 - \frac{1}{p}\right)^p.$$

The expression S is minimized by taking the m_c's as equal as possible, so

$$S \geq (q' - 2)\left(1 - \frac{1}{p}\right)^{\frac{rp}{q'-2}} \geq r\left(1 - \frac{1}{p}\right)^p,$$

since $q' - 2 \geq r$. Thus the smallest lower bound that we get is derived from the expression S with r taking its maximum value of $\Delta - 1$ (so $q' = q$) and the m_c's being taken as nearly equal as possible subject to integrality constraints. The same bound holds for other choices of r and the m_c's.

We therefore define

$$J(q, \Delta) = (q - 2 - v)\left(1 - \frac{1}{p}\right)^u + v\left(1 - \frac{1}{p}\right)^{u+1} = (q-2)\left(1 - \frac{1}{p}\right)^u \left(1 - \frac{v}{(q-2)p}\right),$$

where $u = \lfloor (\Delta - 1)p/(q-2) \rfloor$ and $v = (\Delta - 1)p \bmod (q-2)$, and also define

$$J'(q, \Delta) = (q-2)\left(1 - \frac{1}{p}\right)^{\frac{p(\Delta-1)}{q-2}}.$$

Note that $J \geq J'$, since J and J' are minimizations with and without the constraint of integral m_c's respectively. We have $Z \geq J(q, \Delta) \geq J'(q, \Delta)$. To prove that q is ε-good we need to show that $Z/(\Delta - 1) \geq 1 + \varepsilon - 1/(\Delta - 1)$. For the current lemma we just use the simpler expression J' in the proof. (We observe later that by using the inequality based on J we may obtain a slight improvement for the lower bound on q for some values of Δ. See the remark following Corollary 5.)

Define $x = 1 - (q-2)/(\alpha(\Delta - 1))$, so that

$$\frac{J'(q, \Delta)}{\Delta - 1} = \alpha(1 - x)\left(1 - \frac{1}{p}\right)^{\frac{p}{\alpha(1-x)}}.$$

We use two simple inequalities.

Lemma 17

(i) $-\ln(1 - x) \leq \frac{x}{1-x}$ *for* $-\infty < x < 1$.

(ii) $-p \ln\left(1 - \frac{1}{p}\right) < 1 + \frac{1}{2(p-1)}$ *for* $p > 1$.

Proof. For (i), let $f(x) = x/(1-x) + \ln(1-x)$, so that $f(0) = 0$. Since $df/dx = x/(1-x)^2$ has the sign of x, the inequality holds. For (ii), it is enough to compare the power series expansions in $1/p$. $\square$

Note that
$$p - 1 = q - \Delta > \alpha\Delta - \gamma - \Delta > (\alpha - 1)(\Delta - 1). \tag{3}$$

Applying Lemma 17, the equality $\alpha \ln \alpha = 1$ and inequality (3), we derive:

$$\begin{aligned}
\ln\left(\frac{J'(q,\Delta)}{\Delta-1}\right) &= \ln\alpha + \ln(1-x) + \frac{p}{\alpha(1-x)}\ln\left(1-\frac{1}{p}\right) > \ln\alpha - \frac{x}{1-x} - \frac{1+\frac{1}{2(p-1)}}{\alpha(1-x)} \\
&= \frac{\alpha\ln\alpha\ (1-x) - \alpha x - 1 - \frac{1}{2(p-1)}}{\alpha(1-x)} = \frac{-(\alpha+1)x - \frac{1}{2(p-1)}}{\alpha(1-x)} \\
&> \frac{-(\alpha+1)x - \frac{1}{2(\alpha-1)(\Delta-1)}}{\alpha(1-x)} = \frac{\alpha+1}{\alpha} - \frac{\alpha+1+\frac{1}{2(\alpha-1)(\Delta-1)}}{\alpha(1-x)}.
\end{aligned} \tag{4}$$

Since

$$1 - x = \frac{q-2}{\alpha(\Delta-1)} = \frac{q-\alpha\Delta+\gamma}{\alpha(\Delta-1)} + \frac{\alpha(\Delta-1)-(2+\gamma-\alpha)}{\alpha(\Delta-1)} \geq \varepsilon + 1 - \frac{2+\gamma-\alpha}{\alpha(\Delta-1)},$$

we may write $w = 1/(\Delta - 1)$ and give the following lower bound for the right-hand side of (4)

$$\frac{\alpha+1}{\alpha} - \frac{\frac{\alpha+1}{\alpha} + \frac{w}{2\alpha(\alpha-1)}}{1+\varepsilon-\frac{(2+\gamma-\alpha)w}{\alpha}} = A - \frac{B(\varepsilon)}{C(\varepsilon)-Dw} = F(\varepsilon, w), \text{ say,}$$

where

$$B(\varepsilon) = \frac{\alpha+1}{\alpha} + \frac{1+\varepsilon}{2(\alpha-1)(2+\gamma-\alpha)} > 0,\ A = B(0),\ C(\varepsilon) = 1+\varepsilon, \text{ and } D = \frac{2+\gamma-\alpha}{\alpha}.$$

Our remaining goal in proving that q is ε-good is to show that

$$F(\varepsilon, w) \geq \ln(1+\varepsilon-w), \text{ for } 0 \leq \varepsilon \leq 1, \text{ and } 0 < w \leq 1/2. \tag{5}$$

We first show that

$$\frac{\partial(F - \ln(1+\varepsilon-w))}{\partial w} = -\frac{DB(\varepsilon)}{(C(\varepsilon)-Dw)^2} + \frac{1}{1+\varepsilon-w} > 0.$$

Since $1+\varepsilon-w > 0$, this is equivalent to checking that $(C(\varepsilon)-Dw)^2 - (1+\varepsilon-w)DB(\varepsilon) > 0$. Numerically, this polynomial in ε and w is approximately

$$0.6285\varepsilon^2 + (0.6285 - 0.4305w)\varepsilon + 0.1980w + 0.1608w^2,$$

which is clearly positive[4], since $w \leq 0.5$. The constant term

$$C(0)^2 - DB(0) = \frac{4\alpha^3 - 6\alpha^2 - 3\alpha + 4 - 2\gamma(\alpha^2-1)}{2\alpha^2(\alpha-1)}$$

[4]To verify formally that the expression is positive, one can derive upper and lower bounds on the coefficients using upper and lower bounds on α.

is zero by the choice of γ. It is sufficient therefore to verify the inequality (5) for $w = 0$.

To show that $F(\varepsilon, 0) - \ln(1+\varepsilon) \geq 0$, we first verify that the second derivative is negative, and then merely check the inequality at the extreme values, $\varepsilon = 0, 1$.

However,

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}(F(\varepsilon,0) - \ln(1+\varepsilon)) &= \frac{d^2}{d\varepsilon^2}\left(\frac{\alpha+1}{\alpha}\left(1 - \frac{1}{1+\varepsilon}\right) - \ln(1+\varepsilon)\right) \\ &= -\frac{\alpha+1}{\alpha}\frac{2}{(1+\varepsilon)^3} + \frac{1}{(1+\varepsilon)^2} = \frac{-2(\alpha+1)+\alpha(1+\varepsilon)}{\alpha(1+\varepsilon)^3} < 0,\end{aligned}$$

$$F(0,0) - \ln 1 = 0, \text{ and } F(1,0) - \ln(1+1) = \frac{\alpha+1}{2\alpha} - \ln 2 \approx 0.0904 > 0.$$

This completes the verification. □

We now show how to use Lemma 15 to prove that the effect of a discrepancy at the boundary edge s_X decays exponentially with the distance from s_X.

Let X be a boundary pair. For any $d \geq 1$, let $E_d(X)$ denote the set of level-d edges in T_X. Let $\Gamma_d(X) = \sum_{e \in E_d(X)} \ell(e)$. In Lemma 18 below we show that $\Gamma_d(X)$ is exponentially small in d. Say that a boundary pair X is in $\mathcal{N}_i$ (for $i \in \{0, \ldots, \Delta-1\}$) if exactly i of the neighbours of f_X are in R_X. Let Γ_d be the maximum of $\Gamma_d(X)$, maximized over all boundary pairs X.

Lemma 18 *Suppose that q is ε-good for G. Then for every boundary pair X and any $d \geq 1$, $\Gamma_d(X) \leq (1+\varepsilon)^{-d}$.*

Proof. The proof is by induction on d. For the base case, $d = 1$, note that for any boundary pair X, $\Gamma_1(X) \leq \nu(X)$. Now apply Lemma 13 with $R' = \{f_X\}$ and by the given upper bound on $\mu(X)$ (and the definition of ε-good), we find that $\nu(X) \leq 1/(1+\varepsilon)$. For the inductive step, suppose that $X \in \mathcal{N}_r$. Then using the definition of ε-good again (and Lemma 13), we see that $\Gamma_d(X) \leq \nu(X) \cdot r \cdot \Gamma_{d-1}$. □

Lemma 19 *Suppose that q is ε-good for G. Then for every boundary pair X there exists a coupling Ψ of π_{B_X} and $\pi_{B'_X}$ such that, for all $f \in R_X$,*

$$\mathbb{E}[1_{\Psi,f}] \leq \frac{1}{\varepsilon}(1+\varepsilon)^{-d(f,s_X)+1}.$$

Furthermore,

$$\sum_{f \in R_X} \mathbb{E}[1_{\Psi,f}] \leq \frac{1}{\varepsilon}.$$

Proof. By Lemma 12, $\mathbb{E}[1_{\Psi,f}] \leq \gamma(f, T_X)$. Furthermore, $\gamma(f, T_X) = \sum_{e:n(e)=f} \ell(e)$ which is at most $\sum_{d \geq d(f,s_X)} \Gamma_d(X)$. By Lemma 18, this is at most

$$\sum_{d \geq d(f,s_X)} (1+\varepsilon)^{-d} = (1+\varepsilon)^{-d(f,s_X)} \sum_{d \geq 0} (1+\varepsilon)^{-d} = (1+\varepsilon)^{-d(f,s_X)} \, \frac{1+\varepsilon}{\varepsilon}.$$

Similarly,

$$\sum_{f \in R_X} \mathbb{E}[1_{\Psi,f}] \leq \sum_{f \in R_X} \gamma(f, T_X) = \sum_{f \in R_X} \sum_{e:n(e)=f} \ell(e) = \sum_{e:n(e) \in R_X} \ell(e) = \sum_{d \geq 1} \Gamma_d(X).$$

□

4 Exponential decay, vertex discrepancies and strong spatial mixing

Lemma 19 shows that the effect of a discrepancy at a boundary edge decays exponentially with the distance from that edge. In this section we show that the same holds for a discrepancy at a boundary vertex. This enables us to show that the collection of finite-volume Gibbs measures corresponding to the uniform distribution on proper colourings has strong spatial mixing.

A *vertex-boundary pair* X consists of

- a non-empty finite region R_X of the graph,
- a distinguished vertex v_X in ∂R_X, and
- a pair $(\mathcal{B}_X, \mathcal{B}'_X)$ of colourings of ∂R which differs only on vertex v_X. We require that the two colours $\mathcal{B}_X(v_X)$ and $\mathcal{B}'_X(v_X)$ are both in Q. That is, the two boundary colourings differ on the colour of vertex v_X, but this vertex is not an unconstrained vertex (with colour 0) in either boundary colouring.

Let $d(f, v_X)$ be the distance within R_X from a vertex f to vertex v_X.

Lemma 20 *Suppose that q is ε-good for a graph G with degree at most Δ. For every vertex-boundary pair X there is a coupling Ψ of $\pi_{\mathcal{B}_X}$ and $\pi_{\mathcal{B}'_X}$ such that, for all $f \in R$,*

$$\mathbb{E}[1_{\Psi,f}] \leq \Delta \frac{1}{\varepsilon}(1+\varepsilon)^{-d(f,v_X)+1}.$$

Furthermore,

$$\sum_{f \in R_X} \mathbb{E}[1_{\Psi,f}] \leq \frac{\Delta}{\varepsilon}.$$

Proof. This follows from Lemma 19 using a union bound by breaking the difference in a single vertex into the sum of differences in the edges that bound it. Let $e_1, \ldots, e_k$ be the boundary edges of R_X that are adjacent to v_X. Let X_i be the boundary pair consisting of the region R_X, the distinguished edge e_i, a colouring B of the boundary of R_X that agrees with $\mathcal{B}_X$ except that edges $e_1, \ldots, e_{i-1}$ are coloured with colour $\mathcal{B}'_X(v_X)$ and $e_i \ldots, e_k$ are coloured with colour $\mathcal{B}_X(v_X)$ and a colouring B' that is the same as B except that it colours e_i with colour $\mathcal{B}'_X(v_X)$. We construct a coupling of $\pi_{\mathcal{B}_X}$ and $\pi_{\mathcal{B}'_X}$ by composing couplings $\Psi_1, \ldots, \Psi_k$ of $X_1, \ldots, X_k$. Now

$$\mathbb{E}[1_{\Psi,f}] \leq \sum_{i=1}^{k} \mathbb{E}[1_{\Psi_i,f}].$$

□

Corollary 21 *Suppose that q is ε-good for the graph G, and that the maximum degree, Δ, of G is bounded. Then the system specified by uniform finite-volume Gibbs measures on proper q-colourings of G has strong spatial mixing.*

Proof. Using Definition 1, we wish to show that there are constants β and $\beta' > 0$ such that for any vertex-boundary pair X and any $\Lambda \subseteq R_X$,

$$d_{\text{tv}}(\pi_{\mathcal{B}_X,\Lambda}, \pi_{\mathcal{B}'_X,\Lambda}) \leq \beta|\Lambda| \exp(-\beta' d(v_X, \Lambda)).$$

The total variation distance of $\pi_{\mathcal{B}_X,\Lambda}$ and $\pi_{\mathcal{B}'_X,\Lambda}$ is at most the probability that the induced colourings differ in the coupling Ψ from Lemma 20. This is at most $\sum_{f\in\Lambda}\mathbb{E}[1_{\Psi,f}]$. As in the proof of Lemma 20, Ψ is the composition of $\Psi_1,\ldots,\Psi_k$. Following the proof of Lemma 19,

$$\sum_{f\in\Lambda}\mathbb{E}[1_{\Psi_i,f}] \le \sum_{e:n(e)\in\Lambda}\ell(e) \le \sum_{d\ge d(v_X,\Lambda)}\Gamma_d(X) \le \frac{1}{\varepsilon}(1+\varepsilon)^{-d(v_X,\Lambda)+1}$$

so the total variation distance is at most

$$\frac{\Delta}{\varepsilon}(1+\varepsilon)^{-d(v_X,\Lambda)+1}.$$

Now we can take $\beta = \frac{\Delta(1+\varepsilon)}{\varepsilon}$ and $\beta' = \log(1+\varepsilon)$. □

Remark. Note that the $|\Lambda|$ factor is not crucial in the definition of strong spatial mixing. In particular, our upper bound on the total variation distance does not use this factor.

Combining Corollary 21 with Lemma 15 we get the following result.

Theorem 5 *Let α be the solution to $\alpha^\alpha = e$ (so $\alpha \approx 1.76322$), and $\gamma = \frac{4\alpha^3-6\alpha^2-3\alpha+4}{2(\alpha^2-1)} \approx 0.47031$. Suppose that the graph G is triangle-free and that for some $\Delta \ge 3$ the maximum degree of G is at most Δ. The spin system specified by uniform finite-volume Gibbs measures on proper q-colourings of G has strong spatial mixing if $q > \alpha\Delta - \gamma$.*

Remark. In the proof of Lemma 15 we show that the simple bound of $q \ge \alpha\Delta-\gamma+\alpha\varepsilon(\Delta-1)$ is sufficient to guarantee that $(J'(q,\Delta)+1)/(\Delta-1) \ge 1+\varepsilon$, which yields ε-goodness since $1/\mu_1(X)-1 = Z \ge J(q,\Delta) \ge J'(q,\Delta)$. However, for any particular value of Δ we may use the bounds $(J(q,\Delta)+1)/(\Delta-1)-1 \ge (J'(q,\Delta)+1)/(\Delta-1)-1 \ge \varepsilon$ directly.

For example, with $\Delta = 4$ and $q = 7$, Lemma 15 gives ε-goodness for $\varepsilon \approx 0.079$, whereas the direct use of J' or J give values of about 0.169 or 0.177 respectively.

Of a little more interest are the rather sparse cases where the direct inequalities give a reduced value of q. Strong spatial mixing can be shown for $q = \lfloor \alpha\Delta - \gamma \rfloor$ for $\Delta = 19, 36, 74, 357, 2380, 148264, 686821, \ \ldots$. In the two cases, $\Delta = 19, 74$, the bounds of $q = 33, 130$, respectively, require the use of J (with its integral constraints) rather than just J'.

5 Using the geometry of the lattice

In this section we consider the lattice $\mathbb{Z}^3$. This is a triangle-free graph with degree 6, so Theorem 5 gives strong spatial mixing for $q \ge 11$. We will exploit the geometry of the lattice to show strong spatial mixing for $q = 10$. The idea is to use the geometry to derive a system of recurrences and to use these recurrences to construct the coupling.

We start by recording some upper bounds on $\mu(X)$. Let $\mu' = 125/589$ and let $\mu'' = 625/3121$. The following corollary follows from the proof of Lemma 15.

Corollary 22 *Suppose X is a boundary pair in which R_X consists of a node f_X plus $r \ge 0$ neighbours $y_1,\ldots,y_r$ of f_X. If $r = 0$ then $\mu(X) \le 1/4$. If $r = 4$ then $\mu(X) \le \mu'$. If $r = 5$ then $\mu(X) \le \mu''$.*

Proof. The $r = 0$ case follows from the fact that $\mu(X) \leq 1/(q - \Delta) = 1/4$. For the other cases, we use the same reasoning that we used in the proof of Lemma 15 to determine $J(q, \Delta)$. Let $q' = q - \Delta + 1 + r$, $u = \lfloor \frac{rp}{q'-2} \rfloor$ and $v = rp \bmod (q' - 2)$. (These are the same as the definitions in the proof of Lemma 15 except that there we specialized to $r = \Delta - 1$ so we had $q' = q$.) Let $h(q, \Delta, r)$ be the sum of the $q' - 2$ terms in (2) when we minimize by making the m_c's as equal as possible. Namely,

$$h(q, \Delta, r) = (q' - 2 - v)\left(1 - \frac{1}{p}\right)^{u} + v\left(1 - \frac{1}{p}\right)^{u+1}.$$

It follows from (2) and from the argument in the proof of Lemma 15 that

$$\mu(X) \leq \frac{1}{1 + h(q, \Delta, r)}.$$

The values can then be calculated directly. □

We would like to use the bounds in Corollary 22 to prove that $\Gamma_d(X)$ is exponentially small in d. The proof in Lemma 18 uses the following simple recursive idea. If $X \in \mathcal{N}_r$ then $\Gamma_d(X) \leq \nu(X) \cdot r \cdot \Gamma_{d-1}$. This idea suffices if our upper bound on $\nu(X)$ is less than $1/r$. This is not the case for the bounds in Corollary 22. However, it is a bit pessimistic to assume that all of the r recursive sub-problems correspond to the worst recursive case Γ_{d-1}. Using the geometry of the lattice, we can keep track of the recursion and do better.

We start by defining some sets of boundary pairs.

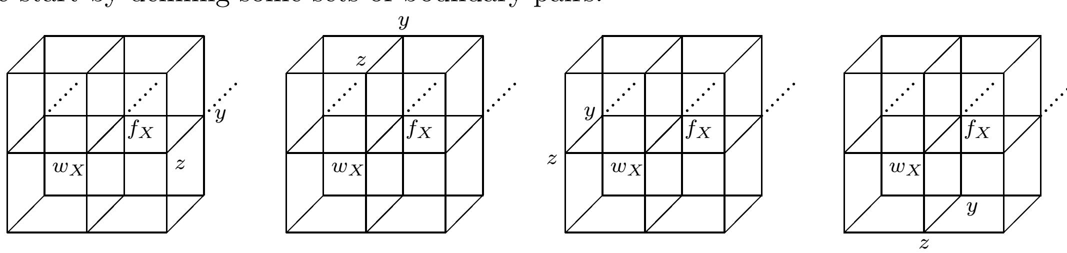

Figure 2: Configurations in U with $y \notin R_X$, or those in V with $z \notin R_X$

We will say that a boundary pair is in the set U if either of the following conditions hold:

- Either $X \in \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$, or
- $X \in \mathcal{N}_4$ and the following is true. Let y be the neighbour of f_X that is not in R_X and is not equal to w_X. We require that the vertices w_X and y differ in exactly two coordinates (in 3-dimensional space). See Figure 2.

We will say that a boundary pair is in the set V if the following condition holds: There is a vertex $z \notin R_X$ and a vertex $y \neq w_X$ such that $z \sim w_X$ (meaning z is adjacent to w_X) and $z \sim y \sim f_X$. See Figure 2 again for the relevant configurations. Note that the subsets U and V of boundary pairs depend only on R_X and s_X (they do not depend on B_X or B'_X). Let $U_d = \max_{X \in U} \Gamma_d(X)$ and $V_d = \max_{X \in V} \Gamma_d(X)$. The next lemma follows from the geometry of the lattice.

Lemma 23 *Suppose that $q = 10$ and G is $\mathbb{Z}^3$. Suppose $d > 1$. Then*

$$\begin{aligned}
\Gamma_d &\leq \max\left(4\mu'\Gamma_{d-1}, \mu''(\Gamma_{d-1} + 4V_{d-1}), \frac{3}{4}\Gamma_{d-1}\right), \\
U_d &\leq \max\left(\mu'(\Gamma_{d-1} + 3V_{d-1}), \frac{3}{4}\Gamma_{d-1}\right), \\
V_d &\leq \max\left(4\mu'\Gamma_{d-1}, \mu''(U_{d-1} + 4\Gamma_{d-1}), \frac{3}{4}\Gamma_{d-1}\right).
\end{aligned}$$

Proof. Consider a boundary pair X. If $X \in \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ then $\Gamma_d(X) \leq \nu(X) \cdot 3 \cdot \Gamma_{d-1}$. By Lemma 13 and Corollary 22 (with $r = 0$), $\nu(X) \leq 1/4$, so $\Gamma_d(X) \leq \frac{3}{4}\Gamma_{d-1}$ and one of the inequalities on Γ_d is satisfied. If $X \in \mathcal{N}_4$ then applying Corollary 22 with $r = 4$ we get $\Gamma_d(X) \leq \mu' 4\Gamma_{d-1}$ and the upper bound on Γ_d is satisfied. Otherwise, X is in $\mathcal{N}_5$. Using the definition of V, we can deduce that

$$\Gamma_d(X) \leq \nu(X)(\Gamma_{d-1} + 4V_{d-1})), \tag{6}$$

which gives us the other bound on Γ_d (using Corollary 22 with $r = 5$). To see that Inequality (6) is satisfied, let y be any of the 4 neighbours of f_X in R_X such that y and w_X differ on two coordinates. Now consider the recursive problem in which f_X becomes the new w_X and y becomes the new f_X. The original w_X becomes the (new) z in the definition of V. So this

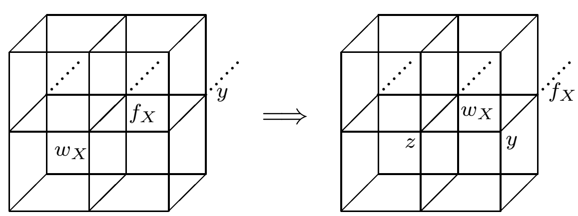

Figure 3: The basic recursion step for a configuration

recursive problem is in V.

Next consider a boundary pair $X \in U$. We wish to show that the inequality on U_d is satisfied. This is straightforward if $X \in \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$, so suppose that X is in $\mathcal{N}_4$. Let y denote the vertex in the definition of U. (So y is not in R_X and y and w_X differ in exactly two coordinates.) Using the definition of V, we can deduce that

$$\Gamma_d(X) \leq \nu(X)(\Gamma_{d-1} + 3V_{d-1}),$$

giving one of the upper bounds on U_d. (To see this, let y' be one of the three neighbours of f_X

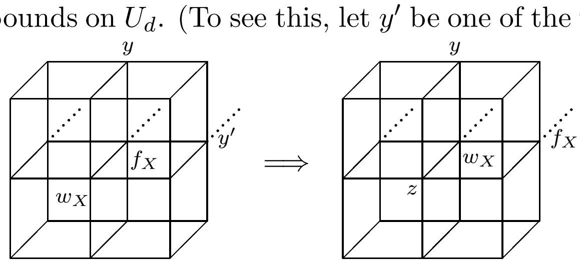

Figure 4: The recursion step for a configuration in U

in R_X that differs from y in two coordinates. Note that the sub-problem moving from f_X to y' is in V. See Figure 4.)

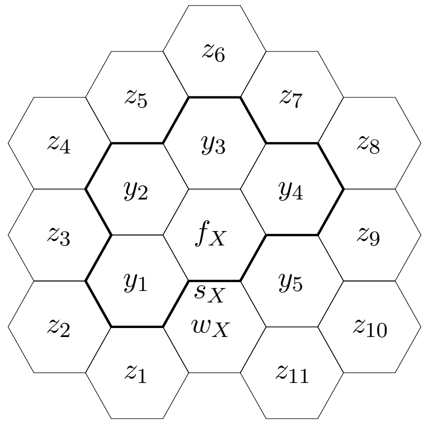

Figure 5: The triangular lattice

Finally, consider a boundary pair $X \in V$. The inequality for V_d is similar to that for Γ_d except when $X \in \mathcal{N}_5$. In this case, let z and y be the vertices in the definition of V. Now note that the sub-problem moving from f_X to y is in U. This gives the third equation. □

Here is the analogy of Lemma 18.

Lemma 24 *Let $\zeta = 0.0001$. Suppose that $q = 10$ and G is $\mathbb{Z}^3$. For every boundary pair X and every $d \geq 1$, $\Gamma_d(X) \leq (1-\zeta)^d$.*

Proof. Let $u = 0.8294$ and $v = 0.968$. We will prove by induction on d that $\Gamma_d \leq (1-\zeta)^d$ and $U_d \leq u(1-\zeta)^d$ and $V_d \leq v(1-\zeta)^d$. For the base case, $d = 1$, note that for any relevant boundary pair X, $\Gamma_1(X) \leq \nu(X)$. Now apply Lemma 13 with $R' = \{f_X\}$ and by Corollary 22 (with $r = 0$), we find that $\nu(X) \leq 1/4$. Thus, $\Gamma_1(x) \leq \min\{u, v, 1\}(1-\zeta)$.

The inductive step follows directly from Lemma 23 since the following inequalities hold. $4\mu' \leq (1-\zeta)\min(1, v)$, $\frac{3}{4} \leq (1-\zeta)\min(1, v, u)$, $\mu''(1+4v) \leq (1-\zeta)$, $\mu'(1+3v) \leq (1-\zeta)u$, and $\mu''(u+4) \leq (1-\zeta)v$. □

The proof of Theorem 6 now follows the argument in Sections 3 and 4. The only difference is that instead of applying Lemma 18, we use Lemma 24 (which gives exactly the same result with $\varepsilon = \zeta/(1-\zeta)$). The analogue of Lemma 20 provides an ε-coupling cover. Since $\mathbb{Z}^3$ is neighbourhood-amenable, we obtain Corollary 10.

6 Computational assistance

In Section 5 we showed strong spatial mixing for $q = 10$ and the degree-6 lattice $\mathbb{Z}^3$. The same argument does not apply to the triangular lattice because we cannot apply Lemma 15 (or its proof) to a graph with triangles. Nevertheless, we can use our method with computational assistance to prove strong spatial mixing for the triangular lattice.

6.1 The lattice

A piece of the triangular lattice is depicted in Figure 5. Each vertex of the lattice is depicted as a hexagonal face in the picture. Every vertex has degree 6. Thus, the 6 neighbours of vertex f_X are w_X, y_1, y_2, y_3, y_4 and y_5.

6.2 Relevant boundary pairs

In order to prove strong spatial mixing, we will need upper bounds on $\mu(X)$ similar to those obtained in Corollary 22. Since we will use computation, we want to restrict the search space as far as possible. We do that by defining the notion of a "relevant" boundary pair. Intuitively, the idea is that these boundary pairs are the ones that are induced by a pair of colourings of the vertex-boundary ∂R_X.

We say that a boundary pair is *relevant* if it is the case that any two *adjacent* edges on the boundary that share a vertex $f \notin R_X$ have the same colour in at least one of the two colourings B_X and B'_X (and so in both of B_X and B'_X except when edge s_X is involved). For example in Figure 5, if R_X consists of the five vertices enclosed by the thicker line (namely f_X, y_1, y_2, y_3, and y_4) and X is a relevant boundary pair, then B_X and B'_X assign the same colour to the edges (y_1, z_3) and (y_2, z_3). Note that our definition of "relevant" is specific to the geometry of the lattice. The edges (y_1, z_2) and (y_1, z_3) are *adjacent* but the edges (y_1, z_3) and (y_1, z_1) are not adjacent.

It is important to observe that our recursive construction preserves "relevance". That is, if X is a relevant boundary pair then all of the boundary pairs in the tree T_X are also relevant. We can ensure this by refining the construction of T_X (see Section 2) as follows. When the edges in E_X are given the names $e_1, \ldots, e_k$, order these edges clockwise around the vertex f_X starting from s_X. This ordering ensures that e_i is not "adjacent" to e_j unless i and j differ by 1. Now note that if X is relevant, then so is the constructed boundary pair $X_i(c, c')$.

Next note that Lemma 13 can be extended as follows. If the boundary pair X is relevant, then the boundary pair X' constructed in the proof is also relevant. Therefore, the set χ can be restricted to relevant boundary pairs. For convenience, we state the extended lemma here.

Lemma 25 *Suppose that X is a relevant boundary pair. Let R' be any subset of R_X which includes f_X. Let χ be the set of relevant boundary pairs $X' = (R_{X'}, s_{X'}, B_{X'}, B'_{X'})$ such that $R_{X'} = R'$, $s_{X'} = s_X$, $B_{X'}$ agrees with B_X on common edges, and $B'_{X'}$ agrees with B'_X on common edges. Then $\nu(X) \leq \max_{X' \in \chi} \mu(X')$.*

6.3 Bounding $\mu(X)$

By analogy to Corollary 22 we will now provide some upper bounds on $\mu(X)$ for the triangular lattice. Let $\mu' = 31/136$ and $\mu'' = 1111/4966$. We now give four lemmas bounding $\mu(X)$ for particular boundary pairs X.

Lemma 26 *Suppose X is a boundary pair with $|R_X| = 1$. Then $\mu(X) \leq 1/4$.*

Proof. The numerator in the definition of $\mu(X)$ is at most 1. The denominator is at least $q - \Delta = 4$. □

Lemma 27 *Suppose X is a boundary pair in which R_X consists of f_X and one neighbour y of f_X. Then $\mu(X) \leq 5/21$.*

Proof. Let E_1 be the set of edges of f_X except for s_X and the edge between f_X and y. Let E_2 be the set of edges of y except for the edge between f_X and y. Let U be the set of colours that B_X (and so also B'_X) assigns to edges in E_1, and similarly let V be the set of colours assigned to E_2. Let $d = B_X(s_X)$ and $d' = B'_X(s_X)$. Let $Q' = Q - \{d, d'\}$. To shorten the notation we will use n_c to denote $n_c(X)$ and N to denote $N(X)$. Let $q_1 = |Q' \setminus U|$ and let $q_2 = |Q \setminus V|$. We see that $q_1 \geq 8 - |U| \geq 4$ and $q_2 \geq 10 - |V| \geq 5$.

For $c \in Q$ we see that

$$\begin{aligned} n_c &= 0 && \text{if } c \in U, \\ &= q_2 && \text{if } c \notin U \text{ and } c \in V, \\ &= q_2 - 1 && \text{if } c \notin U \text{ and } c \notin V. \end{aligned}$$

Thence, $\max(n_d, n_{d'}) \leq q_2$ and

$$N = \sum_{c \in Q'} n_c \geq (q_2 - 1)|Q' \setminus U| = (q_2 - 1)q_1.$$

Since $\mu(X)$ is monotone increasing in $\max(n_d, n_{d'})$ and decreasing in N, we have

$$\mu(X) \leq \frac{q_2}{q_2 + (q_2 - 1)q_1} = \frac{1}{1 + (1 - 1/q_2)q_1} \leq \frac{1}{1 + (1 - 1/5)4} = \frac{5}{21}.$$

□

Lemma 28 *Suppose X is a relevant boundary pair in which $R_X = \{f_X, y_1, y_2, y_3\}$ and f_X is adjacent to each of the y_i's and $w_X \sim y_1 \sim y_2 \sim y_3$. Then $\mu(X) \leq \mu'$.*

Proof. By computation. We considered every such relevant boundary pair X (approximately 2×10^6 of them) and calculated $\mu(X)$. □

Lemma 29 *Suppose X is a relevant boundary pair in which $R_X = \{f_X, y_1, y_2, y_3, y_4\}$ and f_X is adjacent to each of the y_i's and $w_X \sim y_1 \sim y_2 \sim y_3 \sim y_4$. Then $\mu(X) \leq \mu''$.*

Proof. By computation. We considered every such relevant boundary pair X (approximately 16×10^6 of them) and calculated $\mu(X)$. □

6.4 Proving exponential decay

As in Section 5, we will prove that $\Gamma_d(X)$ is exponentially small in d by defining a system of recursive equations. We will restrict attention to relevant boundary pairs Let Rel be the set of relevant boundary pairs. Say that a relevant boundary pair X is in $\mathcal{N}_i^R$ (for $i \in \{0, \ldots, 5\}$) if at most i of the neighbours of f_X are in R_X. Let $\Gamma_d^R = \max_{X \in \text{Rel}} \Gamma_d(X)$.

Lemma 30 *Suppose $d > 1$. If $X \in \mathcal{N}_2^R$ then $\Gamma_d(X) \leq \Gamma_{d-1}^R/2$. If $X \in \mathcal{N}_3^R$ then $\Gamma_d(X) \leq 3\Gamma_{d-1}^R/4$. If $X \in \mathcal{N}_4^R$ then $\Gamma_d(X) \leq 20\Gamma_{d-1}^R/21$.*

Proof. If $X \in \mathcal{N}_2^R$ then by the definition of T_X, $\Gamma_d(X) \leq \nu(X) \cdot 2 \cdot \Gamma_{d-1}^R$. Then by Lemma 25 with $R' = \{f_X\}$ and Lemma 26, $\nu(X) \leq 1/4$. Similarly, if $X \in \mathcal{N}_3^R$ then $\Gamma_d(X) \leq \nu(X) \cdot 3 \cdot \Gamma_{d-1}^R \leq 3\Gamma_{d-1}^R/4$. Finally, suppose that exactly 4 neighbours of f_X are in R_X. Then $\Gamma_d(X) \leq \nu(X) \cdot 4 \cdot \Gamma_{d-1}^R$. Apply Lemma 25 where R' contains f_X and one of its neighbours in R_X. By Lemma 27, $\nu(X) \leq 5/21$. □

Next we define some subsets of Rel. Refer to Figure 5 to clarify these definitions. Let X be a relevant boundary pair.

- X is in U if there is a neighbour y_5 of f_X and of w_X that is not in R_X, and there is a neighbour $y_4 \neq w_X$ of f_X and of y_5 that is not in R_X.
- X is in V if there is a neighbour y_5 of f_X and of w_X that is not in R_X.
- X is in W if there is a neighbour y_5 of f_X and of w_X in R_X, and a neighbour $z_{11} \neq f_X$ of w_X and of y_5 that is not in R_X.

Let $U_d = \max_{X \in U} \Gamma_d(X)$, $V_d = \max_{X \in V} \Gamma_d(X)$, and $W_d = \max_{X \in W} \Gamma_d(X)$. The following lemma follows from the definition of T_X and the geometry of the lattice.

Lemma 31 *Suppose that $q = 10$ and G is the triangular lattice. Suppose $d > 1$. Then*

$$\begin{aligned}
\Gamma_d^R &\leq \max\left(\mu''\left(\Gamma_{d-1}^R + 2V_{d-1} + 2W_{d-1}\right), 20\Gamma_{d-1}^R/21\right), \\
U_d &\leq \max\left(\mu'\left(2V_{d-1} + W_{d-1}\right), \Gamma_{d-1}^R/2\right), \\
V_d &\leq \max\left(\mu''\left(2V_{d-1} + 2W_{d-1}\right), 3\Gamma_{d-1}^R/4\right), \text{ and} \\
W_d &\leq \max\left(\mu''\left(\Gamma_{d-1}^R + V_{d-1} + 2W_{d-1} + U_{d-1}\right), 20\Gamma_{d-1}^R/21\right).
\end{aligned}$$

Proof. Suppose $X \in \text{Rel}$. If $X \in \mathcal{N}_4^R$ then $\Gamma_d(X) \leq 20\Gamma_{d-1}^R/21$ by Lemma 30. Otherwise an examination of Figure 5 reveals that

$$\Gamma_d(X) \leq \nu(X)\left(\Gamma_{d-1}^R + 2V_{d-1} + 2W_{d-1}\right).$$

The two instances of V_{d-1} correspond to y_1 and y_5 in the picture (since w_X is not in R_X), the two instances of W_{d-1} correspond to y_2 and y_4, and the instance of Γ_{d-1}^R corresponds to y_3. Apply Lemma 25 where R' is the set containing f_X and the vertices y_1, y_2, y_3 and y_4 from Figure 5. By Lemma 29, $\nu(X) \leq \mu''$. This proves the upper bound on Γ_d^R.

Suppose $X \in U$. If $X \in \mathcal{N}_2^R$ then $\Gamma_d(X) \leq \Gamma_{d-1}^R/2$ by Lemma 30. Otherwise

$$\Gamma_d(X) \leq \nu(X)\left(2V_{d-1} + W_{d-1}\right).$$

As in the upper bound on Γ_d^R, one instance of V_{d-1} corresponds to y_1 and one instance of W_{d-1} corresponds to y_2. An examination of Figure 5 reveals that, since $X \in U$, y_3 corresponds to V_{d-1}. Apply Lemma 25, where R' is the set containing f_X and the vertices y_1, y_2, and y_3 from Figure 5. By Lemma 28, $\nu(X) \leq \mu'$. This proves the upper bound on U_d.

Suppose $X \in V$. If $X \in \mathcal{N}_3^R$ then $\Gamma_d(X) \leq 3\Gamma_{d-1}^R/4$ by Lemma 30. Otherwise

$$\Gamma_d(X) \leq \nu(X)\left(2V_{d-1} + 2W_{d-1}\right).$$

This is the same as the upper bound on Γ_d^R except that, since $X \in V$, y_3 corresponds to W_{d-1} and y_4 to V_{d-1}. Apply Lemma 25, where R' is the set containing f_X and the vertices y_1, y_2, y_3 and y_4 from Figure 5. By Lemma 29, $\nu(X) \leq \mu''$. This proves the upper bound on V_d.

Suppose $X \in W$. If $X \in \mathcal{N}_4^R$ then $\Gamma_d(X) \leq 20\Gamma_{d-1}^R/21$ by Lemma 30. Otherwise

$$\Gamma_d(X) \leq \nu(X)\left(\Gamma_{d-1}^R + V_{d-1} + 2W_{d-1} + U_{d-1}\right).$$

This is the same as the upper bound on Γ_d^R except that, since $X \in W$, y_5 corresponds to U_{d-1}. Apply Lemma 25, where R' is the set containing f_X and the vertices y_1, y_2, y_3 and y_4 from Figure 5. By Lemma 29, $\nu(X) \leq \mu''$. This proves the upper bound on W_d. □

Here is the analogue of Lemma 18.

Lemma 32 *Let* $\zeta = 0.001$*. Suppose that* $q = 10$ *and* G *is the triangular lattice. For every relevant boundary pair* X *and every* $d \geq 1$, $\Gamma_d(X) \leq (1-\zeta)^d$.

Proof. Let $u = 15/26$, $v = 31/40$, and $w = 21/22$. We will prove by induction on d that $\Gamma_d^R \leq (1-\zeta)^d$ and $U_d \leq u(1-\zeta)^d$ and $V_d \leq v(1-\zeta)^d$ and $W_d \leq w(1-\zeta)^d$. For the base case, $d = 1$, note that for any relevant boundary pair X, $\Gamma_1(X) \leq \nu(X)$. Now apply Lemma 25 with $R' = \{f_X\}$ and by Lemma 26, we find that $\nu(X) \leq 1/4$. Thus, $\Gamma_1(x) \leq \min\{u, v, w, 1\}(1-\zeta)$.

The inductive step follows directly from Lemma 31 since the following inequalities hold. $\mu''(1+2v+2w) \leq 1-\zeta$, $20/21 \leq 1-\zeta$, $\mu'(2v+w) \leq u(1-\zeta)$, $1/2 \leq u(1-\zeta)$, $\mu''(2v+2w) \leq v(1-\zeta)$, $3/4 \leq v(1-\zeta)$, $\mu''(1+v+2w+u) \leq w(1-\zeta)$, and $20/21 \leq w(1-\zeta)$. ☐

Unlike Lemma 18, Lemma 32 applies just to relevant boundary pairs, so we have to finish up the proof of strong spatial mixing. Following the proof of Lemma 19, we obtain the following.

Lemma 33 *Let* $\zeta = 0.001$*. Suppose that* $q = 10$ *and* G *is the triangular lattice. Then for every relevant boundary pair* X *there exists a coupling* Ψ *of* π_{B_X} *and* $\pi_{B'_X}$ *such that, for all* $f \in R_X$*,*

$$\mathbb{E}[1_{\Psi,f}] \leq \frac{1}{\zeta}(1-\zeta)^{d(f,s_X)}.$$

Furthermore,

$$\sum_{f \in R_X} \mathbb{E}[1_{\Psi,f}] \leq \frac{1-\zeta}{\zeta}.$$

6.5 Vertex discrepancies and strong spatial mixing

The proof of strong spatial mixing (Theorem 7) is similar to the proof on Section 4. The only extra problem is showing that the boundary pairs created in Lemma 20 are actually relevant boundary pairs (so that we can apply Lemma 33). This detail complicates the proof of the theorem, so we give a new version of the lemma.

Lemma 34 *Let* $\zeta = 0.001$*. Suppose that* $q = 10$ *and* G *is the triangular lattice. For every vertex-boundary pair* X *there is a coupling* Ψ *of* $\pi_{\mathcal{B}_X}$ *and* $\pi_{\mathcal{B}'_X}$ *such that, for all* $f \in R$*,*

$$\mathbb{E}[1_{\Psi,f}] \leq 10\frac{1}{\zeta(1-\zeta)}(1-\zeta)^{d(f,v_X)}.$$

Furthermore,

$$\sum_{f \in R_X} \mathbb{E}[1_{\Psi,f}] \leq \frac{10(1-\zeta)}{\zeta}.$$

Proof. First suppose that v_X has a neighbour $y \notin R_X$. (This case is straightforward and is like the proof of Lemma 20.) Let $e_1, \ldots, e_k$ be the boundary edges of R_X that are adjacent to v_X. Label these clockwise so that there is at least one non-boundary edge between e_k and e_1. (The point here is that e_i and e_j are only adjacent if i and j differ by 1.) Let X_i be the relevant boundary pair consisting of the region R_X, the distinguished edge e_i, a colouring B

of the boundary of R_X that agrees with $\mathcal{B}_X$ except that edges $e_1, \ldots, e_{i-1}$ are coloured with colour $\mathcal{B}'_X(v_X)$ and $e_i \ldots, e_k$ are coloured with colour $\mathcal{B}_X(v_X)$ and a colouring B' that is the same as B except that it colours e_i with colour $\mathcal{B}'_X(v_X)$. We construct a coupling of $\pi_{\mathcal{B}_X}$ and $\pi_{\mathcal{B}'_X}$ by composing couplings $\Psi_1, \ldots, \Psi_k$ of $X_1, \ldots, X_k$. Now

$$\mathbb{E}[1_{\Psi,f}] \leq \sum_{i=1}^{k} \mathbb{E}[1_{\Psi_i,f}] \leq 5\frac{1}{\zeta}(1-\zeta)^{d(f,v_X)}.$$

Now we must deal with the case in which all neighbours of v_X are in R_X. A technical detail arises here because the natural induced boundary pairs are not all relevant. Let y be any neighbour of v_X. Let Ψ' be any coupling of $\pi_{\mathcal{B}_X}$ and $\pi_{\mathcal{B}'_X}$. Let (C, C') be the random variable corresponding to the pair of colourings in $S(\mathcal{B}_X) \times S(\mathcal{B}'_X)$ drawn from Ψ. We will choose the colour of y in C and C' according to Ψ'. To complete the construction of Ψ, for every pair (c, c'), we will let $\mathcal{B}_X(c)$ denote the vertex-boundary of $R_X - \{y\}$ which agrees with $\mathcal{B}_X$ except that y is coloured c and we will let $\mathcal{B}'_X(c')$ denote the vertex-boundary of $R_X - \{y\}$ which agrees with $\mathcal{B}'_X$ except that y is coloured c'. We will construct a coupling of $\mathcal{B}_X(c)$ and $\mathcal{B}'_X(c')$ by composing the couplings of up to 10 relevant boundary pairs (these boundary pairs correspond to discrepancies on the 5 boundary edges of vertex v_X and the up-to-5 boundary edges on vertex y). The $(1-\zeta)$ in the denominator comes from the fact that the distance from a vertex f to the discrepancy edge may be one less than $d(f, v_X)$. □

Lemma 34 provides an ε-coupling cover for $\varepsilon = \frac{\zeta}{1-\zeta}\frac{6}{10}$. Since the lattice is neighbourhood-amenable, we obtain Corollary 11.

6.6 Extensions

Using techniques similar to those presented in Section 6, we can give an alternative proof to the result of [1] – strong spatial mixing for 6-colourings of the rectangular lattice. The amount of computation in the alternative proof and the proof in Section 6 can be reduced by applying some of the techniques from Lemma 16. Our technique can also be applied to other lattices, for example, some of the others studied by Salas and Sokal [27].

7 Rapid mixing

In this section we prove Theorem 8, showing that for neighbourhood-amenable graphs, our strong spatial mixing proof implies rapid mixing. It is known that strong spatial mixing implies rapid mixing in such cases (see [13, 22, 31]) but existing proofs seem to be written for $\mathbb{Z}^d$ so we add this section for completeness.

Theorem 8 *Let G denote an infinite neighbourhood-amenable graph with maximum degree Δ. Let R be a finite subgraph of G with $|R| = n$ and $\mathcal{B}(R)$ denote a colouring of $\partial(R)$ using the colours $Q \cup \{0\}$. (We assume that $q \geq \Delta + 2$.)*

Suppose there exists $\varepsilon > 0$ such that G has an ε-coupling cover. Then the Glauber dynamics Markov chain on $S(\mathcal{B}(R))$ is rapidly mixing and $\tau(\delta) \in O(n(n + \log \frac{1}{\delta}))$.

We use the method of path coupling to prove this theorem, approaching our result indirectly through the use of Markov chain comparison. We give a brief review of the path-coupling method in the next section, then proceed with the first step in our analysis for graphs that satisfy the hypotheses of Theorem 8. In Section 7.2 we first examine an auxiliary Markov chain which allows recolouring of a slightly larger set of vertices in a single recolouring step. We show this new chain mixes in time $O(n \log n)$. Markov chain comparison is reviewed in Section 7.3, and then the second part of the proof of Theorem 8 is presented in Section 7.4.

7.1 Path coupling

Coupling is a popular method for analysing mixing times of Markov chains. A (Markovian) coupling for a Markov chain $\mathcal{M}$ with state space Ω is a stochastic process (X_t, Y_t) on $\Omega \times \Omega$ such that each of (X_t) and (Y_t), considered marginally, is a faithful copy of $\mathcal{M}$. The coupling lemma (see, for example, Aldous [2]) states that the total variation distance of $\mathcal{M}$ at time t is bounded above by $\Pr(X_t \neq Y_t)$. The *path-coupling* method, introduced in [4], is a powerful method for finding couplings. The idea is that one can find a coupling on a subset U of $\Omega \times \Omega$ and extend this to a coupling on $\Omega \times \Omega$. The following theorem, adapted from [12], summarizes the path-coupling method.

Theorem 35 [4, 12] *Let U be a relation $U \subseteq \Omega^2$ such that U has transitive closure Ω^2. Let $\phi : U \to \{0, 1, 2, \ldots\}$ be a "proximity function" defined on pairs in U. We use ϕ to define a function Φ on Ω^2 as follows: For each pair $(\omega, \omega') \in \Omega^2$, let*

$$\Phi(\omega, \omega') = \min_{\omega_0, \ldots, \omega_k} \sum_{i=0}^{k-1} \phi(\omega_i, \omega_{i+1}),$$

where the minimum is over all paths $\omega = \omega_0, \ldots, \omega_k = \omega'$ such that, for all $i \in [0, k-1]$, $(\omega_i, \omega_{i+1}) \in U$. Let (X_t, Y_t) be a coupling for $\mathcal{M}$ defined over all pairs in U. Suppose for this coupling there is $\beta < 1$ such that for all $(\sigma_1, \sigma_2) \in U$ we have

$$\mathbb{E}[\Phi(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t) = (\sigma_1, \sigma_2)] \leq \beta \Phi(\sigma_1, \sigma_2).$$

Let D be the maximum value that Φ achieves on Ω^2. Then

$$\tau(\delta) \leq \frac{\ln(D/\delta)}{1 - \beta}.$$

7.2 Proof of rapid mixing (Part I)

Our goal is to sample from $S(\mathcal{B})$, the set of proper colourings of R consistent with the boundary colouring , uniformly at random using the single-vertex Glauber dynamics Markov chain. To do this, we first define another Markov chain that corresponds to heat-bath dynamics on small subregions of R. As we defined in Section 1.1, for a vertex $f \in G$ and a non-negative integer d we let $Ball_d(f)$ denote the set of vertices that are at most distance d from f.

Now consider a problem instance of R and $\mathcal{B}$. For a fixed $d \geq 0$ (to be specified later) and a vertex $f \in G$, let $R_f = Ball_d(f) \cap R$. Further, let

$$R^* = \{f \in G \mid R_f \neq \emptyset\}.$$

Then $\mathcal{M}_d$ is the heat-bath Markov chain with state space $S(\mathcal{B})$ and the following transitions: $\mathcal{M}_d$ makes a transition from a state $\sigma \in S(\mathcal{B})$ by choosing a vertex $f \in R^*$ uniformly at random. Let $\mathcal{B}_f$ be the colouring of ∂R_f induced by σ and $\mathcal{B}$. To make the transition from

σ, recolour the vertices in R_f by sampling from $\pi_{\mathcal{B}_f}$, the uniform distribution of colourings on the region R_f induced by $\mathcal{B}_f$.

Since $Ball_0(f) = \{f\}$, Glauber dynamics is $\mathcal{M}_0$. However, in order to prove rapid mixing of $\mathcal{M}_0$ we first demonstrate rapid mixing of the chain $\mathcal{M}_d$ for some constant d. Then we use the comparison method (see Section 7.3 below) to infer rapid mixing of $\mathcal{M}_0$.

Proof of Theorem 8. (Part I)

Path coupling is used to prove rapid mixing of the Markov chain $\mathcal{M}_d$. First we specify the value of d that we use.

Fix an $\varepsilon > 0$ for which G has an ε-coupling cover as guaranteed in the hypothesis of Theorem 8. Recalling Definition 2, since G is neighbourhood-amenable, we can find d such that

$$T_d = \sup_v \frac{|\partial Ball_d(v)|}{|Ball_d(v)|} \leq \frac{\varepsilon}{\varepsilon + \Delta}.$$

In this setting, the distance measure we use is Hamming distance. Because of this, we use the standard approach of taking the set Ω in Theorem 35 to be the set of *all* (proper and improper) q-colourings of the region R. We take U to be the set of pairs of colours that differ at a single vertex. Consider two colourings σ and θ in U with Hamming distance 1, i.e. σ and θ are two (not necessarily proper) colourings of R that disagree at a single vertex v. We describe a coupled transition from the pair (σ, θ) to a new pair of colourings (σ', θ'). In this coupling, we choose the same vertex f for the transition $\sigma \to \sigma'$ that we choose for the transition $\theta \to \theta'$. Note that while σ and θ may not be proper colourings of R, a transition $\sigma \to \sigma'$ is only allowed if σ' is "not more improper" than σ, and similarly for a transition $\theta \to \theta'$. In other words, having chosen a vertex f, we recolour the "window" R_f using a proper colouring of that window (conditioned, of course, on its induced boundary colouring).

First note by construction of R^*, we have $Ball_d(v) \subseteq R^*$. For each vertex $f \in Ball_d(v)$, if f is chosen in the chain $\mathcal{M}_d$, we can ensure that $\sigma' = \theta'$ by choosing the same colouring to recolour R_f. Thus there are $|Ball_d(v)|$ ways to decrease the Hamming distance by one.

If the chosen vertex f is far from v, in the sense that $v \notin \partial R_f$, then we will couple the transitions by again choosing the same recolouring for the region R_f. This ensures that σ' and θ' disagree only at v so they still have Hamming distance 1.

We now calculate an upper bound on how much the distance can increase in one step of the coupling. This can only happen if we choose some vertex f such that $v \in \partial R_f$. With this in mind we define $\mathcal{H}_d(v) = \{f \in R^* : v \in \partial R_f\}$. $|\mathcal{H}_d(v)|$ is an upper bound on the number of vertices whose selection can increase the distance in the new pair (σ', θ'). Let $\mathcal{B}_1$ be the colouring of ∂R_f induced from the colouring σ and $\mathcal{B}_2$ be that induced from θ. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ differ solely at the vertex v. The ε-coupling cover in the hypothesis of the theorem guarantees we can construct a coupling that allows us to choose a pair (σ', θ') of proper colourings so that the expected Hamming distance increases by at most Δ/ε.

Adding it all up, we see the expected Hamming distance between σ' and θ' after one step of the coupling is at most

$$1 - \frac{|Ball_d(v)|}{|R^*|} + \frac{|\mathcal{H}_d(v)|}{|R^*|}\frac{\Delta}{\varepsilon} = 1 - \frac{|Ball_d(v)|\varepsilon - |\mathcal{H}_d(v)|\Delta}{|R^*|\varepsilon}.$$

From the choice of d (using the neighbourhood-amenability property of G), and using $\mathcal{H}_d(v) \subseteq \partial Ball_d(v)$, we have

$$|Ball_d(v)|\varepsilon - |\mathcal{H}_d(v)|\Delta \geq |\partial Ball_d(v)|(\varepsilon + \Delta) - |\partial Ball_d(v)|\Delta = |\partial Ball_d(v)|\varepsilon \geq \varepsilon.$$

Thus we have $\mathbb{E}[\Phi(\sigma', \theta')] \leq 1 - 1/|R^*|$.

So for Theorem 35, we can take $\beta = 1 - 1/|R^*|$. Since we use Hamming distance, we have that $D = n$. Using the facts that $d \in O(1)$ and that $\Delta \in O(1)$ (since Δ is a universal bound on maximum degree for any finite subgraph of G), we see that $1/(1-\beta) = |R^*| \in O(n)$. Using Theorem 35 we conclude $\tau_{\mathcal{M}_d}(\delta) \in O(n \log(\frac{n}{\delta}))$.

The final step to get the desired result about $\mathcal{M}_0$ uses the method of comparing Markov chains. We review this method below, then continue with the proof of Theorem 8 following that. □

7.3 The comparison method

In the previous section we showed rapid mixing of $\mathcal{M}_d$ on the set of all (proper and improper) colourings. This implies that $\mathcal{M}_d$ mixes rapidly on the set of proper colourings. In this section we will compare the mixing times of $\mathcal{M}_d$ and $\mathcal{M}_0$ on the set of proper colourings. We use the method of Diaconis and Saloff-Coste [7]. We provide definitions in the context of these two colouring Markov chains. P_d (respectively, P_0) will be used to denote the transition matrix for the chain $\mathcal{M}_d$ (resp. $\mathcal{M}_0$).

For $i \in \{0, d\}$, let E_i be the set of pairs of distinct colourings (σ, θ) with $P_i(\sigma, \theta) > 0$. We will sometimes refer to the members of E_i as "edges" because they are edges in the transition graph of $\mathcal{M}_i$. For every edge $(\sigma, \theta) \in E_d$, let $\mathcal{P}_{\sigma,\theta}$ be the set of paths from σ to θ using transitions of $\mathcal{M}_0$. More formally, let $\mathcal{P}_{\sigma,\theta}$ be the set of paths $\gamma = (\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \theta)$ such that

1. each (σ_i, σ_{i+1}) is in E_0, and
2. each $(\sigma', \theta') \in E_0$ appears at most once on γ.

We write $|\gamma|$ to denote the length of path γ. So, for example, if $\gamma = (\sigma_0, \ldots, \sigma_k)$ we have $|\gamma| = k$. Let $\mathcal{P} = \cup_{(\sigma,\tau) \in E_d} \mathcal{P}_{\sigma,\tau}$.

A *flow* is a function ϕ from $\mathcal{P}$ to the interval $[0, 1]$ such that for every $(\sigma, \theta) \in E_d$,

$$\sum_{\gamma \in \mathcal{P}_{\sigma,\theta}} \phi(\gamma) = P_d(\sigma, \theta) \pi_{\mathcal{B}}(\sigma). \tag{7}$$

For every $(\sigma', \theta') \in E_0$, the *congestion* of edge (σ', θ') in the flow ϕ is the quantity

$$A_{\sigma',\theta'}(\phi) = \frac{1}{\pi_{\mathcal{B}}(\sigma') P_0(\sigma', \theta')} \sum_{\gamma \in \mathcal{P} : (\sigma', \theta') \in \gamma} |\gamma| \, \phi(\gamma).$$

The *congestion* of the flow is the quantity

$$A(\phi) = \max_{(\sigma', \theta') \in E_0^*} A_{\sigma',\theta'}(\phi).$$

A proof of the following theorem can be found in [11, Observation 13]. This theorem is similar to Proposition 4 of Randall and Tetali [26] except that the latter requires the eigenvalues of the transition matrices to be non-negative. Both results are based closely on the ideas of Aldous [2], Diaconis and Stroock [8], and Sinclair [28].

Theorem 36 *Suppose that ϕ is a flow. Let $c = \min_\sigma P_0(\sigma,\sigma)$ and note that $c \geq 1/q$. Then for any $0 < \delta' < \frac{1}{2}$*

$$\tau(\mathcal{M}_0,\delta) \leq \max\left\{A(\phi)\left[\frac{\tau(\mathcal{M}_d,\delta')}{\ln\frac{1}{2\delta'}}+1\right],\frac{1}{2c}\right\}\ln\frac{1}{\delta\cdot\pi_{\min}},$$

where $\pi_{\min} = \min_\sigma \pi_{\mathcal{B}}(\sigma)$.

We continue with the proof of Theorem 8 in the next section.

7.4 Proof of rapid mixing (Part II)

Suppose we take $\delta' = 1/n$ and use the upper bound from the first part of the proof of Theorem 8. We then have $\tau(\mathcal{M}_d,\delta') \in O(n\log n)$. We now construct a flow ϕ such that $A(\phi) \in O(1)$, and then Theorem 36 gives

$$\tau(\mathcal{M}_0,\delta) \leq O(1)\cdot O(n)\cdot\ln\frac{1}{\delta\cdot\pi_{\min}}.$$

This yields Theorem 8 since $\ln(1/\pi_{\min}) \in O(n)$.

Proof of Theorem 8. (Part II)

Constructing a flow

Consider a problem instance consisting of a non-empty region R with $|R| = n$ and a colouring $\mathcal{B}$ of ∂R. We will now construct a flow ϕ.

For every pair $(\sigma,\theta) \in E_d$, we fix some vertex f such that $Ball_d(f)$ contains all the vertices on which σ and θ differ. Then we fix a canonical ordering on these vertices where they differ, say $v_1,\ldots,v_m$.

Let $\gamma_{\sigma,\theta} \in \mathcal{P}_{\sigma,\theta}$ be the canonical path from σ to θ constructed as follows:

- Update the vertices $v_1,\ldots,v_m$ in order.
- In order to update a given vertex v_i
  - If any neighbours of v_i have colour $\theta(v_i)$, recolour these with the lexicographically first available colour. (Note that these neighbours do not have their final colour in θ.)
  - Recolour v_i with colour $\theta(v_i)$.

Assign all of the flow from σ to θ to path $\gamma_{\sigma,\theta}$. That is, set $\phi(\gamma_{\sigma,\theta}) = P_d(\sigma,\theta)\pi_{\mathcal{B}}(\sigma)$.

Bounding $A(\phi)$

We show that $A(\phi) \in O(1)$, which completes the proof of Theorem 8.

Let σ' and θ', where $(\sigma',\theta') \in E_0$, be colourings that disagree on vertex x. Now

$$A_{\sigma',\theta'}(\phi) \quad = \quad \frac{1}{\pi_{\mathcal{B}}(\sigma')P_0(\sigma',\theta')}\left(\sum_{\substack{(\sigma,\theta)\in E_d\\(\sigma',\theta')\in\gamma_{\sigma,\theta}}}|\gamma_{\sigma,\theta}|\,P_d(\sigma,\theta)\,\pi_{\mathcal{B}}(\sigma)\right).$$

Since $\pi_{\mathcal{B}}$ is uniform and all of the path lengths are $O(1)$, this simplifies to

$$A_{\sigma',\theta'}(\phi) \leq O(1) \times \left(\sum_{\substack{(\sigma,\theta)\in E_d \\ (\sigma',\theta')\in\gamma_{\sigma,\theta}}} \frac{P_d(\sigma,\theta)}{P_0(\sigma',\theta')} \right).$$

To see that this sum is $O(1)$ note that there are only $O(1)$ pairs (σ, θ) in the summation (this holds since σ and θ agree with σ' except in a constant-sized ball around x). Since $\sigma \neq \theta$, $P_d(\sigma, \theta) \in O(1/n)$. Finally, $P_0(\sigma', \theta') \in \Omega(1/n)$.

7.5 Neighbourhood-amenability – How restrictive is it?

Theorem 8 applies to graphs that are neighbourhood-amenable. This condition, while sufficient, is not necessary. The theorem could be extended to a larger class of graphs. The relevant issue is to balance the number of "good" transitions that decrease the distance between the pair with Hamming distance one with the number of "bad" transitions that increase the distance (of course, how much the distance increases from any bad transition also matters). There are other similar conditions that we might require from our graph to prove rapid mixing.

Instead of studying these here, we show that neighbourhood-amenability is fairly widely applicable. We do this by defining an alternative natural condition and showing that it implies neighbourhood-amenability.

Definition 37 *For a vertex v of G, let $N_d(v)$ denote the set of vertices that are at distance d from v, and let $n_d(v) = |N_d(v)|$. (Note that $n_0(v) = 1$.)*

We say that G is uniformly sub-exponential *if there exists a function $\kappa(d)$ such that*

(1) for all $b > 1$, $\kappa(d) \in o(b^d)$, and

(2) there exist $c_2 \geq c_1 > 0$ such that for all $v \in G$, $c_1\kappa(d) \leq n_d(v) \leq c_2\kappa(d)$.

As stated above, the condition of being uniformly sub-exponential implies neighbourhood-amenability as we show below. We first state a lemma that we use to prove this claim.

Lemma 38 *Let $\{a_i\}_{i\geq 0}$ be a sequence of positive numbers. Suppose that $a_d > \alpha(a_{d-1}+a_{d-2}+\cdots+a_0)$ for all $d \geq 1$. Then $a_d > a_0\alpha(1+\alpha)^{d-1}$ for $d \geq 1$.*

Proof. We prove this by induction, where the base case (with $d = 1$) is obvious from the condition imposed on the sequence. Then

$$\begin{aligned}
a_d &> \alpha(a_{d-1} + a_{d-2} + \cdots + a_1 + a_0) \\
&> \alpha(a_0\alpha(1+\alpha)^{d-2} + a_0\alpha(1+\alpha)^{d-3} + \cdots + a_0\alpha + a_0) \quad \text{(by the inductive assumption)} \\
&= a_0\alpha\left(\alpha\left((1+\alpha)^{d-2} + (1+\alpha)^{d-3} + \cdots + 1\right) + 1\right) \\
&= a_0\alpha\left(\alpha\frac{(1+\alpha)^{d-1}-1}{\alpha} + 1\right) \\
&= a_0\alpha(1+\alpha)^{d-1}.
\end{aligned}$$

□

Lemma 39 *Suppose that G is uniformly sub-exponential. Then G is neighbourhood-amenable.*

Proof. We apply the contrapositive of Lemma 38 to the sequence of numbers $\kappa(0), \kappa(1), \ldots$. This means that given $\varepsilon > 0$, there is a d such that $\kappa(d+1) \leq \varepsilon(\kappa(d) + \cdots + \kappa(0))$. (Otherwise, if $\kappa(d+1) > \varepsilon(\kappa(d) + \cdots + \kappa(0))$ for all d, then the lemma says that $\kappa(d+1) > \kappa(0)\varepsilon(1+\varepsilon)^d$ for all $d \geq 0$. Clearly this would violate condition (1) in the definition of "uniformly sub-exponential.")

Thus, for any vertex $v \in G$ we have, using condition (2) in Definition 37,

$$\begin{aligned} |\partial Ball_d(v)| = n_{d+1}(v) &\leq c_2 \kappa(d+1) \\ &\leq c_2 \cdot \varepsilon(\kappa(d) + \cdots + \kappa(0)) \\ &\leq \frac{c_2}{c_1} \cdot \varepsilon(n_d(v) + \cdots + n_0(v)) = \frac{c_2}{c_1} \cdot \varepsilon |Ball_d(v)|. \end{aligned}$$

Hence $T_d = \sup_v \frac{|\partial Ball_d(v)|}{|Ball_d(v)|} \leq \frac{c_2}{c_1} \cdot \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies that G is neighbourhood-amenable. □

One could think of other conditions that would imply neighbourhood-amenability or even different conditions for which a similar proof of rapid mixing such as the one we gave in Theorem 8 could be demonstrated. If we are dealing with a graph that is vertex-transitive, for example, checking whether it is uniformly-subexponential or not provides a relatively straightforward method to determine if it is neighbourhood-amenable.

Readers should consult [30, 31] for further discussion about conditions under which one could demonstrate rapid mixing of Markov chains for sampling proper colourings.

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