# The complexity of choosing an $H$-colouring (nearly) uniformly at random*

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**Abstract**

Cooper, Dyer and Frieze studied the problem of sampling $H$-colourings (nearly) uniformly at random. Special cases of this problem include sampling colourings and independent sets and sampling from statistical physics models such as the Widom-Rowlinson model, the Beach model, the Potts model and the hard-core lattice gas model. Cooper et al. considered the family of "cautious" ergodic Markov chains with uniform stationary distribution and showed that, for every fixed connected "nontrivial" graph $H$, every such chain mixes slowly. In this paper, we give a complexity result for the problem. Namely, we show that for **any** fixed graph $H$ with no trivial components, there is unlikely to be any *Polynomial Almost Uniform Sampler* (PAUS) for $H$-colourings. We show that if there were a PAUS for the $H$-colouring problem, there would also be a PAUS for sampling independent sets in bipartite graphs and, by the self-reducibility of the latter problem, there would be a *Fully-Polynomial Randomised Approximation*

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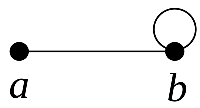


Figure 1: Homomorphisms from $G$ to this graph are independent sets of $G$.

*Scheme* (FPRAS) for #BIS — the problem of counting independent sets in bipartite graphs. Dyer, Goldberg, Greenhill and Jerrum have shown that #BIS is complete in a certain logically-defined complexity class. Thus, a PAUS for sampling $H$-colourings would give an FPRAS for the entire complexity class. In order to achieve our result we introduce the new notion of *sampling-preserving* reduction which seems to be more useful in certain settings than approximation-preserving reduction.

# 1 Introduction

Let $H = (V(H), E(H))$ be any fixed graph. An *$H$-colouring* of a graph $G = (V(G), E(G))$ is just a homomorphism from $G$ to $H$: The vertices of $H$ correspond to "colours" and the edges of $H$ specify which colours may be adjacent. Thus, an $H$-colouring of $G$ is a function $C$ from $V(G)$ to $V(H)$ such that for every edge $(u, v) \in E(G)$, the corresponding edge $(C(u), C(v))$ is in $E(H)$. Informally, colours $C(u)$ and $C(v)$ are allowed to be adjacent in the colouring $C$ of $G$ because the edge $(C(u), C(v))$ is an edge of $H$.

Many combinatorial problems can be viewed as special cases of $H$-colouring. For example, if $H$ is a $k$-clique with no self-loops then $H$-colourings of $G$ correspond to proper $k$-colourings of $G$. (In such a colouring, $k$ colours are available for colouring the vertices of $G$, but no colour may be adjacent to itself.) Here is another example. If $H$ is the graph depicted in Figure 1 then $H$-colourings of $G$ correspond to independent sets of $G$ — vertices which are coloured "$a$" are in the independent set, and vertices which are coloured "$b$" are not. Several models from statistical physics are special cases of $H$-colouring including the Widom-Rowlinson model, the Beach model, and (for weighted $H$-colourings) the Potts model and the hard-core lattice gas model. See [2, 10] for details.

The complexity of $H$-colouring has been well-studied. Many papers considered the following problem: Given a fixed graph $H$, determine, for an input graph $G$, whether $G$ has an $H$-colouring. Hell and Nešetřil [14] completely characterised the set of graphs for which this problem is NP-complete. They observed that the problem is in P if $H$ has a loop or is bipartite and they showed that it is NP-complete for any other fixed $H$. See [14] for references to earlier work on this question and [13] for extensions to the case in which the maximum degree of $G$ is bounded. See [4, 5] for extensions to parameterised complexity.

Dyer and Greenhill [10] considered the problem of *counting* $H$-colourings. Intriguingly, they were able to completely characterise the graphs $H$ for which this problem is #P-complete. A connected component of $H$ is said to be "trivial" if it is a complete graph with all loops present or a complete bipartite graph with no loops present[1]. Dyer and Greenhill showed that counting $H$-colourings is #P-complete if $H$ has a nontrivial component and that it is in P otherwise. They also extended their result to the case in which the maximum degree of $G$ is bounded.

Other work has focused on the complexity of *sampling* $H$-colourings (nearly) uniformly at random[2]. Positive results for particular graphs $H$ (specifically for the case in which $H$-colourings are independent sets and for the case in which $H$-colourings are proper colourings) appear in works such as [9, 15, 18]. A negative result for the independent-set case appears in [6]. The first paper to study the complexity of sampling $H$-colourings in the general case was Cooper, Dyer and Frieze [3]. They focused on connected graphs $H$ for which the decision problem "Is there an $H$-colouring?" is in P, but the counting problem "How many $H$-colourings are there?" is #P-complete. They showed that for any such $H$, $H$-colourings cannot be sampled efficiently using "cautious" Markov chains, which are Markov chains which can change only a constant fraction of the colours of the vertices in a single step. In particular, the mixing time of all such chains is exponential in the number of vertices of $G$. They also give positive results for certain weighted cases, which are extended in [12]. In particular, [12] shows that for every fixed "dismantleable" $H$ and every degree bound $\Delta$, there are positive vertex-weights

[1]Following the usual notation in the area, we will treat self-loops specially, so it makes sense to refer to bipartite graphs with or without loops. The loop-free single-vertex is viewed as a complete bipartite graph.

[2]Some of this work has been motivated by the well-known connection between almost-uniform sampling and approximate counting [17, 8]. For some graphs $H$, it can be shown that the problem of approximately counting $H$-colourings is equivalent to the problem of sampling $H$-colourings (nearly) uniformly at random. See Section 8.

which can be assigned to the vertices of $H$ so that weighted $H$-colourings can be sampled for degree-$\Delta$ graphs $G$. Borgs et al. [1] consider the problem of sampling $H$-colourings on rectangular subsets of the hypercubic lattice. They show that for every nontrivial connected $H$ there is an assignments of weights to colours for which cautious chains are slowly mixing.

In this work, we study the complexity of sampling $H$-colourings. We show that if $H$ has no trivial components then the problem of nearly-uniformly sampling $H$-colourings is intractable in a complexity-theoretic sense. In particular, we show that for any fixed $H$ with no trivial components, there is unlikely to be any *Polynomial Almost Uniform Sampler* (PAUS) for $H$-colourings. We show that if there were a PAUS for the $H$-colouring problem, there would also be a PAUS for sampling independent sets in bipartite graphs and, by the self-reducibility of the latter problem, there would be a *Fully-Polynomial Randomised Approximation Scheme* (FPRAS) for #BIS — the problem of counting independent sets in bipartite graphs. Dyer, Goldberg, Greenhill and Jerrum have shown that #BIS is complete in a certain logically-defined subclass of #P which includes problems such as counting downsets in partial orders and counting satisfying assignments in "restricted Horn" CNF Boolean formulas. Thus, a PAUS for sampling $H$-colourings would give an FPRAS for the entire complexity class. In fact, our result holds even if the input $G$ is restricted to be a connected bipartite graph.

In order to achieve our result we introduce the new notion of *sampling-preserving* reduction. The notion of approximation-preserving reduction (AP-reducibility) from [11] seems to be too demanding. In particular, since AP-reducibility is about *counting* (as opposed to sampling), an AP-reduction is not allowed to inflate the size of the set of structures by a factor which is difficult to compute. Sampling-preserving reductions allow this flexibility while achieving the same final result. The definition of sampling reduction (Section 2) is essentially many-one. Nevertheless the reductions get used in a "Turing reduction" way. In particular, our reduction from SampleBIS to Sample$H$-Col takes an instance of SampleBIS and constructs many Sample$H$-Col instances. Since the resulting maps between $H$-colourings and independent sets are many-one, several reductions can be combined even though they may involve different amounts of inflation of the state space.

The paper is structured as follows. Section 2 gives the relevant definitions including the definition of a sampling-preserving reduction. Section 3 presents some technical lemmas which we need in our proofs. Section 4 outlines a general proof technique for demonstrating the existence of an SP-reduction. Section 5 uses the new proof technique to reduce SampleBIS to a crucial intermediate problem, SampleFixed$H$-Col. Section 6 proves the main result. Sections 7 and 8 discuss extensions.

# 2 Definitions

The total variation distance between two distributions $\pi$ and $\pi'$ on a countable set $\Omega$ is given by

$$\mathrm{d}_{\mathrm{TV}}(\pi, \pi') = \frac{1}{2} \sum_{\omega \in \Omega} |\pi(\omega) - \pi'(\omega)| = \max_{A \subseteq \omega} |\pi(A) - \pi'(A)|.$$

A sampling problem $X$ maps each instance $\sigma$ to a set of structures $X(\sigma)$. The goal is to produce a member of $X(\sigma)$ uniformly at random. The size of each structure in $X(\sigma)$ is at most a polynomial in $|\,\sigma\,|$. For a given graph $H$, the sampling problem Sample$H$-Col will be defined as follows.

*Name.* Sample$H$-Col.

*Instance.* A loop-free graph $G$.

*Output.* An $H$-colouring of $G$ chosen uniformly at random.

We will be particularly interested in the special case of this problem in which the input graph, $G$, is connected and bipartite.

*Name.* SampleB$H$-Col.

*Instance.* A loop-free connected bipartite graph $G$.

*Output.* An $H$-colouring of $G$ chosen uniformly at random.

The problem SampleBIS will be defined as follows.

*Name.* SampleBIS.

*Instance.* A loop-free connected bipartite graph $G$.

*Output.* An independent set of $G$ chosen uniformly at random.

An *almost uniform sampler* [8, 16, 17] for $X$ is a randomised algorithm that takes input $\sigma$ and accuracy parameter $\epsilon \in (0, 1]$ and gives an output such that the variation distance between the output distribution of the algorithm and the uniform distribution on $X(\sigma)$ is at most $\epsilon$. We will say that algorithm is a *polynomial almost uniform sampler (PAUS)* if its running time is bounded from above by a polynomial in the size of the instance $|\sigma|$ and $1/\epsilon$.

A *sampling-preserving reduction* (SP-reduction) from a sampling problem $X$ to a sampling problem $Y$ (denoted $X \leq_{\mathrm{SP}} Y$) consists of

1. A function $f$ which takes an input $(\sigma, \epsilon)$, in which $\sigma$ is an instance of $X$ and $\epsilon \in (0, 1]$ is an accuracy parameter, and produces an instance $f(\sigma, \epsilon)$ of $Y$. If $X(\sigma)$ is non-empty then $Y(f(\sigma, \epsilon))$ must be non-empty.

2. A function $g$ which maps each tuple $(\sigma, \epsilon, y)$ with $y \in Y(f(\sigma, \epsilon))$ to a member of $X(\sigma) \cup \{\perp\}$ where "$\perp$" is an error symbol and for every $(\sigma, \epsilon)$ and every $x \in X(\sigma)$,

$$e^{-\epsilon} \frac{|Y(f(\sigma, \epsilon))|}{|X(\sigma)|} \leq |\{y \in Y(f(\sigma, \epsilon)) \mid g(\sigma, \epsilon, y) = x\}| \leq e^{\epsilon} \frac{|Y(f(\sigma, \epsilon))|}{|X(\sigma)|}. \tag{1}$$

Equation (1) says that for every $x \in X(\sigma)$, the number of $y \in Y(f(\sigma, \epsilon))$ which are mapped to $x$ by $g$ is roughly $\frac{|Y(f(\sigma,\epsilon))|}{|X(\sigma)|}$. Thus, each $x \in X(\sigma)$ is roughly equally represented and the error symbol $\perp$ is represented by only about an $\epsilon$-fraction of $Y(f(\sigma, \epsilon))$.

The functions $f$ and $g$ must be computable in time which is bounded by a polynomial in $|\sigma|$ and $1/\epsilon$.

The motivation for this definition is the following lemma.

**Lemma 1** *If $X \leq_{\mathrm{SP}} Y$ and $Y$ has a PAUS, then $X$ has a PAUS.*

*Proof.* Let $(f, g)$ be the reduction from $X$ to $Y$ and let $\mathcal{A}$ be a PAUS for $Y$. Here is a PAUS for $X$: On input $(\sigma, \epsilon)$, let $y$ be the output of $\mathcal{A}$ when it is run with inputs $f(\sigma, \epsilon/4)$ and $\epsilon/2$; return $g(\sigma, \epsilon/4, y)$. We must show that the variation distance between the output distribution of this algorithm and the uniform distribution on $X(\sigma)$ is at most $\epsilon$. Let $\sigma$ be an input with $|X(\sigma)| \geq 1$. Consider any subset $A_x$ of $X(\sigma)$. Let

$$A_y = \{y \in Y(f(\sigma, \epsilon/4)) \mid g(\sigma, \epsilon/4, y) \in A_x\}.$$

Then the probability that $\mathcal{A}$ gives an output in $A_y$ is at most

$$\begin{aligned}
&\frac{|A_y|}{|Y(f(\sigma,\epsilon/4))|}+\frac{\epsilon}{2}\\
\le\ &\frac{e^{\epsilon/4}|A_x|}{|X(\sigma)|}+\frac{\epsilon}{2}\\
\le\ &\frac{(1+\epsilon/2)|A_x|}{|X(\sigma)|}+\frac{\epsilon}{2}\\
\le\ &\frac{|A_x|}{|X(\sigma)|}+\frac{(\epsilon/2)|A_x|}{|X(\sigma)|}+\frac{\epsilon}{2}\\
\le\ &\frac{|A_x|}{|X(\sigma)|}+\epsilon.
\end{aligned}$$

Also, the probability that $\mathcal{A}$ gives an output in $A_y$ is at least

$$\begin{aligned}
&\frac{|A_y|}{|Y(f(\sigma,\epsilon/4))|}-\frac{\epsilon}{2}\\
\ge\ &\frac{e^{-\epsilon/4}|A_x|}{|X(\sigma)|}-\frac{\epsilon}{2}\\
\ge\ &\frac{(1-\epsilon/2)|A_x|}{|X(\sigma)|}-\frac{\epsilon}{2}\\
\ge\ &\frac{|A_x|}{|X(\sigma)|}-\frac{|A_x|(\epsilon/2)}{|X(\sigma)|}-\frac{\epsilon}{2}\\
\ge\ &\frac{|A_x|}{|X(\sigma)|}-\epsilon.
\end{aligned}$$

□

The problem #BIS is defined as follows.

*Name.* #BIS.

*Instance.* A loop-free bipartite graph $G$.

*Output.* The number of independent sets of $G$.

A component of $H$ is *trivial* if it is a complete graph with all loops present or a complete bipartite graph with no loops present. Recall from Dyer and Greenhill [10] that counting $H$-colourings is in P if $H$ is trivial. The main result of this paper is as follows.

**Theorem 2** *Suppose that $H$ is a fixed graph with no trivial components. If* SampleB$H$-Col *has a PAUS then* SampleBIS *has a PAUS and* #BIS *has an FPRAS. Thus, every problem which is AP-interreducible with* #BIS *(see [11]) has an FPRAS.*

# 3 Technical lemmas

Let $\nu(a,b)$ denote the number of onto functions from a set of size $a$ to a set of size $b$. We need to use the following lemma, which is Lemma 18 of [11].

**Lemma 3** *(DGGJ) If $a$ and $b$ are positive integers and $a \geq 2b \ln b$ then*

$$b^a \left(1 - \exp(-a/(2b))\right) \leq \nu(a,b) \leq b^a.$$

We also need the following technical lemma.

**Lemma 4** *Suppose $c_1$ and $c_2$ are fixed positive reals with $c_1 < c_2$. For any $\delta > 0$ and any non-negative integers $q$ and $a_0$, there are non-negative integers $a$ and $b$ with $a \geq a_0$ which are in $O((a_0+q)/\delta)$ and satisfy*

$$e^{-\delta} c_2^{a+q} \leq c_1^{b+q} \leq e^{\delta} c_2^{a+q}.$$

*Proof.* First, note that it would suffice to find non-negative integers $a'$ and $b'$ which are in $O(q'/\delta)$ and satisfy

$$e^{-\delta} c_2^{a'+q'} \leq c_1^{b'+q'} \leq e^{\delta} c_2^{a'+q'},$$

where $q' = q + a_0$ because we could simply set $a = a' + a_0$ and $b = b' + a_0$ which would imply $a' + q' = a + q$ and $b' + q' = b + q$.

Taking logarithms, what we need is

$$\left| b' - \frac{a' \log c_2 + q' \log(c_2/c_1)}{\log c_1} \right| \leq \frac{\delta}{\log c_1}. \qquad (2)$$

Now let $\rho$ be defined by $c_2 = c_1^{1+\rho}$. Then we want

$$|b' - (a'(1+\rho) + q'\rho)| \leq \frac{\delta}{\log c_1}. \qquad (3)$$

For a positive integer $r$, we will choose $a' = q'r$, so we want

$$|b' - a' - \rho q'(r+1)| \le \frac{\delta}{\log c_1}. \tag{4}$$

Let $R = \lceil 2 \log c_1/\delta \rceil$. Lemma 19 of [11] says: *For any real $z > 0$ and any positive integer $R$ there is an $x \in [1, \ldots, R]$ such that*

$$\min(zx - \lfloor zx \rfloor, \lceil zx \rceil - zx) \le 1/R.$$

Thus, there is an $x \in [1, \ldots, R]$ such that $\rho q' x$ is within $1/R$ of a non-negative integer. If $x > 1$ we will set $r + 1 = x$. If $x = 1$ then note that $\rho q' 2$ is within $2/R$ of a non-negative integer, so we will set $r = 1$.

Now recall that $a' = q'r$, so $a' \in O(q'/\delta)$ as required. □

# 4 Demonstrating the existence of SP-reductions: a proof technique

When we introduce an SP-reduction from a sampling problem $X$ to a sampling problem $Y$, we will need to show that Equation (1) is satisfied. We will typically do this by partitioning $Y(f(\sigma, \epsilon))$ into disjoint sets $Y_0, \ldots, Y_k$. We will show that each of $Y_1, \ldots, Y_k$ is fairly representative of $X(\sigma)$. In particular, for every $x \in X(\sigma)$ and every $i \in [1, k]$,

$$e^{-\epsilon/2} \frac{|Y_i|}{|X(\sigma)|} \le |\{y \in Y_i \mid g(\sigma, \epsilon, y) = x\}| \le e^{\epsilon/2} \frac{|Y_i|}{|X(\sigma)|}. \tag{5}$$

For every $y \in Y_0$, we will have $g(\sigma, \epsilon, y) = \perp$ but we will show that $Y_0$ is a small part of $Y(f(\sigma, \epsilon))$. In particular,

$$\sum_{i=1}^{k} |Y_i| \ge e^{-\epsilon/2} |Y(f(\sigma, \epsilon))|. \tag{6}$$

Together, (5) and (6) imply (1). Note that (6) follows from

$$|Y_0| \le (\epsilon/4)|Y(f(\sigma, \epsilon))|, \tag{7}$$

since (7) implies $|Y| - |Y_0| \ge (1 - \epsilon/4)|Y(f(\sigma, \epsilon))| \ge e^{-\epsilon/2}|Y(f(\sigma, \epsilon))|$.

Quite often the reduction $X \leq_{\mathrm{SP}} Y$ will involve several subproblems $Z_1, Z_2, \ldots$ such that, for each of these, an SP-reduction $(f_i, g_i)$ from $X$ to $Z_i$ is already known. The instance $f(\sigma, \epsilon)$ of $Y$ is then formed by "gluing" together instances $f_1(\sigma, \epsilon/2)$ of $Z_1$, $f_2(\sigma, \epsilon/2)$ of $Z_2$, and so on. $Y_i$ is (roughly) the portion of $Y(f(\sigma, \epsilon))$ for which, within each $y \in Y_i$, we can "zoom in" on a structure $z \in Z_i(f_i(\sigma, \epsilon/2))$. Each structure in $Z_i(f_i(\sigma, \epsilon/2))$ is represented by an equal number of $y \in Y_i$ so we can get (5) by referring to the SP-reduction from $X$ to $Z_i$. Establishing (7) is essentially showing that, although $Y(f(\sigma, \epsilon))$ has some structures which don't allow us to "zoom in" on an appropriate sub-problem to find our sample, these aren't so numerous.

Finally, let $Y_i(x) = \{y \in Y_i \mid g(\sigma, \epsilon, y) = x\}$. Suppose that no $y \in Y_i$ has $g(\sigma, \epsilon, y) = \perp$. In this case we can show (5) by showing that for all $x, x' \in X(\sigma)$,

$$|Y_i(x)| \leq e^{\epsilon/2} |Y_i(x')|. \tag{8}$$

To see this, note that

$$\frac{|Y_i|}{|X(\sigma)|} = \frac{\sum_{x' \in X(\sigma)} |Y_i(x')|}{|X(\sigma)|} \geq e^{-\epsilon/2} \frac{\sum_{x' \in X(\sigma)} |Y_i(x)|}{|X(\sigma)|} = e^{-\epsilon/2} |Y_i(x)|.$$

# 5 Sampling *fixed* $H$-colourings

Suppose that $H$ is connected, loop-free, and bipartite. Denote the vertex partition of $H$ by $(V_L(H), V_R(H))$. We will define the *fixed* $H$-colouring problem as follows.

*Name.* SampleFixed$H$-Col

*Instance.* A loop-free connected bipartite graph $G$ with vertex partition $(V_L(G), V_R(G))$

*Output.* An $H$-colouring of $G$ chosen uniformly at random from the set of $H$-colourings in which vertices of $V_L(G)$ receive colours from $V_L(H)$.

We will study the problem SampleFixed$H$-Col as an intermediate step on the way to the proof of Theorem 2.

A vertex in $V_L(H)$ is said to be *full* if it is adjacent to every vertex in $V_R(H)$. Similarly, a vertex in $V_R(H)$ is said to be *full* if it is adjacent to every vertex in $V_L(H)$. The graph $H$ is said to be *full* if both $V_L(H)$ and $V_R(H)$ contain at least one full vertex. The following lemma is the key ingredient in the proof of Theorem 2.

**Lemma 5** *Suppose that $H$ is a connected nontrivial full loop-free bipartite graph. Then* SampleBIS $\leq_{\mathrm{SP}}$ SampleFixed$H$-Col.

*Proof.* We'll prove the lemma by induction on the number of vertices in $H$. For the base case, suppose that $H$ has at most 4 vertices. The only connected nontrivial full loop-free bipartite graph $H$ with at most 4 vertices is the path of length 3. Let $G$ be an input to SampleBIS. There is a one-to-one correspondence between independent sets of $G$ and fixed $H$-colourings of $G$: The endpoints of $H$ point out the vertices which are in the independent set (see the proof of Theorem 5 of [11]).

We will now move on to the inductive step. The high-level idea is the following. By considering the graph $H$, we will construct several graphs $H_{S_1}, \ldots, H_{S_{j+k}}$, each of which is smaller than $H$ and satisfies certain conditions. By induction, for each $i$, there is an SP-reduction from SampleBIS to SampleFixed$H_{S_i}$-Col. If we apply this reduction to our instance $G$ of SampleBIS, we get an instance $G_i$ of SampleFixed$H_{S_i}$-Col. Our goal is to construct an instance $f(G,\epsilon)$ of SampleFixed$H$-Col. We do this by "gluing together" the various $G_i$'s. Now consider the constructed instance $f(G,\epsilon)$ of SampleFixed$H$-Col. When we sample from the output distribution SampleFixed$H$-Col$(f(G,\epsilon))$, we would like to recover the output distribution of *SampleBIS*$(G)$. Curiously, we can not determine during the reduction itself the relative weights of the sub-instances $G_1, G_2, \ldots$. Nevertheless, once we have an output to SampleFixed$H$-Col$(f(G,\epsilon))$, the output itself tells us which $H_i$ is relevant. From this, we can recover an output to SampleFixed$H_{S_i}$-Col$(G_i)$ and from this we can recover an output to SampleBIS$(G)$. The main technical difficulty lies in showing that the distributions are correct. In particular, since the sub-reductions are SP-reductions (i.e., the equations in Section 4 are satisfied in the construction of $G_1, G_2, \ldots$), the combined reduction is also an SP-reduction.

We now describe the details. Let $F_L$ be the set of full vertices in $V_L(H)$ and let $F_R$ be the set of full vertices in $V_R(H)$. Let $f_L = |F_L|$ and $f_R = |F_R|$ and $v_L = |V_L(H)|$ and $v_R = |V_R(H)|$. For a subset $S$ of $V_R(H)$, let $N(S)$ be the set of mutual neighbours of $S$:

$$N(S) = \{v \in V_L(H) \mid \forall u \in S, (u,v) \in E(H)\}.$$

Note that $F_L \subseteq N(S) \subseteq V_L(H)$. $S$ is said to be *left-reducing* if $F_L \subset N(S) \subset V_L(H)$. If $S$ is left-reducing, let $H_S$ be the subgraph of $H$ induced by vertex partition $(N(S), V_R(H))$. Note that $H_S$ has fewer vertices than $H$. Also, it

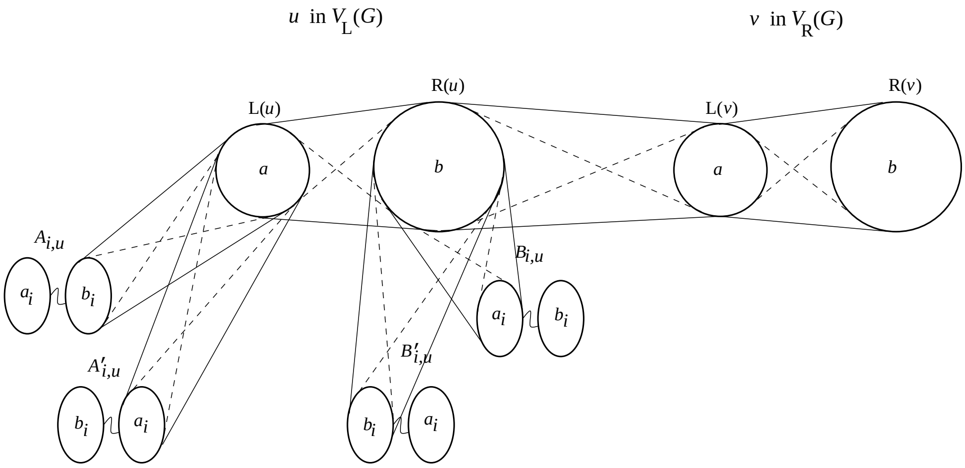


Figure 2: The construction of $f(G, \epsilon)$ in the proof of Lemma 5.

is connected, full and nontrivial: The set of full vertices in $V_L(H_S)$ is $F_L$; the set of full vertices in $V_R(H_S)$ includes all of $F_R$ but it does not equal $V_R(H)$ since $N(S) \supset F_L$.

Similarly, a subset $S$ of $V_L(H)$ is *right-reducing* if $F_R \subset N(S) \subset V_R(H)$. If $S$ is right-reducing, let $H_S$ be the subgraph of $H$ induced by vertex partition $(V_L(H), N(S))$. $H_S$ has fewer vertices than $H$ and is connected, full and nontrivial.

Now, let $S_1, \ldots, S_k$ be the left-reducing subsets of $V_R(H)$ and let $S_{k+1}, \ldots, S_{k+j}$ be the right-reducing subsets of $V_L(H)$. (Either or both of $k$ and $j$ could be zero.) For every $i \in \{1, \ldots, k+j\}$, let $(f_i, g_i)$ be an SP-reduction from SampleBIS to SampleFixed$H_{S_i}$-Col. Take the input $(G, \epsilon)$ to SampleBIS and let $G_i = f_i(G, \epsilon/2)$. Let $a_i = |V_L(G_i)|$ and let $b_i = |V_R(G_i)|$. Let $q = \sum_{i=1}^{k+j}(a_i + b_i)$ and let $n = |V_L(G)| + |V_R(G)|$.

Let $f(G, \epsilon)$ be the graph which is constructed as follows, where $a$ and $b$ will be chosen later to satisfy

$$a \geq 2v_L \lceil q \ln(v_R/f_R) + \ln(16n/\epsilon) \rceil , \qquad (9)$$

and

$$b \geq 2v_R \lceil q \ln(v_L/f_L) + \ln(16n/\epsilon) \rceil . \qquad (10)$$

See Figure 2. For every vertex $u$ of $G$, put a size-$a$ set $L(u)$ into $V_L(f(G, \epsilon))$ and a size-$b$ set $R(u)$ into $V_R(f(G, \epsilon))$. Also, add edges $L(u) \times R(u)$ to

$E(f(G,\epsilon))$. If $u \in V_L(G)$ is connected to $v \in V_R(G)$ by an edge of $G$ then add edges $R(u) \times L(v)$ to $E(f(G,\epsilon))$.

Also, for every vertex $u$ of $G$ and every $i \in [1,\ldots,k+j]$, let $A_{i,u}$ and $B_{i,u}$ be copies of $G_i$ and let $A'_{i,u}$ and $B'_{i,u}$ be copies of $G_i$ in which left-vertices and right-vertices are switched (so the vertices in $V_L(A'_{i,u})$ correspond to the vertices in $V_R(G_i)$ and the vertices in $V_R(A'_{i,u})$ correspond to the vertices in $V_L(G_i)$). Add edges $L(u) \times V_R(A_{i,u})$ and $L(u) \times V_R(A'_{i,u})$ and $R(u) \times V_L(B_{i,u})$ and $R(u) \times V_L(B'_{i,u})$ to $E(f(G,\epsilon))$.

Let

$$V_L(f(G,\epsilon)) = \bigcup_u L(u) \cup \bigcup_{u,i} \{V_L(A_{i,u}) \cup V_L(A'_{i,u}) \cup V_L(B_{i,u}) \cup V_L(B'_{i,u})\}$$

and let $Y$ be the set of fixed $H$-colourings of $f(G,\epsilon)$. We will partition $Y$ into sets $Y_0,\ldots,Y_{k+j+1}$.

For $i \in [1,\ldots,k]$, $Y_i$ is the set of colourings which are not in $Y_1,\ldots,Y_{i-1}$ but in which some $u \in V_L(G)$ has $R(u)$ coloured with (exactly) the colours in $S_i$. For $i \in [k+1,\ldots,k+j]$, $Y_i$ is the set of colourings which are not in $Y_1,\ldots,Y_{i-1}$ but in which some $v \in V_R(G)$ has $L(v)$ coloured with $S_i$.

The high-level structure of our construction is as follows. For every $i \in \{1,\ldots,k+j\}$, we will use the colourings in $Y_i$ by focusing on the induced colourings of the subgraph $G_i$. These are $H_{S_i}$-colourings of $G_i$ and from these we can (by induction) recover a random independent set of $G$. As usual, the colourings in $Y_0$ are not useful for pointing out independent sets, but there are not too many of these. Every colouring in $Y_{k+j+1}$ has a special form — Every vertex $u$ of $G$ either has $R(u)$ coloured $V_R(H)$ or has $L(u)$ coloured $V_L(H)$. These colourings point out independent sets of $G$ in a natural way, and each independent set comes up about the same number of times in this way.

We now look at the details. Note that for any colourings in $Y_0$ or $Y_{k+j+1}$, we have the following property — every vertex $u \in V_L(G)$ has $R(u)$ coloured with a set $S$ of colours such that $N(S)$ is either $F_L$ or $V_L(H)$. Similarly, every vertex $v \in V_R(G)$ has $L(v)$ coloured with a set $S$ of colours such that $N(S)$ is either $F_R$ or $V_R(H)$.

Consider a colouring $y$. Vertex $u \in V_L(G)$ satisfies *Condition (A)* if $R(u)$ is coloured with a set $S$ of colours with $N(S) = F_L$ but $S \subset V_R(H)$. It satisfies *Condition (B)* if $R(u)$ is coloured with a set $S$ of colours with $N(S) = V_L(H)$ but $L(u)$ is coloured with a proper subset of $V_L(H)$. Vertex $v \in V_R(G)$ satisfies *Condition (C)* if $L(v)$ coloured with a set $S$ of colours with $N(S) =$

$F_R$ but $S \subset V_L(H)$. It satisfies *Condition (D)* if $L(v)$ is coloured with a set $S$ of colours with $N(S) = V_R(H)$ but $R(v)$ is coloured with a proper subset of $V_R(H)$.

We now define

$$Y_0 = \{y \in Y - \{Y_1 \cup \cdots \cup Y_{k+j}\} \mid \text{some vertex satisfies Condition A or B or C or D}\}.$$

Now note that colourings in $Y_{k+j+1}$ have the following property. Every vertex $u$ of $G$ either has $R(u)$ coloured $V_R(H)$ or has $L(u)$ coloured $V_L(H)$.

We will first work on establishing Equation (7). Let $Y_{u,A}$ denote the subset of $Y$ in which $u$ satisfies (A). Define $Y_{u,B}$, $Y_{u,C}$ and $Y_{u,D}$ similarly. We will show that the size of each of $Y_{u,A}$, $Y_{u,B}$, $Y_{u,C}$ and $Y_{u,D}$ is at most $(\epsilon/(16n))|Y|$. Equation (7) follows since

$$|Y_0| \leq \sum_{u \in V(G)} |Y_{u,A}| + |Y_{u,B}| + |Y_{u,C}| + |Y_{u,D}|.$$

First, let's show that $|Y_{u,A}| \leq (\epsilon/(16n))|Y|$. Consider the set of colourings in $Y$ in which all neighbours of vertices in $R(u)$ have colours from $F_L$ and let $\psi$ be the number of induced colourings on vertices other than the vertices of $R(u)$. If $\psi = 0$ then $|Y_{u,A}| = 0$, so the claim is trivial. Otherwise, $|Y_{u,A}| \leq \psi(v_R^b - \nu(b, v_R))$ which is at most $\psi v_R^b \exp(-b/(2v_R))$ by Lemma 3. On the other hand, $|Y| \geq \psi v_R^b$, so the claim follows from Equation (10). The proof that $|Y_{u,C}|$ is sufficiently small is similar.

Next, let's show that $|Y_{u,B}| \leq (\epsilon/(16n))|Y|$. Consider the set of colourings in $Y$ in which $R(u)$ is coloured with a subset of $F_R$ and let $\psi$ be the number of induced colourings on all vertices except those in $L(u)$ and $A_{i,u}$ and $A'_{i,u}$ (for $i \in [1, \ldots, j+k]$). If $\psi = 0$ then $|Y_{u,B}| = 0$, so the claim is trivial. Otherwise, $|Y_{u,B}| \leq \psi(v_L^a - \nu(a, v_L))v_R^q v_L^q$ which is at most $\psi v_L^a \exp(-a/(2v_L))v_R^q v_L^q$ by Lemma 3. On the other hand, $|Y| \geq \psi v_L^a f_R^q v_L^q$, so the claim follows from Equation (9). The proof that $|Y_{u,D}|$ is sufficiently small is similar.

We will now work on establishing Equation (5). First consider $i \in [1, \ldots, k]$. Let $Y_{u,i}$ be the set of colourings in $Y_i$ for which $u \in V_L(G)$ is the first vertex in $V_L(G)$ with $R(u)$ coloured $S_i$. Let $\Gamma$ be the set of induced colourings on $B_{i,u}$. Note that $\Gamma$ is the set of fixed $H_{S_i}$-colourings of $G_i = f_i(G, \epsilon/2)$. Also, each colouring in $\Gamma$ comes up $\psi$ times in $Y_{u,i}$ for some $\psi$. (In particular, $\psi$ is the number of colourings of vertices other than $B_{i,u}$ which are induced by colourings in $Y_{u,i}$.) For colouring $y \in Y_{u,i}$ we will let $g(G,\epsilon,y) = g_i(G_i,\epsilon/2,y')$ where $y'$ is the induced colouring on $B_{i,u}$. Then for every independent set $x$ in the set $\mathcal{I}(G)$ of independent sets of $G$,

$$|\{y \in Y_{u,i} \mid g(G,\epsilon,y) = x\}| = \psi\, |\{y' \in \Gamma \mid g_i(G_i,\epsilon/2,y') = x\}|. \tag{11}$$

Since $(f_i, g_i)$ is an SP-reduction, Equation (1) gives

$$e^{-\epsilon/2}\frac{|\Gamma|}{|\mathcal{I}(G)|} \leq |\{y' \in \Gamma \mid g_i(G_i,\epsilon/2,y') = x\}| \leq e^{\epsilon/2}\frac{|\Gamma|}{|\mathcal{I}(G)|} \tag{12}$$

and Equation (5) follows for $Y_{u,i}$ from Equations (11) and (12) since $|Y_{u,i}| = \psi\,|\Gamma|$. Colourings in $Y_{k+1},\ldots,Y_{k+j}$ are handled similarly except that we look at induced colourings of $A_{i,u}$ rather than $B_{i,u}$.

It remains to satisfy Equation (5) for $i = k+j+1$. Note that any colouring $y$ in $Y_{k+j+1}$ points out an independent set of $G$. A vertex $u \in V_L(G)$ is in the independent set if $R(u)$ is coloured $V_R(H)$. A vertex $v \in V_R(G)$ is in the independent set if $L(v)$ is coloured $V_L(H)$. We will define $g(G,\epsilon,y)$ to be this independent set. Let us focus attention on a given independent set containing $w_L$ vertices in $V_L(G)$ and $w_R$ vertices in $V_R(G)$. We will now calculate how many colourings in $Y_{k+j+1}$ correspond to this independent set.

For any bipartite graph $G'$ with vertex partition $(V_L(G'),V_R(G'))$, let $\phi_H(G')$ denote the number of fixed $H$-colourings of $G'$. Then the number of times that this independent set comes up as a colouring in $Y_{k+j+1}$ is the product of the following two quantities.

$$\left(\nu(b,v_R) f_L^a \prod_{i=1}^{k+j} \phi_H(A_{i,u})\phi_H(A'_{i,u}) f_L^{a_i+b_i} v_R^{a_i+b_i}\right)^{w_L+v_R-w_R},$$

$$\left(f_R^b \nu(a,v_L) \prod_{i=1}^{k+j} \phi_H(B_{i,u})\phi_H(B'_{i,u}) v_L^{a_i+b_i} f_R^{a_i+b_i}\right)^{v_L-w_L+w_R}.$$

Now note that $\phi_H(A_{i,u}) = \phi_H(B_{i,u})$ and $\phi_H(A'_{i,u}) = \phi_H(B'_{i,u})$. So if we let

$$Z = \left(\prod_{i=1}^{k+j} \phi_H(A_{i,u})\phi_H(A'_{i,u})\right)^{v_L+v_R} (f_L^a \nu(b,v_R) f_L^q v_R^q)^{v_R} \left(f_R^b \nu(a,v_L) v_L^q f_R^q\right)^{v_L},$$

the contribution of the independent set becomes

$$Z(\nu(b, v_R) f_L^a f_L^q v_R^q)^{w_L - w_R} \left(f_R^b \nu(a, v_L) v_L^q f_R^q\right)^{w_R - w_L},$$

which is

$$Z\left(\frac{\nu(b, v_R) v_L^a}{v_R^b \nu(a, v_L)}\right)^{w_L - w_R} \left(\left(\frac{v_R}{f_R}\right)^{b+q} \left(\frac{f_L}{v_L}\right)^{a+q}\right)^{w_L - w_R}.$$

To get Equation (8) we will show that $a$ and $b$ can be chosen so that

$$e^{-\epsilon/(8n)} \leq \left(\frac{\nu(b, v_R) v_L^a}{v_R^b \nu(a, v_L)}\right) \leq e^{\epsilon/(8n)}, \tag{13}$$

and

$$e^{-\epsilon/(8n)} \leq \left(\frac{v_R}{f_R}\right)^{b+q} \left(\frac{f_L}{v_L}\right)^{a+q} \leq e^{\epsilon/(8n)}. \tag{14}$$

This guarantees that the contribution of this independent set is in the range $[e^{-\epsilon/4} Z, e^{\epsilon/4} Z]$, and Equation (8) follows for $Y_{k+j+1}$. To establish Equation (13), use Lemma 3 to observe that

$$\left(\frac{\nu(b, v_R) v_L^a}{v_R^b \nu(a, v_L)}\right) \leq \frac{1}{1 - \exp(-a/(2v_L))}.$$

Since Equation (9) gives $1 - \exp(-a/(2v_L)) \geq 1 - \epsilon/(16n) \geq e^{-\epsilon/(8n)}$, the right-hand inequality of (13) follows. The left-hand inequality is similar.

We will now show how to choose the values of $a$ and $b$ to satisfy Equation (14). If $v_R/f_R = v_L/f_L$ then simply choose $a = b$ and make them large enough to satisfy Equation (9) and Equation (10). Suppose that $v_R/f_R < v_L/f_L$. Then use Lemma 4 with $c_1 = v_R/f_R$, $c_2 = v_L/f_L$, $\delta = \epsilon/(8n)$, and

$$a_0 = 2v_L \lceil q \ln(v_R/f_R) + \ln(16n/\epsilon) \rceil + 2v_R \lceil q \ln(v_L/f_L) + \ln(16n/\epsilon) \rceil.$$

The lemma gives values of $a$ and $b$ which are in $O((a_0 + q)/\delta)$, which is not too large. Thus, our reduction is sampling-preserving. Note that the reduction can be done in polynomial time — the calculation of $a$ and $b$ does not involve computing $Z$. The case where $v_L/f_L < v_R/f_R$ is similar. $\square$

# 6 The proof of Theorem 2

We start with some definitions. First, for every graph $H$, we will define a loop-free bipartite graph $B[H]$ (this construction was used in [10]). Let the vertices of $H$ be $v_1, \ldots, v_h$. The vertex set of $B[H]$ is $\{x_1, \ldots, x_h\} \cup \{y_1, \ldots, y_h\}$. The edge set of $B[H]$ is

$$\{(x_i, y_j) \mid (v_i, v_j) \in E(H)\}.$$

Thus, a loop $(v_i, v_i)$ in $H$ corresponds to the edge $(x_i, y_i)$ in $B[H]$ and a non-loop $(v_i, v_j)$ in $H$ (for which $i \neq j$) corresponds to two edges $(x_i, y_j)$ and $(y_i, x_j)$ in $B[H]$. For every edge $(v_i, v_j)$ of $H$, let

$$V_L(H_{i,j}) = \{x_\ell \mid (v_\ell, v_j) \in E(H)\}$$

and

$$V_R(H_{i,j}) = \{y_\ell \mid (v_i, v_\ell) \in E(H)\}$$

and let $H_{i,j}$ be the subgraph of $B[H]$ induced by vertex set $V_L(H_{i,j}) \cup V_R(H_{i,j})$. Note that $x_i \in V_L(H_{i,j})$ and $y_j \in V_R(H_{i,j})$ and $x_i$ is adjacent to all of $V_R(H_{i,j})$ in $H_{i,j}$ and $y_j$ is adjacent to all of $V_L(H_{i,j})$. Thus, $H_{i,j}$ is connected and full. Let $\Delta_1(H)$ be the degree of $H$. That is,

$$\Delta_1(H) = \max\{\deg(v) \mid v \in V(H)\}.$$

Similarly, let $\Delta_2(H)$ be the maximum degree amongst neighbours of vertices with degree $\Delta_1(H)$:

$$\Delta_2(H) = \max\{\deg(v) \mid \text{ for some } u \in V(H) \text{ with } \deg(u) = \Delta_1(H),\ (u, v) \in E(H)\}.$$

Let

$$R(H) = \{(v_i, v_j) \mid ((v_i, v_j) \in E(H) \text{ and } \deg(v_i) = \Delta_1(H) \text{ and } \deg(v_j) = \Delta_2(H)\}.$$

We will start with the following lemma.

**Lemma 6** *Let $H$ be any fixed graph with no trivial components. Then $R(H)$ is non-empty and $\Delta_1(H) > 1$ and $\Delta_2(H) > 1$. Also, for all $(v_i, v_j) \in R(H)$, $H_{i,j}$ is connected, loop-free, bipartite, full and nontrivial.*

*Proof.* Since $H$ has no trivial components, $R(H)$ is non-empty and $\Delta_1(H) > 1$ and $\Delta_2(H) > 1$. Suppose $(v_i, v_j) \in R(H)$. Recall that $H_{i,j}$ is connected, loop-free, bipartite and full. Suppose for contradiction that $H_{i,j}$ is a complete bipartite graph (so vertices in $V_L(H_{i,j})$ have degree $\Delta_1(H)$ in $H_{i,j}$ and vertices in $V_R(H_{i,j})$ have degree $\Delta_2(H)$ in $H_{i,j}$).

This assumption guarantees that $H_{i,j}$ is a connected component of $B[H]$: $B[H]$ cannot have an edge with exactly one endpoint in $V_L(H_{i,j})$ — the endpoint would then have degree exceeding $\Delta_1(H)$ in $B[H]$, which is a contradiction; similarly, $B[H]$ cannot have an edge with exactly one endpoint in $V_R(H_{i,j})$.

Thus, for any $x_\ell \in V_L(H_{i,j})$,

$$\{v_r \mid (v_\ell, v_r) \in E(H)\} = \{v_r \mid y_r \in V_R(H_{i,j})\}. \tag{15}$$

Similarly, for any $y_\ell \in V_R(H_{i,j})$,

$$\{v_r \mid (v_\ell, v_r) \in E(H)\} = \{v_r \mid x_r \in V_L(H_{i,j})\}. \tag{16}$$

Now if $H$ has a vertex $v_\ell$ such that $(v_i, v_\ell) \in E(H)$ and $(v_j, v_\ell) \in E(H)$ then $x_\ell \in V_L(H_{i,j})$ and $y_\ell \in V_R(H_{i,j})$ so Equations (15) and (16) imply that

$$\{v_r \mid y_r \in V_R(H_{i,j})\} = \{v_r \mid x_r \in V_L(H_{i,j})\}.$$

Thus, $H_{i,j}$ corresponds to a component of $H$ and that component is a looped clique, which contradicts the fact that $H$ has no trivial component.

On the other hand, if there is no $v_\ell$ with $(v_i, v_\ell) \in E(H)$ and $(v_j, v_\ell) \in E(H)$ then $H_{i,j}$ corresponds to a connected component of $H$ which is a complete bipartite graph, again giving a contradiction. ☐

We can now prove the main lemma.

**Lemma 7** *Suppose that $H$ is a fixed graph with no trivial components. Then* SampleBIS $\leq_{\text{SP}}$ SampleB$H$-Col.

*Proof.* Let $(G, \epsilon)$ be an input to SampleBIS. For each $(v_i, v_j) \in R(H)$, Lemma 6 and Lemma 5 guarantee that there is a sampling-preserving reduction $(f_{i,j}, g_{i,j})$ from SampleBIS to SampleFixed$H_{i,j}$-Col. Let $G_{i,j} = f_{i,j}(G, \epsilon/2)$. Let $f(G, \epsilon)$ be the graph which is constructed as follows. See Figure 3. Let $q = \sum_{(v_i,v_j)\in R(H)} |V_L(G_{i,j})| + |V_R(G_{i,j})|$. Let

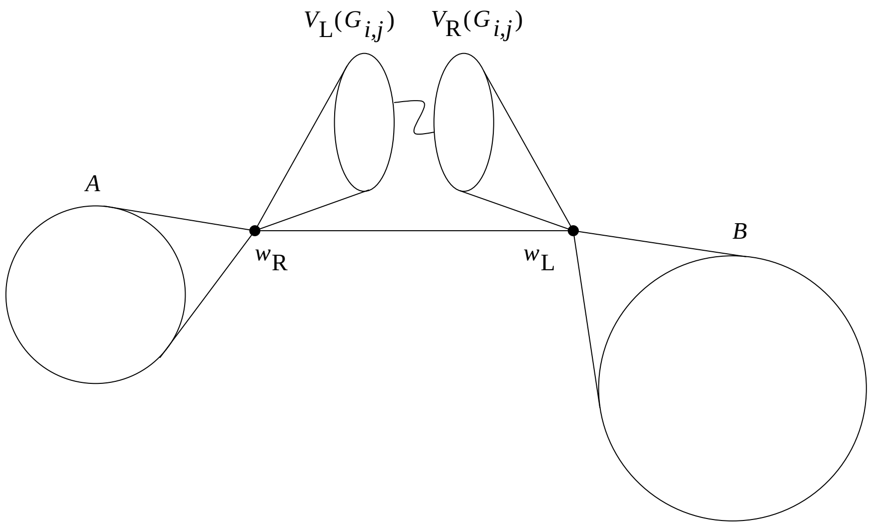


Figure 3: The construction of $f(G,\epsilon)$ in the proof of Lemma 7.

$$V_L(f(G,\epsilon)) = A \cup \{w_L\} \cup \bigcup_{(v_i,v_j)\in R(H)} V_L(G_{i,j}),$$

and

$$V_R(f(G,\epsilon)) = B \cup \{w_R\} \cup \bigcup_{(v_i,v_j)\in R(H)} V_R(G_{i,j}),$$

where $A$ and $B$ are sets of vertices with

$$|A| = \left\lceil \frac{q\ln(|V(H)|) + \ln(8|E(H)|/\epsilon)}{\ln(\Delta_2(H)/(\Delta_2(H)-1))} \right\rceil$$

and

$$|B| = \left\lceil \frac{(q+|A|+1)\ln(|V(H)|) + \ln(8|V(H)|/\epsilon)}{\ln(\Delta_1(H)/(\Delta_1(H)-1))} \right\rceil.$$

Note that there is no division by zero, since $\Delta_1(H)$ and $\Delta_2(H)$ are bigger than one (by Lemma 6). In addition to the edges in the graphs $G_{i,j}$, we add edge $(w_L, w_R)$ and $w_L \times B$ and $w_R \times A$ and, for all $(v_i, v_j) \in R(H)$, we add edges $w_L \times V_R(G_{i,j})$ and $w_R \times V_L(G_{i,j})$.

Let $Y$ be the set of $H$-colourings of $f(G,\epsilon)$. $Y_0$ will be the set of colourings in $Y$ in which $(w_L, w_R)$ is not coloured with an edge $(v_i, v_j)$ from $R(H)$. We will now establish Equation (7). For every $v \in V(H)$ with $\deg(v) < \Delta_1(H)$ let $Y_0(v)$ be the set of colourings in $Y$ in which $w_L$ is coloured $v$. Now

$$|Y_0(v)| \leq (\Delta_1(H) - 1)^{|B|} |V(H)|^{q+|A|+1}.$$

Now consider any $(v_i, v_j) \in R(H)$. There are at least $\Delta_2(H)^{|A|} \Delta_1(H)^{|B|}$ colourings of $f(G, \epsilon)$ with $(w_L, w_R)$ coloured $(v_i, v_j)$ (for example, the colourings in which all of the vertices of the graphs $G_{i,j}$ are coloured with either $v_i$ or $v_j$). Thus, $|Y| \geq \Delta_2(H)^{|A|} \Delta_1(H)^{|B|} \geq \Delta_1(H)^{|B|}$. We conclude that

$$|Y_0(v)| \leq (\epsilon/(8|V(H)|))|Y|. \tag{17}$$

Now consider any edge $(v_i, v_k) \in E(H)$ such that $\deg(v_i) = \Delta_1(H)$ but $\deg(v_k) < \Delta_2(H)$. Let $Y_0(v_i, v_k)$ be the set of colourings in $Y$ in which $(w_L, w_R)$ is coloured $(v_i, v_k)$. Now

$$|Y_0(v_i, v_k)| \leq \Delta_1(H)^{|B|} (\Delta_2(H) - 1)^{|A|} |V(H)|^q.$$

Also, from before $|Y| \geq \Delta_2(H)^{|A|} \Delta_1(H)^{|B|}$ so

$$|Y_0(v_i, v_k)| \leq (\epsilon/(8|E(H)|))|Y|. \tag{18}$$

Equation (17) and (18) imply Equation (7) since $|Y_0| \leq \sum_{v \in V(H)} Y_0(v) + \sum_{(v_i, v_k)} |Y_0(v_i, v_k)|$.

For an edge $(v_i, v_j) \in R(H)$, let $Y_{i,j}$ be the set of colourings of $f(G, \epsilon)$ with $(w_L, w_R)$ coloured $(v_i, v_j)$. Let $\Gamma$ be the set of induced colourings on $G_{i,j}$. Note that $\Gamma$ is the set of fixed $H_{i,j}$-colourings of $G_{i,j}$. Also, each colouring in $\Gamma$ comes up $\psi$ times in $Y_{i,j}$ where $\psi$ is the number of induced colourings on the vertices other than $G_{i,j}$. For a colouring $y \in Y_{i,j}$ we will set $g(G, \epsilon, y) = g_{i,j}(G_{i,j}, \epsilon/2, y')$ where $y'$ is the induced colouring on $G_{i,j}$. Then Equation (5) follows from the fact that $(f_{i,j}, g_{i,j})$ is an SP-reduction. □

Theorem 2 follows from Lemma 1 and Lemma 7 and from Lemma 8 below. Recall the following definitions. A *randomised approximation scheme* (RAS) for a counting problem $F$ is a randomised algorithm that takes input $\sigma$ and accuracy parameter $\epsilon \in (0, 1)$ and produces as output an integer random variable $Y$ satisfying the condition $\Pr(e^{-\epsilon} F(\sigma) \leq Y \leq e^{\epsilon} F(\sigma)) \geq 3/4$. It is a "fully polynomial" randomised approximation scheme (FPRAS) if it runs in time $\mathrm{poly}(|\sigma|, \epsilon^{-1})$. The problem #BIS is "self-reducible" so the following lemma follows from [17].

**Lemma 8** *(JVV) If* SampleBIS *has a PAUS then* #BIS *has an FPRAS.*

*Proof.* The lemma is a special case of Theorem 6.4 of [17]. In order to apply Theorem 6.4 directly we would need to define "self-reducible" formally and to deal with some easy (though annoying) issues:

(i) Inputs to #BIS may be disconnected but inputs to SampleBIS may not.

(ii) In order to apply Theorem 6.4 we technically need a *fully* polynomial almost uniform sampler (FPAUS) for SampleBIS. This can be obtained from a PAUS as [17] explains.

Rather than dealing with these issues, we prefer to simply provide a proof for the lemma. The details given here are from the proof of Proposition 3.4 of [16]. Technically, Jerrum's proof from [16] is about counting *matchings* but the few changes that are needed to yield our lemma are completely routine.

Let $(G, \epsilon)$ be an input to #BIS. Suppose that $G$ has components $G_1, \ldots, G_k$. For each $i$, let the two parts of the vertex set be $V_L(G_i)$ and $V_R(G_i)$ and let the sizes of these parts be $\ell_i$ and $r_i$, respectively. Let $N_i = \ell_i r_i$ and let $E(G_i) = \{e_i(1), \ldots, e_i(m_i)\}$. Denote the non-edges of $G_i$ by $\{e_i(m_i+1), \ldots, e_i(N_i)\}$. For $j \in \{1, \ldots, N_i\}$, let $G_i(j)$ be the graph $(V(G_i), \{e_i(1), \ldots, e_i(j)\})$. For any graph $G'$, let $\mathcal{I}(G')$ denote the set of independent sets of $G'$. Let

$$\rho_i(j) = \frac{|\mathcal{I}(G_i(j+1))|}{|\mathcal{I}(G_i(j))|}.$$

Note that

$$|\mathcal{I}(G_i)| = (\rho_i(m_i)\rho_i(m_i+1)\cdots\rho_i(N_i-1))^{-1}|\mathcal{I}(G_i(N_i))|.$$

Also, the number of independent sets of the complete bipartite graph $G_i(N_i)$ is $2^{\ell_i} + 2^{r_i} - 1$, so

$$|\mathcal{I}(G_i)| = (2^{\ell_i} + 2^{r_i} - 1)\prod_{j=m_i}^{N_i-1} \rho_i(j)^{-1}. \tag{19}$$

Furthermore,

$$|\mathcal{I}(G)| = \prod_{i=1}^{k} |\mathcal{I}(G_i)| = \prod_{i=1}^{k} (2^{\ell_i} + 2^{r_i} - 1)\prod_{j=m_i}^{N_i-1} \rho_i(j)^{-1}. \tag{20}$$

Now let $z = \sum_{i=1}^{k}(N_i - m_i)$. In order to estimate $|\mathcal{I}(G)|$, we need to estimate the $z$ ratios $\rho_i(j)$.

For each ratio $\rho_i(j)$ we can make some observations.

(i) $\rho_i(j) \leq 1$, since $\mathcal{I}(G_i(j+1)) \subseteq \mathcal{I}(G_i(j))$

(ii) $\rho_i(j) \geq 1/2$, since $\mathcal{I}(G_i(j)) \backslash \mathcal{I}(G_i(j+1))$ can be mapped injectively into $\mathcal{I}(G_i(j+1))$ by removing the lexicographically-least endpoint of $e_i(j+1)$.

(iii) Let $\mathcal{A}$ be a PAUS for SampleBIS. For $i \in [1, \ldots, k]$ and $j \in [m_i, \ldots, N_i - 1]$, let $Z_i(j)$ be the indicator variable for the event that, when we run $\mathcal{A}$ with input $G_i(j)$ and accuracy parameter $\delta$, the output is an independent set of $G_i(j+1)$. Note that $\rho_i(j) - \delta \leq E[Z_i(j)] \leq \rho_i(j) + \delta$. This follows immediately from the definition of PAUS, but it is important to note that the input to $\mathcal{A}$, $G_i(j)$, is connected (since all inputs to SampleBIS must be connected).

Let $\overline{Z_i(j)}$ be the result obtained by calling $\mathcal{A}$ $\lceil 74\epsilon^{-2}z \rceil$ times with input $G_i(j)$ and accuracy parameter $\delta = \epsilon/(6z)$ and averaging the value of $Z_i(j)$ which occurs each time. Jerrum shows in his proof that with probability at least 3/4,

$$e^{-\epsilon} \prod_{i=1}^{k} \prod_{j=m_i}^{N_i-1} \rho_i(j) \leq \prod_{i=1}^{k} \prod_{j=m_i}^{N_i-1} \overline{Z_i(j)} \leq e^{\epsilon} \prod_{i=1}^{k} \prod_{j=m_i}^{N_i-1} \rho_i(j).$$

Thus, the quantity

$$\prod_{i=1}^{k} (2^{\ell_i} + 2^{r_i} - 1) \prod_{j=m_i}^{N_i-1} \overline{Z_i(j)}^{-1}$$

is a sufficiently accurate estimate of $|\mathcal{I}(G)|$.

For each of the $z$ pairs $(i, j)$, $O(\epsilon^{-2}z)$ samples were needed, each of which is produced in time $\text{poly}(|G|, z/\epsilon)$. Since $z \leq |V(G)|^2$, the total running time is $\text{poly}(|G|, \epsilon^{-1})$ and we have an FPRAS. □

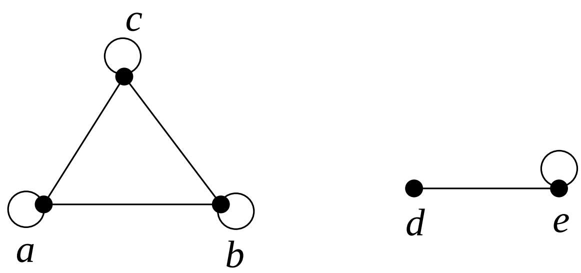


Figure 4: An $H$ with a nontrivial component for which SAMPLE$H$-COL has a PAUS.

# 7 The presence of trivial components

Theorem 2 shows that sampling $H$-colourings is difficult if every component of $H$ is nontrivial. Recall from [10] that exactly counting $H$-colourings is #P-complete if $H$ has even one nontrivial component. Thus, it might appear that Theorem 2 can be improved. In this section, we show that the existence of a single nontrivial component is not enough to make sampling difficult. In particular, we give an example of a graph $H$ with a nontrivial component, for which SAMPLE$H$-COL has a PAUS. Specifically, let $H$ be the graph depicted in Figure 4.

**Observation 9** *Suppose that $H$ is the graph depicted in Figure 4.* SAMPLE$H$-COL *has a PAUS.*

*Proof.* Here is a PAUS for SAMPLE$H$-COL. The input is an instance $(G, \epsilon)$ where $G$ has $n$ vertices and, without loss of generality[3], is connected. If $\epsilon < 2^n/(2^n + 3^n)$ then the algorithm simply runs for $5^n$ steps, constructs all of the $H$-colourings of $G$ (and counts them) and selects one uniformly at random. Note that the running time is at most poly$(1/\epsilon)$ in this case. Otherwise, the algorithm chooses $i$ uniformly at random from $1, \ldots, 3^n + 2^n$. If $i \leq 3^n$, then the algorithm outputs the $i$'th colouring from the $3^n$ colourings with colours "$a$", "$b$", and "$c$". Otherwise, let $C$ be the $(i - 3^n)$th of the $2^n$ (proper and

[3]We can assume that the input is a connected graph without losing generality because we can obtain an $H$-colouring of a $k$-component graph $G$ by independently calling our PAUS for each component, specifying accuracy parameter $\epsilon/k$. The final variation distance (between the output distribution and the uniform distribution on $H$-colourings of $G$) is at most $\epsilon$.

improper) colourings with colours "$d$" and "$e$". If $C$ is a legal $H$-colouring of $G$, then the algorithm outputs it. Otherwise, it outputs the error symbol $\perp$. The variation distance between the output distribution of the algorithm and the uniform distribution on $H$-colourings of $G$ is at most the probability that the algorithm outputs $\perp$, which is at most $2^n/(2^n+3^n) \leq \epsilon$. ☐

# 8 Sampling and Counting

Let #B$H$-COL be defined as follows.

*Name.* #B$H$-COL.

*Instance.* A loop-free connected bipartite graph $G$.

*Output.* The number of $H$-colourings of $G$.

For certain graphs $H$, the problem #B$H$-COL can be expressed as the counting problem associated with a "self-reducible $p$-relation". For such an $H$, Theorem 6.3 of Jerrum, Valiant and Vazirani's paper [17] guarantees that if there is an FPRAS for #B$H$-COL then there is a PAUS for SAMPLEB$H$-COL. If $H$ has no trivial components, this in turn guarantees (by Theorem 2) an FPRAS for #BIS. Dyer and Greenhill [8] have given a more general framework in which these ideas work: If, for a given graph $H$, the problem #B$H$-COL is "self-partitionable" then an FPRAS for #B$H$-COL can be turned into a PAUS for SAMPLEB$H$-COL. It is not clear for which graphs $H$ these ideas can be applied, and this is an interesting open question.

A related problem (which is also open) is to determine for which graphs $H$ an FPRAS for counting $H$-colourings can be turned into a PAUS for SAMPLE$H$-COL. Dyer, Goldberg and Jerrum [7] have shown that for every fixed $H$ a PAUS for SAMPLE$H$-COL can be turned into an FPRAS for counting $H$-colourings.

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# References


[1] C. Borgs, J.T. Chayes, M. Dyer and P. Tetali, On the sampling problem for $H$-colorings on the hypercubic lattice, to appear in the Proceedings of the DIMACS-DIMATIA Workshop on Graphs, Homomorphisms and Statistical Physics, J. Nesetril and P. Winkler, eds., American Mathematical Society.

[2] G.R. Brightwell and P. Winkler, Graph homomorphisms and phase transitions, *Journal of Combinatorial Theory, Series B,* **77** (1999) 221–262.

[3] C. Cooper, M. Dyer and A. Frieze, On Markov chains for randomly $H$-colouring a graph, *Journal of Algorithms,* **39(1)** (2001) 117–134.

[4] J. Diaz, M. Serna and D.M. Thilikos, $(H, C, K)$-coloring: fast, easy, and hard cases, Proccedings 26th International Symposium on Mathematical Foundations of Computer Science, MFCS 2001, August 27 - 31, 2001, Marianske Lazne, Czech Republic, Springer Verlag, Lecture Notes in Computer Science.

[5] J. Diaz, M. Serna and D.M. Thilikos, Problems and conjectures on parameterized H -coloring . Technical report KAM-DIMATIA Series 2001. Technical Report No. LSI-02-27-R. Departament de Llenguatges i Sistemes Informtics, Universitat Politcnica de Catalunya, Barcelona, Spain, 2002

[6] M. Dyer, A. Frieze and M. Jerrum, On counting independent sets in sparse graphs, SIAM J Comput. **33(5)** 1527–1541 (2002)

[7] M. Dyer, L.A. Goldberg and M. Jerrum, Counting and Sampling $H$-colourings, To appear in *Information and Computation.*

[8] M. Dyer and C. Greenhill, Random walks on combinatorial objects. In J.D. Lamb and D.A. Preece, editors, *Surveys in Combinatorics*, volume 267 of *London Mathematical Society Lecture Note Series*, pages 101–136. Cambridge University Press, 1999.

[9] M. Dyer and C. Greenhill, On Markov chains for independent sets, *Journal of Algorithms,* **35** (2000) 17–49.

[10] M.Dyer and C. Greenhill, The complexity of counting graph homomorphisms. *Random Structures and Algorithms,* **17** (2000) 260–289.

[11] M. Dyer, C. Greenhill, L.A. Goldberg, and M. Jerrum, On the relative complexity of approximate counting problems. To appear in *Algorithmica.*

[12] M. Dyer, M. Jerrum and E. Vigoda, Rapidly mixing Markov chains for dismantleable constraint graphs. In J. Nesetril and P. Winkler, editors, *Proceedings of a DIMACS/DIMATIA Workshop on Graphs, Morphisms and Statistical Physics*, March 2001, to appear.

[13] A. Galluccio, P. Hell and J. Nešetřil, The complexity of $H$-coloring of bounded degree graphs, *Discrete Mathematics,* **222** (2000) 101–109.

[14] P. Hell and J. Nešetřil, On the complexity of $H$-coloring, *Journal of Combinatorial Theory, Series B,* **48** (1990) 92–110.

[15] M. Jerrum, A very simple algorithm for estimating the number of $k$-colorings of a low-degree graph, *Random Structures and Algorithms,* **7** (1995) 157–165.

[16] M. Jerrum, Sampling and Counting, Chapter 3 of Counting, sampling and integrating: algorithms and complexity. (Birkhäuser 2003)

[17] M.R. Jerrum, L.G. Valiant, and V.V. Vazirani, Random generation of combinatorial structures from a uniform distribution, *Theoretical Computer Science,* **43** (1986) 169–188.

[18] E. Vigoda, Improved bounds for sampling colorings, *Journal of Mathematical Physics,* **41** (2000) 1555–1569.