A stabilized finite element method for nonlinear poroelasticity

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Abstract

We construct a stabilized finite-element method to compute flow and nonlinear deformations in an incompressible poroelastic medium. We employ a three-field mixed formulation to calculate displacement, fluid flux and pressure directly and introduce a Lagrange multiplier to enforce the no-flux boundary condition. We use a low order approximation, namely, continuous piecewise-linear approximation for the displacements and fluid flux, and piecewise-constant approximation for the pressure. This results in a simple matrix structure with low bandwidth and enables steep pressure gradients, such as those occurring due to rapid changes in material coefficients, boundary conditions or forcing, to be accurately approximated. Three dimensional numerical experiments demonstrate the accuracy of the method and its ability to reliably capture steep pressure gradients, and illustrate its application to modelling of the human lung.

1. Introduction

Poroelasticity is a mixture theory in which a complex fluid-structure interaction is approximated by the superposition of solid and fluid components. Developments of the continuum theory can be found, for example, in [1] and [2]. Poroelastic models have been developed to study numerous geophysical applications ranging from reservoir engineering [3] to earthquake fault zones [4]. Fully saturated, incompressible poroelastic models have been proposed for a variety of biological tissues and processes, including lung parenchyma [5], protein-based hydrogels embedded within cells [6], blood flow in the beating myocardium [7, 8], brain oedema and hydrocephalus [9, 10], and interstitial fluid and tissue in articular cartilage and intervertebral discs [11, 12, 13].

In [14], we developed a stabilized, low-order, mixed finite element method for the fully saturated, incompressible, small deformation case for which a linear elasticity model is sufficient. Low-order finite element methods are relatively easy to implement and allow for efficient preconditioning [15, 16]. We employed the fluid flux as a primary variable resulting in a three-field,

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displacement, fluid flux and pressure formulation. Keeping the fluid flux as a primary variable obviates the need to calculate the flux via post-processing and allows physically meaningful boundary conditions to be applied at the interface when modelling the interaction between a fluid and a poroelastic structure [17]. Further it allows for an easy extension of the fluid model from a Darcy flow to a Brinkman flow, for which there are numerous applications in modelling biological tissues [18]. Rigorous theoretical results for the stability and optimal convergence rate for linear poroelasticity were presented in [14]. The stabilization term requires only a small amount of additional computational work, can be assembled locally on each element using standard finite element information, and leads to a symmetric addition to the original system matrix, thus preserving any existing symmetry. The effect of the stabilization on the conservation of mass is minimal in 3D, and disappears as the mesh is refined, see [14]. Other computational approaches for poroelastic flows are described in [19], [20], [4], [21] and [22].

In section 2, we present a quasi-static incompressible poroelastic model. In section 3 we develop the stabilized nonlinear finite-element method and provide some implementation details in section 4. In section 5, we present a range of 3D numerical experiments to verify the accuracy of the method and illustrate its ability to reliably capture steep pressure gradients. Finally we apply the method to a large scale simulation of the right human lung.

2. Poroelasticity theory

Two complementary approaches have been developed for modelling a deformable porous medium. Mixture theory, also known as the Theory of Porous Media (TPM) [23, 24, 2], has its roots in the classical theories of gas mixtures and makes use of a volume fraction concept in which the porous medium is represented by spatially superposed interacting media. An alternative, purely macroscopic approach is mainly associated with the work of Biot, and a detailed description can be found in the book by Coussy [1]. Relationships between the two theories are explored in [25, 26]. As is most common in biological applications, we use the mixture theory for poroelasticity as outlined in [2] and recently summarized in [22].

2.1. Kinematics

Figure 1: Illustration of the solid deformation.

Let the volume $\Omega(0)$ be the undeformed Lagrangian (material) reference configuration and let $\boldsymbol{X}$ indicate the position of a particle in $\Omega(0)$ at $t = 0$. The position of a particle in the deformed

configuration $\Omega(t)$ at time $t > 0$ is given by $\boldsymbol{x}$, with $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ as shown in Figure 1. The deformation map, $\boldsymbol{\chi}(\boldsymbol{X}, t)$, is a continuously differentiable, invertible mapping from $\Omega(0)$ to $\Omega(t)$. Thus the inverse of the deformation map, $\boldsymbol{\chi}^{-1}(\boldsymbol{x}, t)$, is such that $\boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t)$. The displacement field is given by

$$\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{\chi}(\boldsymbol{X}, t) - \boldsymbol{X}. \tag{1}$$

The deformation gradient tensor is

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}, \tag{2}$$

and the symmetric right Cauchy-Green deformation tensor is

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}. \tag{3}$$

The Jacobian is defined as

$$J = \det(\boldsymbol{F}), \tag{4}$$

and represents the change in an infinitesimal control volume from the reference to the current configuration, i.e.,

$$d\Omega(t) = J d\Omega(0). \tag{5}$$

Note that $J > 0$.

2.2. Volume fractions

We consider saturated porous media only in which the fluid accounts for volume fractions $\phi_0(\boldsymbol{X}, t = 0)$ and $\phi(\boldsymbol{x}, t)$ of the total volume in the reference and current configurations respectively, where ϕ is known as the porosity. The fractions for the solid (or skeleton) are therefore $1 - \phi_0$ and $1 - \phi$ in the reference and current configuration respectively. For the mixture, ρ is the density in the current configuration given by

$$\rho = \rho^s(1 - \phi) + \rho^f \phi \quad \text{in } \Omega(t), \tag{6}$$

where ρ^s and ρ^f are the densities of the fluid and solid, respectively. We assume that both the solid and the fluid are incompressible so that $\rho^s = \rho_0^s$ and $\rho^f = \rho_0^f$. Although both the solid and fluid are assumed to be incompressible, the control volume can expand or contract due to fluid entering or leaving the region, and

$$J = \frac{1 - \phi_0}{1 - \phi}. \tag{7}$$

2.3. The model

We define the boundary $\partial\Omega(t) = \Gamma_d(t) \cup \Gamma_n(t)$ for the mixture and $\partial\Omega(t) = \Gamma_p(t) \cup \Gamma_f(t)$ for the fluid, with an outward pointing unit normal $\boldsymbol{n}$. We seek displacement $\boldsymbol{\chi}(\boldsymbol{X}, t)$, fluid flux $\boldsymbol{z}(\boldsymbol{x}, t)$ and pressure $p(\boldsymbol{x}, t)$ such that

$$\left.\begin{aligned}
-\nabla \cdot (\boldsymbol{\sigma}_e - p\boldsymbol{I}) &= \rho \boldsymbol{f} && \text{in } \Omega(t),\\
\boldsymbol{k}^{-1}\boldsymbol{z} + \nabla p &= \rho^f \boldsymbol{f} && \text{in } \Omega(t),\\
\nabla \cdot (\boldsymbol{\chi}_t + \boldsymbol{z}) &= g && \text{in } \Omega(t),\\
\boldsymbol{\chi}(\boldsymbol{X}, t)|_{X=\chi^{-1}(\boldsymbol{x},t)} &= \boldsymbol{X} + \boldsymbol{u}_D && \text{on } \Gamma_d(t),\\
(\boldsymbol{\sigma}_e - p\boldsymbol{I})\boldsymbol{n} &= \boldsymbol{t}_N && \text{on } \Gamma_n(t),\\
\boldsymbol{z} \cdot \boldsymbol{n} &= q_D && \text{on } \Gamma_f(t),\\
p &= p_D && \text{on } \Gamma_p(t),\\
\boldsymbol{\chi}(\boldsymbol{X}, 0) &= \boldsymbol{X} && \text{in } \Omega(0).
\end{aligned}\right\} \tag{8}$$

The fluid flux $\boldsymbol{z} = \phi(\boldsymbol{v}^f - \boldsymbol{v}^s)$ where $\boldsymbol{v}^f$ and $\boldsymbol{v}^s$ are the velocities of the fluid and solid components respectively, $\boldsymbol{\chi}_t$ denotes $\frac{\partial \boldsymbol{\chi}(\boldsymbol{X},t)}{\partial t}$ and $\boldsymbol{\sigma}_e$ is the stress tensor given by

$$\boldsymbol{\sigma}_e = \frac{1}{J}\boldsymbol{F} \cdot 2\frac{\partial W(\boldsymbol{\chi})}{\partial \boldsymbol{\chi}} \cdot \boldsymbol{F}^T, \tag{9}$$

where $W(\boldsymbol{\chi})$ denotes a strain-energy law (hyperelastic Helmholtz energy functional) dependent on the deformation of the solid. The permeability tensor is given by

$$\boldsymbol{k} = J^{-1}\boldsymbol{F}\boldsymbol{k}_0(\boldsymbol{\chi})\boldsymbol{F}^T, \tag{10}$$

where $\boldsymbol{k}_0(\boldsymbol{\chi})$ is the permeability in the reference configuration, which may be chosen to be some (nonlinear) function dependent on the deformation. Examples of deformation dependent permeability tensors for biological tissues can be found in [27, 12, 28]. Details of the derivation of (8) appear in Appendix A.

It is important to recognize that $\nabla(\cdot) = \partial/\partial\boldsymbol{x}(\cdot)$ denotes the partial derivative with respect to the *deformed* configuration. We will use ∇ to denote the spatial gradient in $\Omega(t)$ rather than the more explicit $\nabla_{\boldsymbol{x}=\boldsymbol{\chi}(\boldsymbol{X},t)}$. The latter more clearly indicates the dependency of the gradient operator on the deformation $\boldsymbol{\chi}(\boldsymbol{X}, t)$ and highlights the inherent nonlinearity that arises due to the fact that the deformation $\boldsymbol{\chi}(\boldsymbol{X}, t)$ is one of the unknowns. Similarly the deformed domain $\Omega(t)$ in which equations (8) pertain, is a function of the deformation map $\boldsymbol{\chi}$, and therefore incorporates another important nonlinearity.

3. The stabilized finite element method

We extend the method of [14] from the linear, small deformation poroelastic case to large deformation poroelasticity. For ease of presentation, we will assume all Dirichlet boundary conditions are homogeneous, ie., $\boldsymbol{u}_D = \boldsymbol{0}, q_D = 0, p_D = 0$.

3.1. Weak formulation

We define the following spaces for the deformed location, fluid flux and pressure respectively,

$$\begin{aligned}
\mathbf{W}^E(\Omega(t)) &= \{\boldsymbol{v} \in (H^1(\Omega(t)))^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_d(t)\},\\
\mathbf{W}^D(\Omega(t)) &= \{\boldsymbol{w} \in H_{div}(\Omega(t)) : \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_f(t)\},\\
\mathcal{L}(\Omega(t)) &= \left\{ \begin{array}{ll} L^2(\Omega(t)) & \text{if } \Gamma_n(t)\cup\Gamma_p(t) \neq \emptyset \\ L^2_0(\Omega(t)) & \text{if } \Gamma_n(t)\cup\Gamma_p(t) = \emptyset, \end{array} \right\},
\end{aligned}$$

where $L^2_0(\Omega(t)) = \left\{q \in L^2(\Omega(t)) : \int_{\Omega(t)} q \, \mathrm{d}\Omega(t) = 0\right\}$.

The continuous weak problem is: Find $\boldsymbol{\chi}(\boldsymbol{X}, t) \in \mathbf{W}^E(\Omega(0))$, $\boldsymbol{z}(\boldsymbol{x}, t) \in \mathbf{W}^D(\Omega(t))$ and $p(\boldsymbol{x}, t) \in \mathcal{L}(\Omega(t))$ for any time $t \in [0, T]$ such that

$$\begin{aligned}
\int_{\Omega(t)} \left[\boldsymbol{\sigma}_e : \nabla^S\boldsymbol{v} - p\nabla\cdot\boldsymbol{v}\right] \mathrm{d}\Omega(t) &= \int_{\Omega(t)} \rho\boldsymbol{f}\cdot\boldsymbol{v} \, \mathrm{d}\Omega(t) + \int_{\Gamma_n(t)} \boldsymbol{t}_N\cdot\boldsymbol{v} \, \mathrm{d}\Gamma_n(t) \quad \forall\boldsymbol{v} \in \mathbf{W}^E(\Omega(t)),\\
\int_{\Omega(t)} \left[\boldsymbol{k}^{-1}\boldsymbol{z}\cdot\boldsymbol{w} - p\nabla\cdot\boldsymbol{w}\right] \mathrm{d}\Omega(t) &= \int_{\Omega(t)} \rho^f\boldsymbol{f}\cdot\boldsymbol{w} \, \mathrm{d}\Omega(t) \quad \forall\boldsymbol{w} \in \mathbf{W}^D(\Omega(t)),\\
\int_{\Omega(t)} \left[q\nabla\cdot\boldsymbol{\chi}_t + q\nabla\cdot\boldsymbol{z}\right] \mathrm{d}\Omega(t) &= \int_{\Omega(t)} gq \, \mathrm{d}\Omega(t) \quad \forall q \in \mathcal{L}(\Omega(t)).
\end{aligned} \tag{11}$$

Here $\nabla^S\boldsymbol{v} = \frac{1}{2}\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right)$ for some vector $\boldsymbol{v}$.

3.2. The fully discrete model

Let $\mathcal{T}^h$ be a quasi-uniform partition of $\Omega(t)$ into non-overlapping elements K, where h denotes the size of the largest element in $\mathcal{T}^h$. We then define the following finite element spaces,

$$\begin{aligned}
\mathbf{W}_h^E(\Omega(t)) &= \left\{\boldsymbol{v}_h \in C^0(\Omega(t)) : \boldsymbol{v}_h|_K \in P_1(K)\ \forall K \in \mathcal{T}^h, \boldsymbol{v}_h = 0 \text{ on } \Gamma_d(t)\right\},\\
\mathbf{W}_h^D(\Omega(t)) &= \left\{\boldsymbol{w}_h \in C^0(\Omega(t)) : \boldsymbol{w}_h|_K \in P_1(K)\ \forall K \in \mathcal{T}^h, \boldsymbol{w}_h \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_f(t)\right\},\\
Q_h(\Omega(t)) &= \begin{cases} \left\{q_h : q_h|_K \in P_0(K)\ \forall K \in \mathcal{T}^h\right\} & \text{if } \Gamma_n(t) \cup \Gamma_p(t) \neq \emptyset \\ \left\{q_h : q_h|_K \in P_0(K), \int_{\Omega(t)} q_h = 0\ \forall K \in \mathcal{T}^h\right\} & \text{if } \Gamma_n(t) \cup \Gamma_p(t) = \emptyset \end{cases},
\end{aligned}$$

where $P_0(K)$ and $P_1(K)$ are the spaces of constant and linear polynomials on K respectively. We define the combined solution space $\mathcal{U}_h(t) = \mathbf{W}_h^E(\Omega(0)) \times \mathbf{W}_h^D(\Omega(t)) \times Q_h(\Omega(t))$.

The discretization in time is given by partitioning $[0, T]$ into N evenly spaced non-overlapping regions $(t_{n-1}, t_n], n = 1, 2, \ldots, N$, where $t_n - t_{n-1} = \Delta t$. For any sufficiently smooth function $v(t, x)$ we define $v^n(x) = v(t_n, x)$ and the discrete time derivative by $v^n_{\Delta t} := \frac{v^n - v^{n-1}}{\Delta t}$.

The fully discrete weak problem is: For $n = 1, \ldots, N$, find $\boldsymbol{\chi}_h^n \in \mathbf{W}_h^E(\Omega(0))$, $\boldsymbol{z}_h^n \in \mathbf{W}_h^D(\Omega(t_n))$ and $p_h^n \in Q_h(\Omega(t_n))$ such that

$$\begin{aligned}
&\int_{\Omega(t_n)} \left[\boldsymbol{\sigma}^n_{e,h} : \nabla^S \boldsymbol{v}_h - p_h^n \nabla \cdot \boldsymbol{v}_h\right] \mathrm{d}\Omega(t_n) = \int_{\Omega(t_n)} \rho \boldsymbol{f}^n \cdot \boldsymbol{v}_h \,\mathrm{d}\Omega(t_n) + \int_{\Gamma_n(t_n)} \boldsymbol{t}_N^n \cdot \boldsymbol{v}_h \,\mathrm{d}\Gamma_n(t_n)\\
&\qquad\qquad \forall \boldsymbol{v}_h \in \mathbf{W}_h^E(\Omega(t_n)),\\
&\int_{\Omega(t_n)} \left[\boldsymbol{k}^{-1} \boldsymbol{z}_h^n \cdot \boldsymbol{w}_h - p_h^n \nabla \cdot \boldsymbol{w}_h\right] \mathrm{d}\Omega(t_n) = \int_{\Omega(t_n)} \rho^f \boldsymbol{f}^n \cdot \boldsymbol{w}_h \,\mathrm{d}\Omega(t_n) \quad \forall \boldsymbol{w}_h \in \mathbf{W}_h^D(\Omega(t_n)),\\
&\int_{\Omega(t_n)} \left[q_h \nabla \cdot \boldsymbol{\chi}^n_{h,\Delta t} + q_h \nabla \cdot \boldsymbol{z}_h^n\right] \mathrm{d}\Omega(t_n) + J(p^n_{h,\Delta t}, q_h) = \int_{\Omega(t_n)} g^n q_h \,\mathrm{d}\Omega(t_n) \quad \forall q_h \in Q_h(\Omega(t_n)).
\end{aligned} \tag{12}$$

The stabilization term is given by

$$J(p, q) = \Upsilon \sum_{K \in \mathcal{T}^h} \int_{\partial K \setminus \partial\Omega(t_n)} h_{\partial K} [\![p]\!][\![q]\!] \, ds,$$

where Υ is a stabilization parameter that is independent of h and Δt. Here $h_{\partial K}$ denotes the size (diameter) of an element edge in 2D or face in 3D, and $[\![\cdot]\!]$ is the jump across an edge or face (taken on the interior edges only). The stabilization term has been introduced here to add stability and ensure a well-posed fully-discrete model. It has been shown in [14] that the convergence is insensitive to Υ.

3.3. Solution via Newton iteration at $t_n, n = 1, \ldots, N$.

Let $\mathfrak{u}_h^n = \{\boldsymbol{\chi}_h^n, \boldsymbol{z}_h^n, p_h^n\} \in \mathcal{U}_h(t_n)$ denote the solution vector at a particular time step, $\delta\mathfrak{u}_h = \{\delta\boldsymbol{v}, \delta\boldsymbol{z}, \delta p\}$ denote the solution increment vector, and $\mathfrak{v}_h = \{\boldsymbol{v}_h, \boldsymbol{w}_h, q_h\} \in \mathcal{V}_h(t)$ where $\mathcal{V}_h(t) = \mathbf{W}_h^E(\Omega(t)) \times \mathbf{W}_h^D(\Omega(t)) \times Q_h(\Omega(t))$. The nonlinear system of equations (12) can be recast in the form: Find $\mathfrak{u}_h^n \in \mathcal{U}_h(t_n)$ such that

$$G^n(\mathfrak{u}_h^n, \mathfrak{v}_h) = 0 \ \forall \mathfrak{v}_h \in \mathcal{V}_h(t_n), \tag{13}$$

where

$$G^n(\mathfrak{u}_h^n, \mathfrak{v}_h) = \int_{\Omega(t_n)} \Big[\boldsymbol{\sigma}_{e,h}^n : \nabla^S \boldsymbol{v}_h - p_h^n \nabla \cdot \boldsymbol{v}_h + \boldsymbol{k}^{-1} \boldsymbol{z}_h^n \cdot \boldsymbol{w}_h - p_h^n \nabla \cdot \boldsymbol{w}_h + q_h \nabla \cdot (\boldsymbol{v}_{\Delta t,h}^n + \boldsymbol{z}_h^n) \\ -\rho \boldsymbol{f}^n \cdot \boldsymbol{v}_h + \rho^f \boldsymbol{f}^n \cdot \boldsymbol{w}_h + g q_h \Big] \, \mathrm{d}\Omega(t_n) \\ - \int_{\Gamma_n(t_n)} \boldsymbol{t}_N^n \cdot \boldsymbol{v}_h \, \mathrm{d}\Gamma_n(t_n). \tag{14}$$

Given an approximate solution $\overline{\mathfrak{u}}_h^n$, we approximate (13) by

$$G^n(\overline{\mathfrak{u}}_h^n, \mathfrak{v}_h) + DG^n(\overline{\mathfrak{u}}_h^n, \mathfrak{v}_h)[\delta \mathfrak{u}_h] = 0 \ \forall \mathfrak{v}_h \in \mathcal{V}_h(t_n),$$

and solve

$$DG^n(\overline{\mathfrak{u}}_h^n, \mathfrak{v}_h)[\delta \mathfrak{u}_h] = -G(\overline{\mathfrak{u}}_h^n, \mathfrak{v}_h) \ \forall \mathfrak{v}_h \in \mathcal{V}_h(t_n), \tag{15}$$

for the Newton step $\delta \mathfrak{u}_h$, where DG is the directional derivative of G, at $\overline{\mathfrak{u}}_h^n$, in the direction $\delta \mathfrak{u}_h$.

3.3.1. Approximation of DG^n.

In biphasic tissue problems, it is common to approximate directional derivative of G by assuming the nonlinear elasticity term is the dominant nonlinearity and ignoring the other nonlinearities [29, 4]. Let

$$E^n((\boldsymbol{\chi}_h^n, p_h^n), \boldsymbol{v}_h) = \int_{\Omega(t_n)} \Big[\boldsymbol{\sigma}_{e,h}^n : \nabla^S \boldsymbol{v}_h - p_h^n \nabla \cdot \boldsymbol{v}_h\Big] \, \mathrm{d}\Omega(t_n). \tag{16}$$

For Newton's method we require the directional derivative of $E^n((\boldsymbol{\chi}_h^n, p_h^n), \boldsymbol{v}_h)$ at a particular trial solution $(\overline{\boldsymbol{\chi}_h^n}, \overline{p_h^n})$ in the direction $\delta \boldsymbol{\chi}_h$, given by (see [30, section 3.5.3])

$$DE^n((\overline{\boldsymbol{\chi}_h^n}, \overline{p_h^n}), \boldsymbol{v}_h)[\delta \boldsymbol{\chi}_h] = \int_{\overline{\Omega}(t_n)} \Big[\nabla^S \boldsymbol{v}_h : \overline{\boldsymbol{\Theta}_h^n} : \nabla^S \delta \boldsymbol{\chi}_h + \overline{\boldsymbol{\sigma}_{e,h}^n} : \Big((\nabla \delta \boldsymbol{v})^T \cdot \nabla \boldsymbol{v}_h\Big)\Big] \, \mathrm{d}\Omega(t_n), \tag{17}$$

where $\overline{\boldsymbol{\Theta}_h^n}$ is a fourth-order tensor and $\overline{\boldsymbol{\sigma}_{e,h}^n}$ is the effective (elastic) stress tensor, both evaluated at a trial solution $\overline{\boldsymbol{\chi}_h^n}$. Further, any variable with a bar above it will correspond to it being evaluated at a trial solution. The fourth-order spatial tangent modulus tensor $\boldsymbol{\Theta}$ is described in Appendix B. For a detailed explanation and derivation see [30, 31]. The approximate linearization of the nonlinear problem is thus given by

$$DG^n(\overline{\mathfrak{u}}_h^n, \mathfrak{v}_h)[\delta \mathfrak{u}_h] \approx \\ \int_{\overline{\Omega}(t_n)} \Big[\nabla^S \boldsymbol{v}_h : \overline{\boldsymbol{\Theta}_h^n} : \nabla^S \delta \boldsymbol{\chi}_h + \overline{\boldsymbol{\sigma}_{e,h}} : \Big((\nabla \delta \boldsymbol{\chi}_h)^T \cdot \nabla \boldsymbol{v}_h\Big) - \delta p_h \nabla \cdot \boldsymbol{v}_h \\ + \bar{\boldsymbol{k}}^{-1} \delta \boldsymbol{z}_h \cdot \boldsymbol{w}_h - \delta p_h \nabla \cdot \boldsymbol{w}_h + q_h \nabla \cdot \left(\frac{\delta \boldsymbol{\chi}_h}{\Delta t} + \delta \boldsymbol{z}_h\right)\Big] \, \mathrm{d}\Omega(t_n), \tag{18}$$

Using (14), (18) and equation (15) the Newton solve becomes: Find $\delta\boldsymbol{\chi}_h \in \mathbf{W}_h^E(\Omega(0))$, $\delta z_h \in \mathbf{W}_h^D(\Omega(t_n))$ and $\delta p_h \in Q_h(\Omega(t_n))$ such that

$$
\begin{aligned}
&\int_{\overline{\Omega}(t_n)} \left[\nabla^S \boldsymbol{v}_h : \overline{\boldsymbol{\Theta}_h^n} : \nabla^S \delta\boldsymbol{\chi}_h + \overline{\boldsymbol{\sigma}_{e,h}^n} : \left((\nabla\delta\boldsymbol{\chi}_h)^T \cdot \nabla \boldsymbol{v}_h\right) - \delta p_h \nabla\cdot\boldsymbol{v}_h\right] \mathrm{d}\overline{\Omega}(t_n) \\
&\quad = \int_{\overline{\Omega}(t_n)} \left[\overline{\boldsymbol{\sigma}_{e,h}^n} : \nabla^S \boldsymbol{v}_h - \overline{p_h^n}\nabla\cdot\boldsymbol{v}_h - \overline{\rho}\boldsymbol{f}^n\cdot\boldsymbol{v}_h\right] \mathrm{d}\overline{\Omega}(t_n) - \int_{\overline{\Gamma}_n(t_n)} \boldsymbol{t}_N^n \cdot \boldsymbol{v}_h \, \mathrm{d}\overline{\Gamma}_n(t_n) \quad \forall \boldsymbol{v}_h \in \mathbf{W}_h^E(\Omega(t_n)), \\
&\int_{\overline{\Omega}(t_n)} \left[\bar{\boldsymbol{k}}^{-1}\delta z_h \cdot \boldsymbol{w}_h - \delta p_h \nabla\cdot\boldsymbol{w}_h\right] \mathrm{d}\overline{\Omega}(t_n) \\
&\quad = \int_{\overline{\Omega}(t_n)} \left[\bar{\boldsymbol{k}}^{-1}\overline{\boldsymbol{z}_h^n}\cdot\boldsymbol{w}_h - \overline{p_h^n}\cdot\nabla\boldsymbol{w}_h - \overline{\rho^f}\boldsymbol{f}^n\cdot\boldsymbol{w}_h\right] \mathrm{d}\overline{\Omega}(t_n) \qquad \forall \boldsymbol{w}_h \in \mathbf{W}_h^D(\Omega(t_n)), \\
&\int_{\overline{\Omega}(t_n)} \left[q_h \nabla\cdot\left(\frac{\delta\boldsymbol{\chi}_h}{\Delta t} + \delta z_h\right)\right] \mathrm{d}\overline{\Omega}(t_n) + J\left(\frac{\delta p_h}{\Delta t}, q_h\right) \\
&\quad = \int_{\overline{\Omega}(t_n)} \left[q_h \nabla\cdot(\overline{\boldsymbol{\chi}_{\Delta t,h}} + \overline{z_h}) - g q_h\right] \mathrm{d}\overline{\Omega}(t_n) \; + J\left(\overline{p_{h,\Delta t}}, q_h\right) \quad \forall q_h \in Q_h(\Omega(t_n)).
\end{aligned}
\tag{19}
$$

4. Implementation details

4.1. Matrix assembly for the Newton iteration

Let $\boldsymbol{\phi}_k$ denote a vector-valued linear basis function for the $(P1)^d$ space, and

$$
\boldsymbol{\chi}_i^n = \sum_{k=1}^{n_u} \boldsymbol{\chi}_{i,k}^n \boldsymbol{\phi}_k \in \mathbf{W}_h^E(\Omega(0)), \qquad \boldsymbol{z}_i^n = \sum_{k=1}^{n_z} \boldsymbol{z}_{i,k}^n \boldsymbol{\phi}_k \in \mathbf{W}_h^D(\Omega(t_n)).
$$

Similarly let ψ_i denote a basis function for the space P0, hence

$$
\boldsymbol{p}_i^n = \sum_{k=1}^{n_p} p_{i,k}^n \psi_k \in Q_h(\Omega(t_n)).
$$

Now let $\mathrm{u}_i^n := (\boldsymbol{\chi}_i^n, \boldsymbol{z}_i^n, p_i^n) \in \mathbb{R}^{n_u+n_z+n_p}$ denote the fully discrete solution at the ith step within the Newton method at time t_n. The Newton algorithm at a particular time step n, is given in Algorithm 1.

Algorithm 1 Newton algorithm at t_n

$i = 0$
$\mathrm{u}_0^n = \{\boldsymbol{\chi}^{n-1}, \boldsymbol{z}^{n-1}, p^{n-1}\}$
while $\|\boldsymbol{R}(\mathrm{u}_i^n, \mathrm{u}^{n-1})\| >$ TOL & $i <$ ITEMAX **do**
  Assemble $\boldsymbol{R}(\mathrm{u}_i^n, \mathrm{u}^{n-1})$ and $\boldsymbol{K}(\mathrm{u}_i^n)$ on $\Omega(t_n)_i$
  Solve $\boldsymbol{K}(\mathrm{u}_i^n)\delta\mathrm{u}_{i+1}^n = -\boldsymbol{R}(\mathrm{u}_i^n, \mathrm{u}^{n-1})$
  Compute $\mathrm{u}_{i+1}^n = \mathrm{u}_i^n + \delta\mathrm{u}_{i+1}^n$
  Update the mesh, $\Omega(t_n)_{i+1} = \boldsymbol{\chi}_i^n$
  $i = i + 1$
end while

At each Newton iteration we are required to solve the linear system

$$\boldsymbol{K}(\mathrm{u}_i^n)\delta\mathrm{u}_{i+1}^n = -\boldsymbol{R}(\mathrm{u}_i^n, \mathrm{u}^{n-1}). \tag{20}$$

This system can be expanded as

$$\begin{bmatrix} \boldsymbol{K}^e & 0 & \boldsymbol{B}^T \\ 0 & \boldsymbol{M} & \boldsymbol{B}^T \\ -\boldsymbol{B} & -\Delta t\boldsymbol{B} & \boldsymbol{J} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\chi}_{i+1}^n \\ \delta\boldsymbol{z}_{i+1}^n \\ \delta p_{i+1}^n \end{bmatrix} = - \begin{bmatrix} \boldsymbol{r}_1(\boldsymbol{\chi}_i^n, p_i^n) \\ \boldsymbol{r}_2(\boldsymbol{\chi}_i^n, \boldsymbol{z}_i^n, p_i^n) \\ \boldsymbol{r}_3(\boldsymbol{\chi}_i^n, \boldsymbol{\chi}^{n-1}, \boldsymbol{z}_i^n, p_i^n) \end{bmatrix}, \tag{21}$$

where the elements in the matrices in (21) are given by

$$\begin{aligned}
\boldsymbol{k}_{kl}^e &= \int_{\Omega(t_n)_i} \left[\boldsymbol{E}_k^T \boldsymbol{D}(\boldsymbol{\chi}_i^n)\boldsymbol{E}_l + (\nabla\boldsymbol{\phi}_k)^T \boldsymbol{\sigma}_e(\boldsymbol{\chi}_i^n)\nabla\boldsymbol{\phi}_l \right] \mathrm{d}\Omega(t_n)_i, \\
\boldsymbol{m}_{kl} &= \int_{\Omega(t_n)_i} \boldsymbol{k}^{-1}(\boldsymbol{\chi}_i^n)\boldsymbol{\phi}_k \cdot \boldsymbol{\phi}_l \, \mathrm{d}\Omega(t_n)_i, \\
\boldsymbol{b}_{kl} &= -\int_{\Omega(t_n)_i} \psi_k \nabla \cdot \boldsymbol{\phi}_l \, \mathrm{d}\Omega(t_n)_i, \\
\boldsymbol{j}_{kl} &= \Upsilon \sum_{K\in\mathcal{T}_i^h} \int_{\partial K\setminus\partial\Omega(t_n)_i} h_{\partial K} [\![\psi_k]\!][\![\psi_l]\!] \, \mathrm{d}s. \\
\boldsymbol{r}_{1i} &= \int_{\Omega(t_n)_i} \left[(\boldsymbol{\sigma}_e(\boldsymbol{\chi}_i^n) - p_i^n \boldsymbol{I}) : \nabla\boldsymbol{\phi}_i - \rho(\boldsymbol{\chi}_i^n)\boldsymbol{\phi}_i \cdot \boldsymbol{f} \right] \mathrm{d}\Omega(t_n)_i - \int_{\Gamma_n(t_n)_i} \boldsymbol{\phi}_i \cdot \boldsymbol{t}_N(\boldsymbol{\chi}_i^n) \, \mathrm{d}\Gamma_n(t_n)_i, \\
\boldsymbol{r}_{2i} &= \int_{\Omega(t_n)_i} \left[\boldsymbol{k}^{-1}(\boldsymbol{\chi}_i^n)\boldsymbol{\phi}_i \cdot \boldsymbol{z}_i^n - p_i^n \nabla \cdot \boldsymbol{\phi}_i - \rho^f \boldsymbol{\phi}_i \cdot \boldsymbol{f} \right] \mathrm{d}\Omega(t_n)_i, \\
\boldsymbol{r}_{3i} &= \int_{\Omega(t_n)_i} \psi_i \left[\nabla \cdot \left(\boldsymbol{\chi}_i^n - \boldsymbol{\chi}^{n-1} \right) + \Delta t \psi_i \nabla \cdot \boldsymbol{z}_i^n - \Delta t \psi_i g \right] \mathrm{d}\Omega(t_n)_i \\
&\qquad + \Upsilon \sum_{K\in\mathcal{T}_i^h} \int_{\partial K\setminus\partial\Omega(t_n)_i} h_{\partial K} [\![\psi_i]\!][\![p_i^n - p^{n-1}]\!] \, \mathrm{d}s.
\end{aligned}$$

Details of the matrices $\boldsymbol{D}$ and $\boldsymbol{E}$ appear in Appendix B

4.2. Stabilization matrix assembly

Let $K \in \mathcal{T}_h$ be an element and $\mathcal{D}(K)$ be the pressure degree of freedom associated with element K. We define $\mathcal{A}(K)$ to be the set of elements $L \in \mathcal{T}_h$ neighboring K.

Algorithm 2 Stabilization matrix $\boldsymbol{J}$ assembly

for every $K \in \mathcal{T}_h$ **do**
  for every $L \in \mathcal{A}(K)$ **do**
    Calculate $h_{\partial K}$
    $i \leftarrow \mathcal{D}(K)$
    $j \leftarrow \mathcal{D}(L)$
    $\boldsymbol{J}_{ii} \leftarrow \boldsymbol{J}_{ii} + (\delta h_{\partial K}$ in 2D, $\delta h_{\partial K}^{3/2}$ in 3D)
    $\boldsymbol{J}_{ij} \leftarrow \boldsymbol{J}_{ij} - (\delta h_{\partial K}$ in 2D, $\delta h_{\partial K}^{3/2}$ in 3D)
  end for
end for

4.3. Fluid-flux boundary condition

When solving the equations for Darcy flow using the Raviart-Thomas element (RT-P0), the fluid-flux boundary condition is enforced naturally by this divergence free element. Unfortunately this is not possible using our proposed P1-P1-P0-stabilized element. However, solving the poroelastic equations (8) using a piecewise linear approximation for the deformation and Raviart-Thomas element for the fluid (P1-RT-P0) does not satisfy the discrete inf-sup condition and can yield spurious pressure oscillations, see [32, 33] for details.

To enforce the no-flux boundary condition $\mathbf{z} \cdot \boldsymbol{n} = q_D$ along the boundary $\Gamma_f(t)$ we introduce a Lagrange multiplier Λ_h, where $\Lambda_h \in W_h^f(t)$, the discrete space of piecewise constant functions defined on all element surfaces with non-zero intersection with $\Gamma_f(t)$. The resulting modified continuous weak-form is

$$\begin{aligned} G((\boldsymbol{\chi}_h, \boldsymbol{z}_h, p_h), (\boldsymbol{v}_h, \boldsymbol{w}_h, q_h)) + (\Lambda_h, \boldsymbol{w}_h \cdot \boldsymbol{n})_{\Gamma_f} = q_D, \forall (\boldsymbol{v}_h, \boldsymbol{w}_h, q_h) \in \mathcal{V}_h(t), \\ (\boldsymbol{z} \cdot \boldsymbol{n}, \boldsymbol{l})_{\Gamma_f} = q_D, \ \forall \boldsymbol{l} \in W_h^f(t). \end{aligned} \tag{22}$$

The discretization and implementation of this additional constraint is straightforward and results in a discrete system with additional degrees of freedom for every node on Γ_f. The terms $(\Lambda_h, \ \boldsymbol{w} \ \cdot \ \boldsymbol{n})_{\Gamma_f}$ and $(\boldsymbol{z} \ \cdot \ \boldsymbol{n}, \ \boldsymbol{l})_{\Gamma_f}$ are nonlinear since the normal is a function of the (nonlinear) displacement. Note, within all the simulations we have undertaken, we found that treating these terms as linear terms did not prevent the convergence of the Newton algorithm. Alternatively these terms could be linearized as has been described in detail for the traction boundary condition, see [30, section 4.2.5] and [20].

5. Numerical results

We present four numerical examples to test the performance of the proposed stabilized finite element method. The first two examples are from mechanobiology and geotechnical applications, the third is a swelling example that undergoes significant large deformations and the fourth is an application from respiratory physiology. For the implementation we used the C++ library libmesh [34], and the multi-frontal direct solver mumps [35] to solve the resulting linear systems. For the strain energy law we chose a Neo-Hookean law taken from [30, eqn. (3.119)], with the penalty term chosen such that $0 \leq \phi < 1$, namely

$$W(\boldsymbol{\chi}) = \frac{\mu}{2}(\mathrm{tr}(\boldsymbol{C}) - 3) + \frac{\lambda}{4}(J^2 - 1) - (\mu + \frac{\lambda}{2})\ln(J - 1 + \phi_0). \tag{23}$$

For further discussion of strain energy laws for porelasticity we refer to [26] and [22]. The material parameters μ and λ in (23) can be related to the Young's modulus E and the Poisson ratio ν by $\mu = E/(2(1+\nu))$ and $\lambda = (E\nu)/((1+\nu)(1-2\nu))$. Details of the effective stress tensor and fourth-order spatial tangent modulus for this particular law can be found in Appendix B. For the permeability law we chose

$$\boldsymbol{k}_0(\boldsymbol{\chi}) = k_0 \boldsymbol{I}. \tag{24}$$

5.1. 3D unconfined compression stress relaxation

In this test, a cylindrical specimen of porous tissue is subjected to a prescribed displacement in the axial direction while left free to expand radially. The original experiment involved a specimen of articular cartilage being compressed via impervious smooth plates as shown in Figure

2a. After loading the tissue, the displacement is held constant and the tissue is allowed to relax in the radial direction. The fluid pressure is constrained to be zero at the outer radial surface. The outer radial boundary is permeable and free-draining, the upper and lower fluid boundaries are impermeable and frictionless. The outer radius and height of the cylinder is $5mm$, whereas the axial compression is $0.01mm$. The parameters used for the simulation can be found in Table 1.

(a) (b)

Figure 2: (a) The test problem. (b) Pressure field at $t = 200s$ using a mesh with 3080 tetrahedra.

Parameter	Description	Value
k	Dynamic permeability	10^{-3} m^3 s kg^{-1}
ν	Poisson's ratio	0.15
E	Young's modulus	1000 kg m^{-1} s^{-2}
Δt	Time step used in the simulation	4 s
T	Final time of the simulation	1000 s
Υ	Stabilization parameter	10^{-3}

Table 1: Parameters used for the unconfined compression test problem.

For the special case of an axisymmetric solution, [36] provides the following closed-form analytical solution for the radial displacement in response to a step loading function,

$$\frac{u}{a}(a,t) = \epsilon_0 \left[\nu + (1-2\nu)(1-\nu) \sum_{n=1}^{\infty} \frac{\exp\left(-\alpha_n^2 \frac{Mkt}{a^2}\right)}{\alpha_n^2(1-\nu)^2 - (1-\nu)} \right], \tag{25}$$

where ϵ_0 is the amplitude of the applied axial strain and a is the radius of the cylinder. Here t_g is the characteristic time of diffusion given by $t_g = a^2/Mk$, where $M = \lambda + 2\mu$ is the P-wave modulus of the elastic solid skeleton, k is the permeability and α_n are the solutions to the characteristic equation, given by $J_1(x) - (1-\nu)xJ_0(x)/(1-2\nu) = 0$, where J_0 and J_1 are Bessel functions. The computed radial displacement shown in Figure 3 is in good agreement with the analytical solution. The effect of the stabilization parameter on the numerical solution was investigated in [14], and shown to have a negligible effect since the stabilization parameter can be chosen to be very small in 3D. Similar unconfined compression problems have been used to test other large deformation poroelastic software such as FEBio [37].

Figure 3: Radial expansion versus time comparing the analytical and numerical solutions with $\Upsilon = 0.001$.

5.2. Terzaghi's problem

This is a common test problem from geomechanics with an analytical solution, and has been used to investigate the origins of non-physical pressure oscillations that arise in some finite element solutions near the boundary [38, 4]. The domain consists of a porous column of unit height, bounded at the sides and bottom by rigid and impermeable walls. The top is free to drain ($p_D = 0$) and has a downward traction force, p_0, applied to it. The boundary and initial conditions for this 1D problem can be written as

$$\begin{aligned} t_N = -p_0, \qquad & p_D = 0 \quad \text{for } x = 0,\ t > 0 \\ u = 0,\ \ z = 0 \qquad & \quad\ \ \text{for } x = 1,\ t > 0 \\ u = 0,\ \ z = 0, \quad & p = 0 \quad \text{for } x \in [0, 1],\ t = 0. \end{aligned} \tag{26}$$

The analytical pressure solution, in non-dimensional form is given by

$$p^* = \sum_n^\infty \frac{2}{\pi(n + 1/2)} \sin(\pi(n + 1/2)) \exp^{-\pi(n+1/2)(\lambda+2\mu)kt}. \tag{27}$$

For a detailed explanation and derivation of this solution see [1, section 5.2.2].

We discretized the column using 60 hexahedra elements and solved the problem using both the stabilized low-order finite element method and a higher-order inf-sup stable finite element method with piecewise linear pressure approximation. The material parameters used for the simulation can be found in Table 2. The simulation results of the pressure for the two methods at $t = 0.01s$ and $t = 1s$ are shown in Figure 4. At $t = 0.01s$ the piecewise linear (continuous) approximation fails to approximate the thin boundary layer in the pressure field and suffers from overshooting (Figure 4a). The stabilized low-order method does not suffer from this problem and accurately captures the pressure field near the boundary (Figure 4b). At $t = 1s$ the boundary layer has grown and both the piecewise linear pressure approximation (Figure 4c) and the piecewise constant pressure approximations (Figure 4d) yield satisfactory results.

Parameter	Description	Value
k_0	Dynamic permeability	10^{-5} m^3 s kg^{-1}
ν	Poisson ratio	0.25
E	Young's modulus	100 kg m^{-1} s^{-2}
Δt	Time step used in the simulation	0.01 s
T	Final time of the simulation	1 s
Υ	Stabilization parameter	2×10^{-5}

Table 2: Parameters used for Terzaghi's problem.

(a) (b)

(c) (d)

Figure 4: (a) Pressure at $t = 0.01s$ using a continuous linear pressure approximation. (b) Pressure at $t = 0.01s$ using a discontinuous piecewise constant approximation. (c) Pressure at $t = 1s$ using a continuous linear pressure approximation. (d) Pressure at $t = 1s$ using a discontinuous piecewise constant approximation.

5.3. Swelling test

This problem is similar to the one in [7] and highlights the method's ability to reliably capture steep gradients in the pressure solution due to rapid changes in material parameters. Given a unit cube of material, a fluid pressure gradient is imposed between the two opposite faces at $X = 0$ and $X = 1$. The pressure p_D on the inlet face $X = 0$ is increased very rapidly from zero to a limiting value of 10kPa, i.e., $p_D = 10^4(1 - \exp(-t^2/0.25))$ Pa). On the outlet face $X = 1$, the pressure is fixed to be zero, $p_D = 0$. There are no sources of sinks of fluid. A zero flux

condition is applied for the fluid velocity on the four other faces ($Y = 0, 1,\ Z = 0, 1$). Normal displacements are required to be zero on the planes $X = 0,\ Y = 0$ and $Z = 0$. The permeability of the cube $0 < X < 0.5,\ 0.5 < Y < 1,\ 0 < Z < 0.5$, i.e., 1/8 of the volume, is smaller than in the rest of the unit cube by a factor of 500. The computational domain is shown in Figure 5a, highlighting the region of reduced permeability. The parameters chosen for this test problem are given in Table 3.

Fluid enters the region from the inlet face and the material swells like a sponge, undergoing large deformation as shown in Figure 5b. The evolution of the pressure and the Jacobian at the points at $(0, 0, 1),\ (0.5, 0, 1)$ and $(1, 0, 1)$ in the reference configuration are shown in Figures 6a and 6b respectively. These position are indicated by the red, blue and green balls in Figure 5a. The pressure and volume change at the point $(0, 1, 0)$ (black ball in Figure 5a) is also shown in Figures 6a and 6b. Due to its reduced permeability, this region is much slower to swell and achieve its ultimate equilibrium state and the fluid mainly flows around the region of reduced permeability, see Figure 5b. The steep pressure gradients at the boundary of the less permeable region seen in Figure 5b are well approximated by the piecewise constant (discontinuous) pressure space even on this relatively coarse discretization, and the no-flux boundary condition is enforced correctly along the deformed boundary. Continuous pressure spaces would require a much finer discretization in this region.

(a) (b)

Figure 5: (a) Initial simulation setup. The grey cube represents the region of reduced permeability. The colored balls indicate the position of the points used for tracking the pressure and volume changes shown in Figures 6a and 6b. (b) The deformed cube after 1s showing the pressure solution and fluid flux.

Parameter	Value
k_0	10^{-5} m^3 s kg^{-1}
ν	0.3
E	8000 kg m^{-1} s^{-2}
Δt	0.02 s
T	20 s
Υ	10^{-4}

Table 3: Parameters used for the swelling test problem.

(a) (b)

Figure 6: (a) Pressure, p, (b) volume change, J (b) plotted against time at locations $\boldsymbol{X} = (0, 0, 1)$ [red], $\boldsymbol{X} = (0.5, 0, 1)$ [blue], $\boldsymbol{X} = (1, 0, 1)$ [green] and $\boldsymbol{X} = (1, 0, 1)$ [black]. (These locations are shown using the colored balls in Figure 5a).

5.4. Ventilation of the right lung lobe

In this example we derive a whole organ model of the right lung. We use high-resolution CT images taken at total lung capacity (TLC) and functional residual capacity (FRC) to provide a rough estimate of the deformation of the lung surface from expiration to inspiration. To simulate breathing we assume a sinusoidal breathing cycle and expand the lobe surface from FRC to 40% of TLC (normal inspiratory volume). The lung is first segmented from the CT data and then meshed using standard techniques. The conducting airways are also segmented from the CT data at TLC, and a centerline with radial information is calculated. To approximate the remaining airways we use a volume filling airway generation algorithm [39]. We assume that a simple Poissiuelle flow model can describe the air flow in the airways and that the lung parenchyma can be modelled as a poroelastic medium [40, 27].

We couple the airway network to the parenchyma by adding the flow contribution from each distal (terminal) airway as a source term in the poroelastic mass conservation equation for the segment of poroelastic tissue supplied by that distal branch. We also set the average pressure in each of these regions to be equal to the distal pressure in the airway supplying that region. The flow in the airway network, the poroelastic equations and the coupling conditions are solved in a monolithic fashion. For details see [41]. Figure 7 shows the pressure in the airway and on the outer surface of the lung. Heterogeneity in the airway radii and lengths results in heterogeneity in airway resistance and produces the patchy pressure distribution seen in Figure 7b. This

phenomenon has also been observed in [42] and is well approximated by the (discontinuous) piecewise constant pressure elements. Once again for this example, a continuous pressure field would require a finer discretization.

(a) (b)

Figure 7: Pressure in the airway tree (a) and on the lung surface (b) at 80% of full inspiration level.

6. Conclusions

Stabilized low-order methods can offer significant computational advantages over higher order approaches. In particular, one can employ meshes with fewer degrees of freedom, fewer Gauss points, and simpler data structures. The additional stabilization terms can also improve the convergence properties of iterative solvers.

The main contribution of this paper is to extend the local pressure jump stabilization method [43] already applied to three-field linear poroelasticity in [14], to the large deformation case. Thus, the proposed scheme is built on an existing scheme for which rigorous theoretical results addressing the stability and optimal convergence have been proven, and for which numerical experiments have demonstrated its ability to overcome spurious pressure oscillations. Owing to the discontinuous pressure approximation, sharp pressure gradients due to changes in material coefficients or boundary layers can be captured reliably, circumventing the need for severe mesh refinement. The addition of the stabilization term introduces minimal additional computational work, can be assembled locally on each element using standard element information, and leads to a symmetric addition to the original system matrix, thus preserving any existing symmetry. As the numerical examples have demonstrated, the stabilization scheme is robust and leads to high-quality solutions.

Appendix A. Model derivation

Appendix A.1. Conservation of mass

The mass balance for the solid and fluid phases respectively can be expressed as

$$\frac{d}{dt}\int_{\Omega(t)}(1-\phi)\rho^s \,\mathrm{d}\Omega(t) = \int_{\Omega(t)}\left(\frac{\partial(1-\phi)\rho^s}{\partial t} + \nabla\cdot((1-\phi)\rho^s\boldsymbol{v}^s)\right)\mathrm{d}\Omega(t) = 0, \tag{A.1}$$

$$\frac{d}{dt}\int_{\Omega(t)}\phi\rho^f \,\mathrm{d}\Omega(t) = \int_{\Omega(t)}\left(\frac{\partial\phi\rho^f}{\partial t} + \nabla\cdot(\phi\rho^f\boldsymbol{v}^f)\right)\mathrm{d}\Omega(t) = \int_{\Omega(t)}\rho^f g \,\mathrm{d}\Omega(t), \tag{A.2}$$

where $\boldsymbol{v}^f$ is the velocity of the fluid and $\boldsymbol{v}^s$ is the velocity of the solid given by

$$\boldsymbol{v}^s(\boldsymbol{x},t)|_{\boldsymbol{x}=\boldsymbol{\chi}(\boldsymbol{X},t)} = \frac{\partial \boldsymbol{\chi}(\boldsymbol{X},t)}{\partial t}\,. \tag{A.3}$$

and g is a general source or sink term. In differential form,

$$\frac{\partial(1-\phi)\rho^s}{\partial t} + \nabla\cdot((1-\phi)\rho^s\boldsymbol{v}^s) = 0 \quad \text{in } \Omega(t), \tag{A.4}$$

$$\frac{\partial(\phi\rho^f)}{\partial t} + \nabla\cdot(\phi\rho^f\boldsymbol{v}^f) = \rho^f g \quad \text{in } \Omega(t), \tag{A.5}$$

or,

$$\frac{\partial\hat{\rho}^s}{\partial t} + \nabla\cdot(\hat{\rho}^s\boldsymbol{v}^s) = 0 \quad \text{in } \Omega(t), \tag{A.6}$$

$$\frac{\partial\hat{\rho}^f}{\partial t} + \nabla\cdot(\hat{\rho}^f\boldsymbol{v}^f) = \rho^f g \quad \text{in } \Omega(t), \tag{A.7}$$

where $\hat{\rho}^s = \rho^s(1-\phi)$ and $\hat{\rho}^f = \rho^f\phi$.

First noting that ρ^s and ρ^f are constants and can be factored out and then adding equations (A.4) and (A.5), provides the mass balance or continuity equation of the mixture,

$$\nabla\cdot((1-\phi)\boldsymbol{v}^s) + \nabla\cdot(\phi\boldsymbol{v}^f) = g \quad \text{in } \Omega(t). \tag{A.8}$$

Appendix A.2. Conservation of momentum

For $\alpha = s, f$ the conservation of linear momentum for solid and fluid components is given by

$$\frac{d}{dt}\int_{\Omega(t)}\hat{\rho}^\alpha\boldsymbol{v}^\alpha d\Omega(t) = \int_{\Omega(t)}\nabla\cdot\boldsymbol{\sigma}^\alpha + \hat{\rho}^\alpha\boldsymbol{f} + \hat{\boldsymbol{p}}^\alpha + \beta^\alpha\boldsymbol{v}^\alpha\ \mathrm{d}\Omega(t), \tag{A.9}$$

where $\boldsymbol{\sigma}^\alpha$ is the Cauchy stress tensor for the $\alpha = s, f$, $\boldsymbol{f}$ is a body force, $\hat{\boldsymbol{p}}^\alpha$ are interaction forces representing frictional interactions between the solid and fluid (see Appendix A.3) and β^α is the constituent source term. Here $\beta^s = 0$ and $\beta^f = \rho^f g$. Applying Reynolds Transport Theorem, we rewrite the integral conservation law in differential form and obtain

$$\frac{\partial(\hat{\rho}^\alpha\boldsymbol{v}^\alpha)}{\partial t} + (\boldsymbol{v}^\alpha\cdot\nabla)(\hat{\rho}^\alpha\boldsymbol{v}^\alpha) + \hat{\rho}^\alpha\boldsymbol{v}^\alpha(\nabla\cdot\boldsymbol{v}^\alpha) = \nabla\cdot\boldsymbol{\sigma}^\alpha + \hat{\rho}^\alpha\boldsymbol{f} + \hat{\boldsymbol{p}}^\alpha + \beta^\alpha\boldsymbol{v}^\alpha \quad \text{in } \Omega(t). \tag{A.10}$$

Expanding the LHS,

$$\begin{aligned}&\hat{\rho}^\alpha\left(\frac{\partial\boldsymbol{v}^\alpha}{\partial t} + (\boldsymbol{v}^\alpha\cdot\nabla)\boldsymbol{v}^\alpha\right) + \left(\frac{\partial\hat{\rho}^\alpha}{\partial t} + (\boldsymbol{v}^\alpha\cdot\nabla)\hat{\rho}^\alpha + \hat{\rho}^\alpha(\nabla\cdot\boldsymbol{v}^\alpha)\right)\boldsymbol{v}^\alpha\\ &= \hat{\rho}^\alpha\frac{D\boldsymbol{v}^\alpha}{Dt} + \left(\frac{\partial\hat{\rho}^\alpha}{\partial t} + \nabla\cdot(\hat{\rho}^\alpha\boldsymbol{v}^\alpha)\right)\boldsymbol{v}^\alpha.\end{aligned}$$

Using (A.6) and (A.7) to replace the second term above,

$$\hat{\rho}^\alpha\boldsymbol{a}^\alpha = \nabla\cdot\boldsymbol{\sigma}^\alpha + \hat{\rho}^\alpha\boldsymbol{f} + \hat{\boldsymbol{p}}^\alpha \quad \text{in } \Omega(t), \tag{A.11}$$

where

$$\boldsymbol{a}^s(\boldsymbol{x},t)|_{\boldsymbol{x}=\boldsymbol{\chi}(\boldsymbol{X},t)} = \frac{\partial^2\boldsymbol{\chi}(\boldsymbol{X},t)}{\partial t^2}\,, \tag{A.12}$$

$$\boldsymbol{a}^f = \frac{\partial\boldsymbol{v}^f}{\partial t} + (\boldsymbol{v}^f\cdot\nabla)\boldsymbol{v}^f. \tag{A.13}$$

Appendix A.3. Constitutive relationships

Constitutive relationships for the interaction forces, permeability tensor and solid and fluid stress tensors are provided below.

Appendix A.3.1. Interaction forces

The interaction force is given by

$$\hat{\boldsymbol{p}}^s = -\hat{\boldsymbol{p}}^f = -p\nabla\phi + \phi^2 \boldsymbol{k}^{-1} \cdot (\boldsymbol{v}^f - \boldsymbol{v}^s), \tag{A.14}$$

where $\boldsymbol{k}$ is the (dynamic) permeability tensor and p is the fluid pressure [1]. The first term, $p\nabla\phi$, accounts for the pressure effect resulting from the variation of the section offered to the fluid flow, and the second term, $\phi^2 \boldsymbol{k} \cdot (\boldsymbol{v}^f - \boldsymbol{v}^s)$, describes the viscous resistance opposed by the shear stress to the fluid flow from the drag at the internal walls of the porous network. This particular choice for the interaction force means that the momentum balance for the fluid flow can later be reduced to the well known Darcy law.

Appendix A.3.2. Permeability tensor

The permeability tensor is given by

$$\boldsymbol{k} = J^{-1} \boldsymbol{F} \boldsymbol{k}_0(\boldsymbol{\chi}) \boldsymbol{F}^T, \tag{A.15}$$

where $\boldsymbol{k}_0(\boldsymbol{\chi})$ is the permeability in the reference configuration, which may be chosen to be some (nonlinear) function dependent on the deformation. Examples of deformation dependent permeability tensors for biological tissues can be found in [27, 12, 28]. A common isotropic assumption is

$$\boldsymbol{\kappa} = \kappa_0 \Pi(J)\, \boldsymbol{I}, \tag{A.16}$$

where κ_0 is the permeability in the reference configuration and $\Pi(J)$ is some function dependent on the volume change. For example, in [27], the following isotropic constitutive law for the permeability of lung tissue is proposed

$$\boldsymbol{\kappa} = \kappa_0 \left(J \frac{\phi}{\phi_0} \right)^{2/3} \boldsymbol{I}, \tag{A.17}$$

where κ_0 is the permeability in the reference configuration.

Appendix A.3.3. Solid stress tensor

The solid stress tensor is given by [2],

$$\boldsymbol{\sigma}^s = \boldsymbol{\sigma}_e^s - (1-\phi) p \boldsymbol{I}, \tag{A.18}$$

where $\boldsymbol{\sigma}_e^s$ is the effective stress tensor given by

$$\boldsymbol{\sigma}_e^s = \frac{1}{J} \boldsymbol{F} \cdot 2 \frac{\partial W(\boldsymbol{\chi})}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^T. \tag{A.19}$$

Here $W(\boldsymbol{\chi})$ denotes a strain-energy law (hyperelastic Helmholtz energy functional) dependent on the deformation of the solid.

Appendix A.3.4. Fluid stress tensor

The fluid stress tensor can be written as [2],

$$\boldsymbol{\sigma}^f = \boldsymbol{\sigma}^f_{vis} - \phi p \boldsymbol{I}, \tag{A.20}$$

where $\boldsymbol{\sigma}^f_{vis}$ denotes the viscous stress tensor of the fluid, given by

$$\boldsymbol{\sigma}^f_{vis} = \mu_f \phi \left((\nabla \boldsymbol{v}_f) + (\nabla \boldsymbol{v}_f)^T - \frac{2}{3} \nabla \cdot \boldsymbol{v}_f \right), \tag{A.21}$$

where μ_f is the dynamic viscosity of the fluid.

Appendix A.4. The general poroelasticity model

Summing the conservation laws for solid and fluid and applying the constitutive relations, the conservation of linear momentum for the mixture is

$$\hat{\rho}^s \boldsymbol{a}^s + \hat{\rho}^f \boldsymbol{a}^f = \nabla \cdot (\boldsymbol{\sigma}_e + \boldsymbol{\sigma}_{vis} - p\boldsymbol{I}) + \rho \boldsymbol{f} \quad \text{in } \Omega(t). \tag{A.22}$$

The momentum equation for the fluid flow alone is

$$\hat{\rho}^f \boldsymbol{a}^f = \nabla \cdot (\boldsymbol{\sigma}^f_{vis} - \phi p \boldsymbol{I}) + \hat{\rho}^f \boldsymbol{f} + p \nabla \phi - \phi^2 \boldsymbol{k}^{-1} (\boldsymbol{v}^f - \boldsymbol{v}^s) \quad \text{in } \Omega(t). \tag{A.23}$$

We define the boundary $\partial\Omega(t) = \Gamma_d(t) \cup \Gamma_n(t)$ for the mixture and $\partial\Omega(t) = \Gamma_p(t) \cup \Gamma_f(t)$ for the fluid, with an outward pointing unit normal $\boldsymbol{n}$. The problem for the mixture theory model is: Find $\boldsymbol{\chi}(\boldsymbol{X}, t)$, $\boldsymbol{v}^f(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t)$ such that

$$\left.\begin{aligned}
\hat{\rho}^s \boldsymbol{a}^s + \hat{\rho}^f \boldsymbol{a}^f &= \nabla \cdot (\boldsymbol{\sigma}_e + \boldsymbol{\sigma}_{vis} - p\boldsymbol{I}) + \rho \boldsymbol{f} && \text{in } \Omega(t),\\
\hat{\rho}^f \boldsymbol{a}^f &= \nabla \cdot (\boldsymbol{\sigma}^f_{vis} - \phi p \boldsymbol{I}) + p \nabla \phi - \phi^2 \boldsymbol{k}^{-1} (\boldsymbol{v}^f - \boldsymbol{v}^s) + \hat{\rho}^f \boldsymbol{f} && \text{in } \Omega(t),\\
\nabla \cdot ((1-\phi)\boldsymbol{v}^s) + \nabla \cdot (\phi \boldsymbol{v}^f) &= g && \text{in } \Omega(t),\\
\boldsymbol{\chi}(\boldsymbol{X}, t)|_{X = \chi^{-1}(\boldsymbol{x}, t)} &= \boldsymbol{X} + \boldsymbol{u}_D && \text{on } \Gamma_d(t),\\
(\boldsymbol{\sigma}_e + \boldsymbol{\sigma}_{vis} - p\boldsymbol{I})\boldsymbol{n} &= \boldsymbol{t}_N, && \text{on } \Gamma_n(t),\\
\boldsymbol{v}^f &= \boldsymbol{v}^f_D && \text{on } \Gamma_f(t),\\
\boldsymbol{n} \cdot \boldsymbol{\sigma}^f_{vis} \cdot \boldsymbol{n} - \phi p \boldsymbol{I} \cdot \boldsymbol{n} &= \boldsymbol{s}_D && \text{on } \Gamma_p(t),\\
\boldsymbol{\chi}(\boldsymbol{X}, 0) = \boldsymbol{X}, \quad \boldsymbol{v}^s(\boldsymbol{X}, 0) &= \boldsymbol{v}^{s0}, \quad \boldsymbol{v}^f(\boldsymbol{X}, 0) = \boldsymbol{v}^{f0} && \text{in } \Omega(0).
\end{aligned}\right\} \tag{A.24}$$

Appendix A.5. Simplification of the model

We assume accelerations $\boldsymbol{a}^\alpha$ and the viscous shear stress in the fluid $\boldsymbol{\sigma}^f_{vis}$ are small, and define the fluid flux variable

$$\boldsymbol{z} = \phi(\boldsymbol{v}^f - \boldsymbol{v}^s). \tag{A.25}$$

The resulting problem is: Find $\boldsymbol{\chi}(\boldsymbol{X},t)$, $\boldsymbol{z}(\boldsymbol{x},t)$ and $p(\boldsymbol{x},t)$ such that

$$\left.\begin{aligned} -\nabla\cdot(\boldsymbol{\sigma}_e - p\boldsymbol{I}) &= \rho\boldsymbol{f} && \text{in } \Omega(t),\\ \boldsymbol{k}^{-1}\boldsymbol{z} + \nabla p &= \rho^f\boldsymbol{f} && \text{in } \Omega(t),\\ \nabla\cdot(\boldsymbol{v}^s + \boldsymbol{z}) &= g && \text{in } \Omega(t),\\ \boldsymbol{\chi}(\boldsymbol{X},t)|_{X=\chi^{-1}(\boldsymbol{x},t)} &= \boldsymbol{X} + \boldsymbol{u}_D && \text{on } \Gamma_d(t),\\ (\boldsymbol{\sigma}_e - p\boldsymbol{I})\boldsymbol{n} &= \boldsymbol{t}_N && \text{on } \Gamma_t(t),\\ \boldsymbol{z}\cdot\boldsymbol{n} &= q_D && \text{on } \Gamma_f(t),\\ p &= p_D && \text{on } \Gamma_p(t),\\ \boldsymbol{\chi}(\boldsymbol{X},0) &= \boldsymbol{X} && \text{in } \Omega(0). \end{aligned}\right\} \tag{A.26}$$

We note that, for example

$$\int_{\Omega_t}\nabla_{\boldsymbol{x}}\cdot\sigma(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\Omega_t}\nabla_{\boldsymbol{X}}\cdot\boldsymbol{F}^{-1}\sigma(\boldsymbol{\chi}(\boldsymbol{X},t))\,\mathrm{d}\boldsymbol{x} = \int_{\Omega_0}\nabla_{\boldsymbol{X}}\cdot\boldsymbol{F}^{-1}J\sigma(\boldsymbol{\chi}(\boldsymbol{X},t))\,\mathrm{d}\boldsymbol{X} = \int_{\Omega_0}\nabla_{\boldsymbol{X}}\cdot(\boldsymbol{S}\boldsymbol{F}^T)\,\mathrm{d}\boldsymbol{X}, \tag{A.27}$$

where $\boldsymbol{S}$ is the second Piola-Kirchoff stress tensor.

Appendix B. The fourth-order spatial tangent modulus tensor $\boldsymbol{\Theta_{ijkl}}$

The fourth-order spatial tangent modulus tensor Θ_{ijkl} can be written as (in component form, see [31, section 5.3.2] and [44, section 6.6])

$$\Theta_{ijkl} = \frac{1}{J}F_{iI}F_{jJ}F_{kK}F_{lL}\mathbf{C}_{IJKL}, \tag{B.1}$$

where $\mathbf{C}$ is the associated tangent modulus tensor in the reference configuration, given by

$$\mathbf{C}_{IJKL} = \frac{4\partial^2 W}{\partial C_{IJ}\partial C_{KL}} + pJ\frac{\partial C^{-1}_{IJ}}{\partial C_{KL}}. \tag{B.2}$$

For the numerical examples we have used the following Neo-Hookean strain-energy law

$$W(\boldsymbol{C}) = \frac{\mu}{2}(\mathrm{tr}(\boldsymbol{C}) - 3) + \frac{\Lambda}{4}(J^2 - 1) - (\mu + \frac{\Lambda}{2})\ln(J - 1 + \phi_0). \tag{B.3}$$

Thus, the resulting effective stress tensor is given by

$$\boldsymbol{\sigma}_e = \frac{\Lambda}{2}\left(J - \frac{1}{J - 1 + \phi_0}\right)\boldsymbol{I} + \mu\left(\frac{\boldsymbol{C}^T}{J} - \frac{\boldsymbol{I}}{J - 1 + \phi_0}\right), \tag{B.4}$$

and the spatial tangent modulus tensor is given as

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}_e + p(\boldsymbol{I}\otimes\boldsymbol{I} - 2\mathcal{Z}), \tag{B.5}$$

where

$$\begin{aligned}\boldsymbol{\Theta}_e = \left[\Lambda J - 2\mu\left(\frac{1}{2(J - 1 + \phi_0)} - \frac{J}{2(J - 1 + \phi_0)^2}\right)\right]\boldsymbol{I}\otimes\boldsymbol{I}\\ + \left[\frac{2\mu}{J - 1 + \phi_0} - \Lambda(J - \frac{1}{J - 1 + \phi_0})\right]\mathcal{B},\end{aligned} \tag{B.6}$$

and

$$\mathcal{B}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \mathcal{Z}_{ijkl} = \delta_{ik}\delta_{jl}, \quad \boldsymbol{I} \otimes \boldsymbol{I} = \delta_{ij}\delta_{kl}. \tag{B.7}$$

See [31, chapter 5] and [30, chapter 3] for further details.

To simplify the implementation of the spatial tangent modulus we make use of matrix Voigt notation. The matrix form of β is given by $\boldsymbol{D}$, which can be written as (see [31, section 7.4.2])

$$\boldsymbol{D} = \frac{1}{2}\begin{pmatrix} 2\boldsymbol{\Theta}_{1111} & 2\boldsymbol{\Theta}_{1122} & 2\boldsymbol{\Theta}_{1133} & \boldsymbol{\Theta}_{1112}+\boldsymbol{\Theta}_{1121} & \boldsymbol{\Theta}_{1113}+\boldsymbol{\Theta}_{1131} & \boldsymbol{\Theta}_{1123}+\boldsymbol{\Theta}_{1132} \\ & 2\boldsymbol{\Theta}_{2222} & 2\boldsymbol{\Theta}_{2233} & \boldsymbol{\Theta}_{2212}+\boldsymbol{\Theta}_{2221} & \boldsymbol{\Theta}_{2213}+\boldsymbol{\Theta}_{2231} & \boldsymbol{\Theta}_{2223}+\boldsymbol{\Theta}_{2232} \\ & & 2\boldsymbol{\Theta}_{3333} & \boldsymbol{\Theta}_{3312}+\boldsymbol{\Theta}_{3321} & \boldsymbol{\Theta}_{3313}+\boldsymbol{\Theta}_{3331} & \boldsymbol{\Theta}_{3323}+\boldsymbol{\Theta}_{3332} \\ & & & \boldsymbol{\Theta}_{1212}+\boldsymbol{\Theta}_{1221} & \boldsymbol{\Theta}_{1213}+\boldsymbol{\Theta}_{1231} & \boldsymbol{\Theta}_{1223}+\boldsymbol{\Theta}_{1232} \\ & \text{sym.} & & & \boldsymbol{\Theta}_{1313}+\boldsymbol{\Theta}_{1331} & \boldsymbol{\Theta}_{1323}+\boldsymbol{\Theta}_{1332} \\ & & & & & \boldsymbol{\Theta}_{2323}+\boldsymbol{\Theta}_{2332} \end{pmatrix}. \tag{B.8}$$

We also make use of the following implementation friendly matrix notation for $\nabla^S \boldsymbol{\phi}_k$

$$\boldsymbol{E}_k = \begin{bmatrix} \phi_{k,1} & 0 & 0 \\ 0 & \phi_{k,2} & 0 \\ 0 & 0 & \phi_{k,3} \\ \phi_{k,2} & \phi_{k,1} & 0 \\ 0 & \phi_{k,3} & \phi_{k,2} \\ \phi_{k,3} & 0 & \phi_{k,1} \end{bmatrix}. \tag{B.9}$$

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