Two-Level Game Semantics, Intersection Types, and Recursion Schemes

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Abstract. We introduce a new cartesian closed category of *two-level* arenas and innocent strategies to model intersection types that are refinements of simple types. Intuitively a property (respectively computation) on the upper level refines that on the lower level. We prove *Subject Expansion*—any lower-level computation is closely and canonically tracked by the upper-level computation that lies over it—which is a measure of the robustness of the two-level semantics. The game semantics of the type system is *fully complete*: every winning strategy is the denotation of some derivation. To demonstrate the relevance of the game model, we use it to construct new semantic proofs of non-trivial algorithmic results in higher-order model checking.

1 Introduction

The development of *higher-order model checking*—the model checking of trees generated by higher-order recursion schemes (HORS)—has benefitted much from ideas and methods in semantics. Ong's proof [1] of the decidability of the monadic second-order (MSO) theories of trees generated by HORS was based on game semantics [2]. Using HORS as an intermediate model of higher-order computation, Kobayashi [3] showed that safety properties of functional programs can be verified by reduction to the model checking of HORS against trivial automata (i.e. Büchi tree automata with a trivial acceptance condition). His model checking algorithm is based on an intersection type-theoretic characterisation of the trivial automata acceptance problem of trees generated by HORS. This type-theoretic approach was subsequently refined and extended to characterise alternating parity tree automata [4], thus yielding a new proof of Ong's MSO decidability result. (Several other proofs of the result are now known. Hague et al. [5] developed a new hierarchy of *collapsible pushdown automata* and proved that they are equi-expressive with HORS for generating trees. Salvati and Walukiewicz's proof [6] uses a Krivine machine formulation of the operational semantics of HORS.)

This paper was motivated by a desire to understand the connexions between the game-semantic proof [1] and the type-based proof [3,4] of the MSO decidability result. As a first step in clarifying their relationship, we construct a *two-level game semantics* to model intersection types that are refinements of simple types.

Given a set Q of colours (modelling the states of an automaton), we introduce a cartesian closed category whose objects are triples (A, U, K) called *two-level arenas*, where A is a *Q-coloured arena* (modelling intersection types), K is a standard arena (modelling simple types), and U is a colour-forgetting function from A-moves to K-moves which preserves the justification relation. A map of the category from (A, U, K) to (A', U', K') is a pair of innocent and colour-reflecting strategies, $\sigma : A \longrightarrow A'$ and $\bar{\sigma} : K \longrightarrow K'$, such that the induced colour-forgetting function maps plays of σ to plays of $\bar{\sigma}$. This captures the intuition that the upper-level computation represented by σ refines (or is more constrained than) the lower-level computation represented by $\bar{\sigma}$, a semantic framework reminiscent of two-level denotational semantics in abstract interpretation as studied by Nielson [7]. Given triples $\mathcal{A}_1 = (A_1, U_1, K)$ and $\mathcal{A}_2 = (A_2, U_2, K)$ that have the same base arena K, their *intersection* $\mathcal{A}_1 \wedge \mathcal{A}_2$ is $(A_1 \times A_2, [U_1, U_2], K)$. Building on the two-level game semantics, we make the following contributions.

(i) How good is the two-level game semantics? Our answer is *Subject Expansion* (Theorem 3), which says intuitively that any computation (reduction) on the lower level can be closely and canonically tracked by the higher-level computation that lies over it. Subject Expansion clarifies the relationship between the two levels; we think it is an important measure of the robustness (and, as we shall see, the reason for the usefulness) of the game semantics.

(ii) We put the two-level game model to use by modelling Kobayashi's intersection type system [3]. Derivations of intersection-type judgements, which we represent by the terms of a new proof calculus, are interpreted by *winning strategies* i.e. compact and total (in addition to innocent and colour-reflecting). We prove that the interpretation is *fully complete* (Theorem 5): every winning strategy is the denotation of some derivation.

(iii) Finally, to demonstrate the usefulness and relevance of the two-level game semantics, we apply it to construct new semantic proofs of three non-trivial *algorithmic* results in higher-order model checking: (a) characterisation of trivial automata acceptance (existence of an accepting run-tree) by a notion of typability [3], (b) minimality of the type environment induced by traversal tree [1], and (c) completeness of GTRecS, a game-semantics based practical algorithm for model checking HORS against trivial automata [9].

Outline [4] We introduce (coloured) arenas, innocent strategies and related game-semantic notions in Section 2. In Section 3 we present two-level games, culminating in the Subject Expansion Theorem. In Section 4 we construct a fully complete two-level game model of Kobayashi's intersection type system. Finally, Section 5 applies the game model to reason about algorithmic problems in higher-order model checking.

[4] See http://www.cs.ox.ac.uk/people/luke.ong/personal/publications/icalp12.pdf for a full version of this submission with proofs.

2 Coloured Arenas and Innocent Strategies

This section defines coloured arenas, innocent strategies and related notions.

2.1 Definition and Constructions of Coloured Arenas

For sets A and B, we write $A + B$ for the disjoint union and $A \times B$ for the Cartesian product.

Definition 1 (Coloured Arena). For a set Q of symbols, a Q-*coloured arena* A is a quadruple $(M_A, \vdash_A, \lambda_A, c_A)$, where

- M_A is a set of *moves*,
- $\vdash_A \subseteq M_A + (M_A \times M_A)$ is a *justification relation*,
- $\lambda_A : M_A \to \{P, O\}$, and
- $c_A : M_A \to Q$ is a *colouring*.

We write $\vdash_A m$ for $m \in (\vdash_A)$ and $m \vdash_A m'$ for $(m, m') \in (\vdash_A)$. The justification relation must satisfy the following conditions:

- For every move $m \in M_A$, either $\vdash_A m$ or $m' \vdash_A m$ for a unique move $m' \in M_A$.
- If $\vdash_A m$, then $\lambda_A(m) = O$. If $m \vdash_A m'$, then $\lambda_A(m) \neq \lambda(m')$.

Fer a Q-coloured arena A, the set $\mathit{Init}_A \subseteq M_A$ of *initial moves* of A is $\{m \in M_A \mid \vdash_A m\}$. A move $m \in M_A$ is called an *O-move* if $\lambda_A(m) = O$ and a *P-move* if $\lambda_A(m) = P$.

2.2 Views and Innocent Strategies

A *justified sequence* of a Q-coloured arena A is a sequence of moves of which each element except the first is equipped with a pointer to some previous move. We call the pointer a *justification pointer*. For a justified sequence s and moves m and m' in s, we say m' is *hereditary justified* by m if there exists a sequence of moves $m_0, m_1, \ldots, m_n$ in s that starting from m and ending with m' such that m_i is justified by m_{i-1} $(1 \leq i \leq n)$.

A *well-formed sequence* over A is a justified sequence $s = m_0 \cdot m_1 \cdot \cdots \cdot m_n$ that has the following properties:

Well-opened $\vdash_A m_0$,
Alternation For all $i < n$, $\lambda_A(m_i) \neq \lambda_A(m_{i+1})$, and
Justification If m_i points m_j $(j < i)$, then $m_j \vdash_A m_i$.

For well-formed sequences s and s', we say s is a *prefix* of s' if the underlying sequence of moves of s is a prefix of that of s' and their justification pointers coincide.

For a well-formed sequence s, its *P-view* $\ulcorner s \urcorner$ and O-view $\llcorner s \lrcorner$ are defined inductively as follows:

$$\begin{aligned} \ulcorner \epsilon \urcorner &= \epsilon \\ \ulcorner m \urcorner &= m \\ \ulcorner s \cdot m \urcorner &= \ulcorner s \urcorner \cdot m \qquad (\text{if } \lambda(m) = P) \\ \ulcorner s \cdot m \cdot s' \cdot m' \urcorner &= \ulcorner s \urcorner \cdot m \cdot m' \qquad (\text{if } \lambda(m) = O \text{ and } m' \text{ is justified by } m) \end{aligned}$$

$$\begin{aligned} \llcorner \epsilon \lrcorner &= \epsilon \\ \llcorner m \lrcorner &= m \\ \llcorner s \cdot m \lrcorner &= \llcorner s \lrcorner \cdot m \qquad (\text{if } \lambda(m) = O) \\ \llcorner s \cdot m \cdot s' \cdot m' \lrcorner &= \llcorner s \lrcorner \cdot m \cdot m' \qquad (\text{if } \lambda(m) = P \text{ and } m' \text{ is justified by } m) \end{aligned}$$

A *play* of an arena A is a well-formed sequence s satisfying the following conditions:

Visibility For every prefix $s' \cdot m \preceq s$ ending with a P-move (resp. an O-move that is not initial), m is justified by a move in $\ulcorner s' \urcorner$ (resp. $\llcorner s' \lrcorner$).

A *P-strategy* (or a *strategy*) σ of an arena A is a prefix-closed subset of plays of A satisfying the following conditions:

Determinacy If $s{\cdot}m \in \sigma$ and $s{\cdot}m' \in \sigma$ for P-moves m and m' then $s{\cdot}m = s{\cdot}m'$.
Contingent Completeness If $s \in \sigma$, m is an O-move and $s \cdot m$ is a justified sequence, then $s \cdot m \in \sigma$.
Colour Reflecting Only the opponent can change the colour, i.e. for every P-move m_1^P and O-move m_2^O, if $s \cdot m_1^O \cdot m_2^P \in \sigma$, then $c(m_1^O) = c(m_2^P)$.

For arenas A_1, A_2 and A_3, an *interaction sequence* is a play of $(A_1 \Rightarrow A_2) \Rightarrow A_3$. We write $\mathrm{Int}(A_1, A_2, A_3)$ for the set of all interaction sequences. For an interaction sequence $s \in \mathrm{Int}(A_1, A_2, A_3)$, a component of s is either (A_2, A_3) or (A_1, A_2, b) where b is an initial move occurring in s. The projection $s \upharpoonright_X$ of an interaction sequence s into a component X is defined by:

- $s \upharpoonright_{(A_2,A_3)}$ is a subsequence of s consisting of all A_2 moves and A_3 moves in s.
- $s \upharpoonright_{(A_1,A_2,b)}$ is a subsequence of s consisting of all moves that are hereditary justified by b.

The projection into (A_1, A_3) is defined by a similar way: $s \upharpoonright_{(A_1,A_3)}$ is a subsequence of s consisting of all A_1 moves and A_3 moves, in which initial A_1 moves are justified by a (unique) initial A_3 move occurring in s. For an interaction sequence $s \in \mathrm{Int}(A_1, A_2, A_3)$, $s \upharpoonright_{(A_2,A_3)}$ is a play of $A_2 \Rightarrow A_3$, $s \upharpoonright_{(A_1,A_2,b)}$ is of $A_1 \Rightarrow A_2$ (for every initial A_2 move b occurring in s) and $s \upharpoonright_{(A_1,A_3)}$ is of $A_1 \Rightarrow A_3$.

For strategies (or just sets of plays) $\sigma_1 : A_1 \Rightarrow A_2$ and $\sigma_2 : A_2 \Rightarrow A_3$, the set $\mathbf{Int}(\sigma_1, \sigma_2) \subseteq \mathrm{Int}(A_1, A_2, A_3)$ of interaction sequences that are consistent with σ_1 and σ_2 is give by:

$$\mathbf{Int}(\sigma_1, \sigma_2) = \{s \in \mathrm{Int}(A_1, A_2, A_3) \mid s \upharpoonright_{(A_2,A_3)} \in \sigma_2 \text{ and for every initial } A_2 \text{ move } b,\ s \upharpoonright_{(A_1,A_2,b)} \in \sigma_1\}.$$

The *composition* $(\sigma_1; \sigma_2) : A_1 \Rightarrow A_3$ is defined as $\{s \upharpoonright_{(A_1,A_3)} \mid s \in \mathbf{Int}(\sigma_1, \sigma_2)\}$. For each $s \in (\sigma_1; \sigma_2)$, the *uncovering* of s is the *minimum* interaction sequence $u \in \mathbf{Int}(\sigma_1, \sigma_2)$ (with respect to the prefix ordering) such that $s = u \upharpoonright_{(A_1,A_3)}$.[5]

A strategy σ is *innocent* if for every pair of plays $s \cdot m,\ s' \cdot m' \in \sigma$ ending with P-moves m and m', $\ulcorner s \urcorner = \ulcorner s' \urcorner$ implies $\ulcorner s \cdot m \urcorner = \ulcorner s' \cdot m' \urcorner$. A strategy σ is *colour-reflecting* if $s \cdot m \cdot m' \in \sigma$ and $\lambda_A(m) = O$ and $\lambda_A(m') = P$ implies $c_A(m) = c_A(m')$.

We say an innocent and colour-reflecting strategy σ is a *winning strategy* if all of the following conditions hold:

Compact The domain $\mathrm{dom}(f_\sigma)$ of the view function of σ is a finite set.
Total If $s \cdot m \in \sigma$ for an O-move m, then $s \cdot m \cdot m' \in \sigma$ for some P-move m'.

2.3 Category of Coloured Arenas and Innocent Strategies

We defined the category of Q-coloured games. Objects are Q-coloured arenas. Arrows from A to B are innocent and colour-reflecting strategies of the arena $A \Rightarrow B$. The category of Q-coloured games is a CCC, and thus is a model of the simply-typed lambda calculus with (indexed) products.

We define three constructions of arenas: a binary product, an indexed product and a function space.

Product For Q-coloured arenas A and B, we define $A \times B$ by:

- $M_{A\times B} = M_A + M_B$,
- $\vdash_{A\times B} m \iff \vdash_A m$ or $\vdash_B m$,
- $m \vdash_{A\times B} m' \iff m \vdash_A m'$ or $m \vdash_B m'$,
- $\lambda_{A\times B}(m) = \begin{cases} \lambda_A(m) & (\text{if } m \in M_A) \\ \lambda_B(m) & (\text{if } m \in M_B), \end{cases}$
- $c_{A\times B}(m) = \begin{cases} c_A(m) & (\text{if } m \in M_A) \\ c_B(m) & (\text{if } m \in M_B). \end{cases}$

For an indexed set $\{A_i\}_{i\in I}$ of Q-coloured arenas, their product $\prod_{i\in I} A_i$ is defined similarly.

[5] The definition here differs from the one in [2]: in [2], the uncovering is the *maximum* interaction sequence.

Function Space For Q-coloured arenas A and B, we define $A \Rightarrow B$ by:

- $M_{A \Rightarrow B} = M_A \times \mathit{Init}_B + M_B$,
- $\vdash_{A \Rightarrow B} m \iff \vdash_B m$,
- $m \vdash_{A \Rightarrow B} m' \iff$
  - $m \vdash_B m'$, or
  - $\vdash_B m$ and $m' = (m'_A, m)$ and $\vdash_A m_A$, or
  - $m = (m_A, m_B)$ and $m' = (m'_A, m_B)$ and $m_A \vdash_A m'_A$,
- $\lambda_{A \Rightarrow B}(m) = \begin{cases} \lambda_A(m_A) & \text{(if } m = (m_A, m_B) \in M_A \times \mathit{Init}_B\text{)} \\ \lambda_B(m) & \text{(if } m \in M_B\text{),} \end{cases}$
- $c_{A \times B}(m) = \begin{cases} c_A(m) & \text{(if } m = (m_A, m_B) \in M_A \times \mathit{Init}_B\text{)} \\ c_B(m) & \text{(if } m \in M_B\text{).} \end{cases}$

Theorem 1. *For every set Q, the category of Q-coloured games is a CCC with the Cartesian product $A \times B$ and the exponent $A \Rightarrow B$.*

2.4 Definability

3 Two-level Arena Games

3.1 Definitions and Constructions

Definition 2 (Two-Level Arenas). An *two-level arena* based on Q is a triple $\mathcal{A} = (A, U, K)$, where A is a Q-colored arena, K is a $\{o\}$-colored arena (i.e. an ordinary arena, which we call the *base arena* of $\mathcal{A}$) and U is a map from M_A to M_K that satisfies: (i) $\lambda_A(m) = \lambda_K(U(m))$ (ii) If $m \vdash_A m'$ then $U(m) \vdash_K U(m')$; and if $\vdash_A m$ then $\vdash_K U(m)$.

For a justified sequence $s = m_1 \cdot m_2 \cdots m_k$, we write $U(s)$ to mean the justified sequence $U(m_1) \cdot U(m_2) \cdots U(m_k)$ whose justification pointers are induced by those of s.

Lemma 1. *Let $\mathcal{A} = (A, U, K)$ be a two-level arena and s be a play of A. Then $U(s)$ is a play of K.*

Proof. Easy. □

For a strategy σ of A, $U(\sigma) := \{U(s) \mid s \in \sigma\}$ is a set of plays of K, which is not necessarily a strategy.

Definition 3 (Strategy of Two-Level Games). A *strategy* of a two-level arena (A, U, K) is a pair $(\sigma, \bar{\sigma})$ of strategies of A and K respectively such that $U(\sigma) \subseteq \bar{\sigma}$.

Lemma 2. *Let $\mathcal{A}_i = (A_i, U_i, K_i)$ be a two-level arena for $i = 1, 2, 3$. Then for every interaction sequence s of (A_1, A_2, A_3)*

(i) $U(\ulcorner s \urcorner) = \ulcorner U(s) \urcorner$

(ii) for any component C of s, $U(s \upharpoonright_C) = U(s) \upharpoonright_{U(C)}$ (here subscripts of U should be chosen appropriately).

In the above, the forgetful function U (whose definition we omit) has a natural extension to a forgetful function on components.

Proof. Because U does not change the structure of justification pointers. □

3.2 Category of Two-Level Arenas and Innocent Strategies

Definition 4 (Innocent Strategies). A strategy $(\sigma, \bar{\sigma})$ of $\mathcal{A} = (A, U, K)$ is *innocent* just if σ and $\bar{\sigma}$ are innocent as strategies of A and K respectively.

Let $\mathcal{A}_i = (A_i, U_i, K_i)$ where $i = 1, 2$ be two-level arenas. We define product, function space and intersection constructions as follows.

Product $\mathcal{A}_1 \times \mathcal{A}_2 := (A_1 \times A_2, U, K_1 \times K_2)$, where $U : (M_{A_1} + M_{A_2}) \to (M_{K_1} + M_{K_2})$ is defined as $U_1 + U_2$.

Function Space $\mathcal{A}_1 \Rightarrow \mathcal{A}_2 := (A_1 \Rightarrow A_2, U, K_1 \Rightarrow K_2)$, where $U : ((M_{A_1} \times \mathit{Init}_{A_2}) + M_{A_2}) \to ((M_{K_1} \times \mathit{Init}_{K_2}) + M_{K_2})$ is defined as $U_1 \times U_2 + U_2$.

Intersection *Provided* $K_1 = K_2 = K$, define $\mathcal{A}_1 \wedge \mathcal{A}_2 := (A_1 \times A_2, U, K)$, where $U : (M_{A_1} + M_{A_2}) \to M_K$ is defined as $[U_1, U_2]$.

Definition 5 (Category of Two-Level Arenas and Innocent Strategies).

- Objects: Two-Level Arenas $\mathcal{A}$.
- Arrows of $\mathcal{A}_1 \to \mathcal{A}_2$: innocent strategies of $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$.
- Composition is defined component-wise. I.e. composition of $(\sigma_1, \bar{\sigma}_1) : \mathcal{A}_1 \Rightarrow \mathcal{A}_2$ and $(\sigma_2, \bar{\sigma}_2) : \mathcal{A}_2 \Rightarrow \mathcal{A}_3$ is defined as $(\sigma_1; \sigma_2,\ \bar{\sigma}_1; \bar{\sigma}_2) : \mathcal{A}_1 \Rightarrow \mathcal{A}_3$.

We first show that the composition of strategies is well-defined (Lemma 4 and Lemma 5).

Lemma 3. *Let $\mathcal{A}_i = (A_i, U_i, K_i)$ be a two-level arena for $i = 1, 2, 3$, $\sigma_1 : A_1 \Rightarrow A_2$ and $\sigma_2 : A_2 \Rightarrow A_3$ be strategies of coloured arenas. Then $U_{\mathcal{A}_1 \Rightarrow \mathcal{A}_3}(\sigma_1; \sigma_2) \subseteq U_{\mathcal{A}_1 \Rightarrow \mathcal{A}_2}(\sigma_1); U_{\mathcal{A}_2 \Rightarrow \mathcal{A}_3}(\sigma_2)$.*

Proof. Let $\bar{s} \in U(\sigma_1; \sigma_2)$. By definition, we have $s \in (\sigma_1; \sigma_2)$ such that $U(s) = \bar{s}$. Let $u \in \mathbf{Int}(\sigma_1, \sigma_2)$ be the uncovering of s, i.e. an interaction sequence that satisfies the following properties.

(i) $u \restriction_{(A_1, A_3)} = s$.
(ii) $u \restriction_{(A_2, A_3)} \in \sigma_2$.
(iii) For any initial A_2 move b in u, $u_A \restriction_{(A_1, A_2, b)} \in \sigma_1$.

Let $U_{(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)}$ be the forgetful map on interaction sequences and $\bar{u} = U_{(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)}(u)$. The following argument shows that $\bar{u} \in \mathbf{Int}(U(\sigma_1), U(\sigma_2))$ (here we use Lemma 2. Subscripts of U should be chosen appropriately).

(i) $\bar{u} \restriction_{(K_1, K_3)} = \bar{s}$, since

$$\bar{u} \restriction_{(K_1, K_3)} = U(u) \restriction_{U(\,(A_1, A_3)\,)} = U(u \restriction_{(A_1, A_3)}) = U(s) = \bar{s}.$$

(ii) $\bar{u} \restriction_{(K_2, K_3)} \in U\sigma_2$, since

$$\bar{u} \restriction_{(K_2, K_3)} = U(u) \restriction_{U(\,(A_2, A_3)\,)} = U(u \restriction_{(A_2, A_3)}) \in U(\sigma_2).$$

(iii) Let $(K_1, K_2, \bar{b})$ be a component of $\bar{u}$. There is a component (A_1, A_2, b) such that $U(\,(A_1, A_2, b)\,) = (K_1, K_2, \bar{b})$. Then

$$\bar{u} \upharpoonright_{(K_1,K_2,\bar{b})} = U(u) \upharpoonright_{U(\,(A_1,A_2,b)\,)} = U(u \upharpoonright_{(A_1,A_2,b)}) \in U(\sigma_1).$$

Therefore $\bar{s} = \bar{u} \upharpoonright_{(K_1,K_3)} \in (U(\sigma_1); U(\sigma_2))$. □

Lemma 4. *Let $\mathcal{A}_i = (A_i, U_i, K_i)$ be a two-level arena for $i = 1, 2, 3$, $(\sigma_1, \bar{\sigma_1}) : \mathcal{A}_1 \Rightarrow \mathcal{A}_2$ and $(\sigma_2, \bar{\sigma_2}) : \mathcal{A}_2 \Rightarrow \mathcal{A}_3$ be strategies. Then $(\sigma_1, \bar{\sigma}_1); (\sigma_2, \bar{\sigma}_2) = (\sigma_1; \sigma_2,\ \bar{\sigma}_1; \bar{\sigma}_2)$ is a strategy of $\mathcal{A}_1 \Rightarrow \mathcal{A}_3$.*

Proof. Obviously, $\sigma_1; \sigma_2$ is a strategy of $A_1 \Rightarrow A_3$ and $\bar{\sigma}_1; \bar{\sigma}_2$ be a strategy of $K_1 \Rightarrow K_3$. So it suffices to show that $U_{\mathcal{A}_1 \Rightarrow \mathcal{A}_3}(\sigma_1; \sigma_2) \subseteq (\bar{\sigma}_1; \bar{\sigma}_2)$. Since $(\sigma_1, \bar{\sigma}_1)$ is a strategy of $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$, we have $U_{\mathcal{A}_1 \Rightarrow \mathcal{A}_2}(\sigma_1) \subseteq \bar{\sigma}_1$. Similarly, $U_{\mathcal{A}_2 \Rightarrow \mathcal{A}_3}(\sigma_2) \subseteq \bar{\sigma}_2$. By Lemma 3 and monotonicity of composition, we have

$$U_{\mathcal{A}_1 \Rightarrow \mathcal{A}_3}(\sigma_1; \sigma_2) \subseteq (U_{\mathcal{A}_1 \Rightarrow \mathcal{A}_2}(\sigma_1); U_{\mathcal{A}_2 \Rightarrow \mathcal{A}_3}(\sigma_2)) \subseteq (\bar{\sigma}_1; \bar{\sigma}_2)$$

as required. □

Lemma 5. *If σ_1 and σ_2 are innocent, so is $\sigma_1; \sigma_2$.*

Theorem 2. *The category of two-level arenas and innocent strategies is cartesian closed.*

Proof. Trivial. □

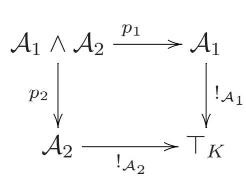
If two two-level arenas share the same base arena, then we can construct their *intersection*.

Definition 6 (Intersection of Two-Level Arenas). Let $\mathcal{A}_i = (A_i, U_i, K)$ for $i = 1, 2$ be two-level arenas that share the same base arena K. Their *intersection* $\mathcal{A}_1 \wedge \mathcal{A}_2$ is defined as $(A_1 \times A_2, U, K)$, where $U : (M_{A_1} + M_{A_2}) \to M_K$ is defined as $[U_1, U_2]$.

For every base arena K, we define $\top_K$ as the two-level arena $(\top, \emptyset, K)$, where $\top$ is the empty arena, which is the terminal object in the category of Q-coloured arenas.

For a Q-coloured arena A, we write $!_A$ for the unique strategy of $A \Rightarrow \top$. For a two-level arena $\mathcal{A} = (A, U, K)$, we define $!_{\mathcal{A}} : \mathcal{A} \Rightarrow \top_K$ as $(!_A, \mathrm{id}_K)$.

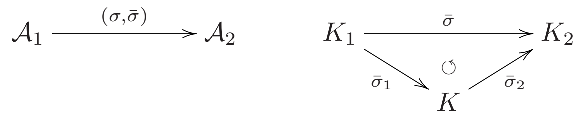
Lemma 6. *Let $\mathcal{A}_1 = (A_1, U_1, K)$ and $\mathcal{A}_2 = (A_2, U_2, K)$ be two-level arenas that share the same base arena K. The arena $\mathcal{A}_1 \wedge \mathcal{A}_2$ is the pullback of $\mathcal{A}_1$ and $\mathcal{A}_2$, i.e. there are innocent strategies p_1 and p_2 of two-level arenas that make the following diagram a pullback square.*



Proof. Taking $p_1 = (\pi_1, \mathrm{id}_K)$ and $p_2 = (\pi_2, \mathrm{id}_K)$. □

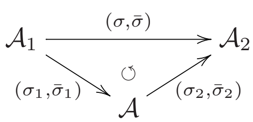
3.3 Subject Expansion

Theorem 3 (Subject Expansion). *Let $\mathcal{A}_i = (A_i, U_i, K_i)$ be a two-level arena for $i = 1, 2$ and K be a base arena. If*



(in two-level arenas) *(in base arenas)*

then there are a two-level arena $\mathcal{A}$ whose underlying kind arena is K and strategies $\sigma_1 : A_1 \to A$ and $\sigma_2 : A \to A_2$ such that



Moreover, there is a canonical triple $(\sigma_1, \mathcal{A}, \sigma_2)$: for every triple $(\sigma'_1, \mathcal{A}', \sigma'_2)$ that satisfies $\sigma'_1; \sigma'_2 = \sigma$, there exists a mapping φ from moves of $\mathcal{A}$ to moves of $\mathcal{A}'$ such that $[\mathrm{id}_{\mathcal{A}_1}, \varphi](\sigma_1) \subseteq \sigma'_1$ and $[\varphi, \mathrm{id}_{\mathcal{A}_2}](\sigma_2) \subseteq \sigma'_2$.

The key observation of the proof is that *innocent* strategies can (mostly) be reconstructed from their interaction sequences. Let σ_1 and σ_2 be innocent strategies of $A \Rightarrow B$ and $B \Rightarrow C$. Observe that since σ_2 is innocent, it is determined by the set of P-views in σ_2. Using $\mathbf{Int}(\sigma_1, \sigma_2)$ we define a set of P-views by $\varphi'_2 = \{\ulcorner s \upharpoonright_{B \Rightarrow C} \urcorner \mid s \in \mathbf{Int}(\sigma_1, \sigma_2)\}$. Then φ'_2 can be regarded as a view function, which determines an innocent strategy σ'_2. Similarly, we can construct a view function φ'_1 and an innocent strategy σ'_1. Then the resulting strategies σ'_1 and σ'_2 are respective *under-approximations* of σ_1 and σ_2 i.e. $\sigma'_1 \subseteq \sigma_1$ and $\sigma'_2 \subseteq \sigma_2$ and $\sigma'_1; \sigma'_2 = \sigma_1; \sigma_2$.

Now the goal is to construct "interaction sequences" of σ_1 and σ_2. There are two conditions that the set $\mathbf{Int}(\sigma_1, \sigma_2)$ of all interaction sequences must satisfy.

- $u \in \mathbf{Int}(\sigma_1, \sigma_2)$ implies $U(u) \in \mathbf{Int}(\bar{\sigma}_1, \bar{\sigma}_2)$.
- $u \in \mathbf{Int}(\sigma_1, \sigma_2)$ implies $u \upharpoonright_{\mathcal{A}_1 \Rightarrow \mathcal{A}_2} \in \sigma$.

These requirements give basic patterns of interaction sequences. Let $s \in \sigma$ and $\bar{u} \in \mathcal{A}(\bar{\sigma}_1, \bar{\sigma}_2)$ and $\bar{u} = \bar{m}_1 \cdot \bar{m}_2 \cdots \bar{m}_k$, such that $U(s) = \bar{u} \upharpoonright_{K_1 \Rightarrow K_2}$. Then a justified sequence of pairs of moves of base arenas and Q-coloured arenas

$$\begin{bmatrix} \bar{m}_1 \\ m_1 \end{bmatrix} \cdot \begin{bmatrix} \bar{m}_2 \\ m_2 \end{bmatrix} \cdots \begin{bmatrix} \bar{m}_k \\ m_k \end{bmatrix}$$

is called *annotated interaction sequences* generated by s and u if (i) $\bar{m}_i \in K_1 \cup K_2$ implies $U(m_i) = \bar{m}_i$, (ii) $(m_1 \cdot m_2 \ldots m_k) \upharpoonright_{A_1 \Rightarrow A_2} = s$, (iii) $\bar{m}_i \in K$ implies $m_i = \star$. An interaction sequence over σ_1 and σ_2 can be constructed by replacing $\star$ with appropriate moves.

Example 1. Let $\bar{\sigma}_1$ and $\bar{\sigma}_2$ be strategies defined by

$$\bar{\sigma}_1 = [\![c : o^1, a : o^2 \to o^3 \ \vdash \ (\lambda x.a(a(x)),\ c) : (o^4 \to o^5) \times o^6]\!]$$
$$\bar{\sigma}_2 = [\![f : o^4 \to o^5, x : o^6 \ \vdash \ f(f(x)) : o^7]\!],$$

where $[\![\cdot]\!]$ is the standard interpretation of the simply-typed lambda calculus. Their composition is equivalent to

$$[\![c : o^1, a : o^2 \to o^3 \ \vdash \ f(f(x))[(\lambda x.a(a(x)))/f, c/x] : o^7]\!]$$
$$= [\![c : o^1, a : o^2 \to o^3 \ \vdash \ a(a(a(a(c)))) : o^7]\!],$$

which have a derivation of a judgement

$$[\![c : q_0^P, a : (q_3^O \to q_4^P) \wedge (q_2^O \to q_3^P) \wedge (q_1^O \to q_2^P) \wedge (q_0^O \to q_1^P) \ \vdash \ a(a(a(a(c)))) : q_4^O]\!].$$

(Here o^1 and o^2 are different occurrences of the same kind, q_1 and q_2 are different types and q_1^P and q_1^O are different occurrences of the same type q_1.) Let σ be the strategy corresponding to the derivation. Then σ contains a play

$$s = q_4^O \cdot q_4^P \cdot q_3^O \cdot q_3^P \cdot q_2^O \cdot q_2^P \cdot q_1^O \cdot q_1^P \cdot q_0^O \cdot q_0^P$$

that is mapped by U to $U(s) = \bar{s} = o^7 \cdot o^3 \cdot o^2 \cdot o^3 \cdot o^2 \cdot o^3 \cdot o^2 \cdot o^3 \cdot o^2 \cdot o^1$; and $\mathbf{Int}(\bar{\sigma}_1, \bar{\sigma}_2)$ contains $\bar{u} = o^7 \cdot o^5 \cdot o^3 \cdot o^2 \cdot o^3 \cdot o^2 \cdot o^4 \cdot o^5 \cdot o^3 \cdot o^2 \cdot o^3 \cdot o^2 \cdot o^4 \cdot o^6 \cdot o^1$. Note that $U(s) = \bar{u} \upharpoonright_{K_1 \Rightarrow K_2}$. The annotated interaction sequence generated by s and $\bar{u}$ is

$$\begin{bmatrix} o^7 \\ q_4^O \end{bmatrix} \cdot \begin{bmatrix} o^5 \\ \star \end{bmatrix} \cdot \begin{bmatrix} o^3 \\ q_4^P \end{bmatrix} \cdot \begin{bmatrix} o^2 \\ q_3^O \end{bmatrix} \cdot \begin{bmatrix} o^3 \\ q_3^P \end{bmatrix} \cdot \begin{bmatrix} o^2 \\ q_2^O \end{bmatrix} \cdot \begin{bmatrix} o^4 \\ \star \end{bmatrix} \cdot \begin{bmatrix} o^5 \\ \star \end{bmatrix} \cdot \begin{bmatrix} o^3 \\ q_2^P \end{bmatrix} \cdots \begin{bmatrix} o^1 \\ q_0^P \end{bmatrix}.$$

The set of moves with which $\star$ is replaced should satisfy competing requirements. Occurrences of $\star$ should be distinguished as much as possible in order to fulfil the universal property, but distinguishing them too much makes σ_1 and σ_2 non-innocent strategies. A coloured arena $A = (M, \vdash, \lambda, c)$ is defined as follows.

- $M = \{ \binom{\bar{u}}{p} \mid \bar{u} \in \mathbf{Int}(\bar{\sigma}_1, \bar{\sigma}_2), \bar{u}$ ends with K-move$, p \in \sigma, \ulcorner p \urcorner = p, U(p) = \bar{u} \upharpoonright_{K_1 \Rightarrow K_2} \}$
- $\vdash \binom{\bar{u}}{p}$ iff the last move of $\bar{u}$ is an initial move of K; $\binom{\bar{u}}{p} \vdash \binom{\bar{u}'}{p'}$ iff (i) p is a prefix of p', and (ii) The last move of $\bar{u}'$ is justified by the last move of $\bar{u}$.
- $\lambda(\binom{\bar{u}}{p}) = \lambda_K(\bar{m})$ where $\bar{m}$ is the last move of $\bar{u}$; and $c(\binom{\bar{u}}{p}) = c_{A_1 \Rightarrow A_2}(m)$ where m is the last move of p.

The two-level arena $\mathcal{A}$ is defined as (A, U, K) where $U(\binom{\bar{u}}{p}) = \bar{m}$ (here $\bar{m}$ is the last move of $\bar{u}$).

Definitions of $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$ For each pair $(p, \bar{u}) \in \sigma \times \mathbf{Int}(\bar{\sigma}_1, \bar{\sigma}_2)$ such that p is a P-view and $U(p) = \bar{u} \upharpoonright_{K_1 \Rightarrow K_2}$, we construct an interaction sequence of $\mathrm{Int}(\mathcal{A}_1, \mathcal{A}, \mathcal{A}_2)$, written $\langle \bar{u}, p \rangle$. Basically, $\langle \bar{u}, p \rangle$ is generated by replacing $\star$ in the annotated interaction sequence with appropriate moves of $\mathcal{A}$. $\langle \bar{u}, p\rangle$ is defined by induction on $\bar{u}$ as follows:

$$\langle \bar{u}\cdot\bar{m}, p\rangle = \langle \bar{u}, p\rangle \cdot \begin{pmatrix} \bar{u}\cdot\bar{m} \\ p \end{pmatrix} \qquad (\text{if } \bar{m} \in K)$$
$$\langle \bar{u}\cdot\bar{m}, p\cdot m\rangle = \langle \bar{u}, p\rangle \cdot m \qquad (\text{if } \bar{m} \in K_1 \Rightarrow K_2)$$

where justification pointers are induced from $\bar{u}\cdot\bar{m}$.

Lemma 7. *Let p be a P-view of $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$ and $\bar{u} \in \mathrm{Int}(K_1, K, K_2)$ be an interaction sequence and assume that $U(p) = \bar{u} \restriction_{K_1 \Rightarrow K_2}$. Then $\langle \bar{u}, p\rangle \in \mathrm{Int}(\mathcal{A}_1, \mathcal{A}, \mathcal{A}_2)$.*

Proof. By induction on the length of $\bar{u}$.

We define $I \subseteq \mathrm{Int}(\mathcal{A}_1, \mathcal{A}, \mathcal{A}_2)$ by

$$I = \{\, \langle \bar{u}, p\rangle \mid \bar{u} \in \mathbf{Int}(\bar{\sigma}_1, \bar{\sigma}_2),\ p \in \sigma,\ \ulcorner p \urcorner = p,\ U(p) = u \restriction_{K_1 \Rightarrow K_2} \}\,.$$

Now we define strategies. Let φ_1 be an view function of an arena $A_1 \Rightarrow A$ determined by a set of P-views $\{\ulcorner s \restriction_{\mathcal{A}_1 \Rightarrow \mathcal{A}} \urcorner \mid s \in I\}$ and φ_2 be a view function of an arena $A \Rightarrow A_2$ determined by $\{\ulcorner s \restriction_{\mathcal{A} \Rightarrow \mathcal{A}_2} \urcorner \mid s \in I\}$. The strategy σ_1 is induced from φ_1 and σ_2 from φ_2.

We show that σ_1 and σ_2 are well-defined. We need an auxiliary lemma.

Lemma 8. *Let p be a P-view of $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$ and $\bar{u} \in \mathrm{Int}(K_1, K, K_2)$ be an interaction sequence and assume that $U(p) = \bar{u} \restriction_{K_1 \Rightarrow K_2}$.*

(i) If $\bar{u}$ ends with an O-move of $K_1 \Rightarrow K$, then $\langle \bar{u}, p\rangle$ can be determined by $\ulcorner \langle \bar{u}, p\rangle \restriction_{A_1 \Rightarrow A} \urcorner$.
(ii) If $\bar{u}$ ends with an O-move of $K \Rightarrow K_2$, $\langle \bar{u}, p\rangle$ can be determined by $\ulcorner \langle \bar{u}, p\rangle \restriction_{A \Rightarrow A_2} \urcorner$.

Proof. We prove (i). (ii) is shown by a similar way.

If $\bar{u}$ ends with a move of $\mathcal{A}$, then the last move contains as much information as the pair $(\bar{u}, p)$. Assume that $\bar{u}$ ends with a move of K_1. Then there are moves $\bar{m}_1^P$ and $\bar{m}_2^O$ and some justified sequence $\bar{v}$ such that $\bar{u} = \bar{u}' \cdot \bar{m}_1^P \cdot \bar{v} \cdot \bar{m}_2^O$, where $\bar{m}_1^P$ justifies $\bar{m}_2^O$. Note that $K_1 \Rightarrow K_2$ component of $\bar{u}$ is a P-view by the assumption. Thus $\bar{v}$ contains no moves in $K_1 \Rightarrow K_2$. Since $U(p) = \bar{u} \restriction_{K_1 \Rightarrow K_2} = \bar{u}' \cdot \bar{m}_1^P \cdot \bar{v} \cdot \bar{m}_2^O \restriction_{K_1 \Rightarrow K_2}$, there are some moves $m_1^P, m_2^O \in A_1 \Rightarrow A_2$ and a P-view p' of $A_1 \Rightarrow A_2$ such that $p = p' \cdot m_1^P \cdot m_2^O$ and $U(m_1^P) = \bar{m}_1^P$ and $U(m_2^O) = \bar{m}_2^O$. Therefore we have

$$\langle \bar{u}, p\rangle = \langle \bar{u}', p'\rangle \cdot m_1^P \cdot m_2^O.$$

Since m_2^O is an O-move of A_1,

$$\ulcorner \langle \bar{u}, p\rangle \urcorner = \ulcorner \langle \bar{u}', p'\rangle \cdot m_1^P \cdot m_2^O \urcorner = \ulcorner \langle \bar{u}', p'\rangle \urcorner \cdot m_1^P \cdot m_2^O\,.$$

By induction hypothesis, we can compute $\langle \bar{u}', p'\rangle$ from $\ulcorner \langle \bar{u}', p'\rangle \urcorner$. Thus (i) holds. □

Lemma 9. *φ_1 and φ_2 are well-defined view functions.*

Proof. We prove that φ_1 is well-defined. Well-definedness of φ_2 is shown by the same way.

Let $s \in \varphi_1$ be an play ending with an O-move. What we should show are:

(*i*) $s \cdot m \in \varphi_1$ for some m.
(*ii*) If $s \cdot m \in \varphi_1$ and $s \cdot m' \in \varphi_1$, then $s \cdot m = s \cdot m'$.

(i) is easy to show because for every $u \in I$ ending with an O-move of $A_1 \Rightarrow A$, we have $u \cdot m$ for some move m of $A_1 \Rightarrow A$. We prove (ii). Assume that $s \cdot m \in \varphi_1$ and $s \cdot m' \in \varphi_1$. By definition of φ_1, we have $u \cdot m$, $u' \cdot m' \in I$ such that $\ulcorner (u \cdot m) \upharpoonright_{A_1 \Rightarrow A} \urcorner = s \cdot m$ and $\ulcorner (u' \cdot m') \upharpoonright_{A_1 \Rightarrow A} \urcorner = s \cdot m'$. Since s is ending with an O-move of $A_1 \Rightarrow A$, by Lemma 8, s completely determines u and u'. Thus $u = u'$ and $u \cdot m' \in \varphi_1$. By determinacy of σ, $\bar{\sigma}_1$ and $\bar{\sigma}_2$, we have $u \cdot m = u \cdot m'$ as required. □

Thanks to Lemma 9, σ_1 and σ_2 are well-defined innocent strategies. Trivially, $U(\sigma_1) \subseteq \bar{\sigma}_1$ and $U(\sigma_2) \subseteq \bar{\sigma}_2$.

The following property of σ_1 and σ_2 are easy to show.

Lemma 10. *If σ is colour-reflecting, then so are σ_1 and σ_2.*

Lemma 11. $(\sigma_1; \sigma_2) = \sigma$.

Proof. We first prove that $\sigma \subseteq (\sigma_1; \sigma_2)$. Since σ is innocent, it suffices to show that $\sigma_1; \sigma_2$ contains every P-view $p \in \sigma$. Let $p \in \sigma$ be a P-view. Then $U(p) \in \bar{\sigma} = (\bar{\sigma}_1; \bar{\sigma}_2)$. Thus there is $\bar{u} \in \mathbf{Int}(\bar{\sigma}_1, \bar{\sigma}_2)$ such that $U(p) = \bar{u} \upharpoonright_{K_1 \Rightarrow K_2}$. By definition, $\langle \bar{u}, p \rangle \in I$. We can prove $\langle \bar{u}, p \rangle \in \mathbf{Int}(\sigma_1, \sigma_2)$ by induction on the length of $\bar{u}$. So $p = \langle \bar{u}, p \rangle \upharpoonright_{A_1 \Rightarrow A_2} \in (\sigma_1; \sigma_2)$.

Second we prove that $(\sigma_1; \sigma_2) \subseteq \sigma$. It suffices to show that $p \in \sigma$ for every P-view $p \in (\sigma_1; \sigma_2)$. Since $p \in (\sigma_1; \sigma_2)$, we have its uncovering $u \in \mathbf{Int}(\sigma_1, \sigma_2)$ (so $p = u \upharpoonright_{A_1 \Rightarrow A_2}$). Then by induction on the length of u, we can prove that $u \upharpoonright_{A_1 \Rightarrow A_2} \in \sigma$. □

Thanks to Lemma 11, we have finished to construct an object $\mathcal{A}$ and strategies σ_1 and σ_2, required by Theorem 3.

Canonicity of $\mathcal{A}$ Let $(\sigma'_1, \mathcal{A}', \sigma'_2)$ be another triple that satisfies the requirement of subject expansion except for canonicity. So $\sigma'_1 : \mathcal{A}_1 \to \mathcal{A}'$, $\sigma'_2 : \mathcal{A}' \to \mathcal{A}_2$, $U(\sigma'_i) \subseteq \bar{\sigma}_i$ (for $i = 1, 2$) and $\sigma = (\sigma'_1; \sigma'_2)$.

What we should do is construction of a function from moves of A to moves of A'. Let $\binom{\bar{u}}{p}$ be a move of A. Note that by definition p ends with an O-move of $A_1 \Rightarrow A_2$. Let m be the move such that $p \cdot m \in \sigma$. Since $p \cdot m \in \sigma = (\sigma'_1; \sigma'_2)$, we have the uncovering of $p \cdot m$ over σ'_1 and σ'_2, say u'. Since $U(\sigma'_1) \subseteq \bar{\sigma}_1$ and $U(\sigma'_2) \subseteq \bar{\sigma}_2$, we have $U(u') \in (\bar{\sigma}_1; \bar{\sigma}_2)$. So $\langle \bar{u'}, p \cdot m \rangle$ is an postfix of $\langle \bar{u}, p \rangle$. Thus $\langle \bar{u'}, p \cdot m \rangle$ contains the move $\binom{\bar{u}}{p}$, say, as the kth move. Let m' be the kth move of u'. We map $\binom{\bar{u}}{p}$ to m'.

Let φ be the mapping defined below. It is easy to prove that φ is well-defined. $[\mathrm{id}_{K_1}, \varphi](\sigma_1) \subseteq \sigma'_1$ (resp. $[\varphi, \mathrm{id}_{K_2}](\sigma_2) \subseteq \sigma'_2$) can be show by induction on the length of plays in σ_1 (resp. σ_2).

4 Interpretation of Intersection Types

In this section, we interpret Kobayashi's intersection type system [3] in the two-level game model. In [3], an important notion is a well-formedness of an intersection type with respect to a simple type. Intersection types such as $(q_1 \wedge q_2) \to p$ and $(q_1 \to p_1) \wedge (q_2 \to p_2)$ are well-formed and their underlying structures are represented by a simple type $o \to o$. In contrast, the type $(q_1 \to q_2) \wedge p$ is ill-formed since $\wedge$ connects types of different shape: the left argument has the shape $o \to o$ and the right has o.

We interpret a derivation of a sequent $\vdash t : \tau :: \kappa$ as a (winning) strategy $\sigma = (\mathtt{C}(\sigma), \mathtt{K}(\sigma))$ of a two-level arena. The components $\mathtt{K}(\sigma)$ and $\mathtt{C}(\sigma)$ represent the term t and the derivation respectively; the condition $U(\mathtt{C}(\sigma)) \subseteq \mathtt{K}(\sigma)$ ensures coherence of the structures, corresponding to the side condition of the introduction rule of the intersection. We show that the interpretation is *fully complete* i.e. every winning strategy is the interpretation of some derivation.

Lastly, we briefly see how the subtyping relation can be characterised in this framework.

4.1 An Intersection Type System

We consider the standard Church-style simply-typed lambda calculus defined by the following grammar:

$$\begin{array}{lrcl} \textit{Sorts} & \kappa & ::= & o \mid \kappa_1 \to \kappa_2 \\ \textit{Terms} & t & ::= & x \mid \lambda x^{\kappa}.t \mid t_1 t_2 \end{array}$$

We refer to simple types as *kinds* to avoid confusion with intersection types. Let Δ be a *kind environment* i.e. a set of variable-kind bindings, $x : \kappa$. We write $\Delta \vdash t :: \kappa$ to mean t has kind κ under the environment Δ. Fix a set Q of symbols, ranged over by q. The set of *intersection pre-types* is defined by the following grammar where $n \geq 0$:

$$\textit{Intersection Pre-Types} \qquad \tau, \sigma \quad ::= \quad q \;\mid\; \tau \to \sigma \;\mid\; \bigwedge_{i \in I} \tau_i$$

The *well-kindedness relation* $\tau :: \kappa$ is defined by the following rules.

$$\frac{}{q :: o}$$

$$\frac{\tau_i :: \kappa \quad (\text{for all } i \in I) \qquad \sigma :: \kappa'}{(\bigwedge_{i \in I} \tau_i) \to \sigma :: \kappa \to \kappa'}$$

An *intersection type* is an intersection pre-type τ such that $\tau :: \kappa$ for some κ.

An *(intersection) type environment* Γ is a set of variable-type bindings, $x : \bigwedge_{i \in I} \tau_i$. We write $\Gamma :: \Delta$ just if $x : \bigwedge_{i \in I} \tau_i \in \Gamma$ implies that for some κ, $x : \kappa \in \Delta$

and $\tau_i :: \kappa$ for all $i \in I$. *Valid typing sequents* are defined by induction over the following rules.

$$\overline{\Gamma, x : \bigwedge_{i\in I} \tau_i \vdash x : \tau_i}$$

$$\frac{\Gamma \vdash t_1 : (\bigwedge_{i\in I} \tau_i) \to \sigma \qquad \Gamma \vdash t_2 : \tau_i \quad (\text{for all } i \in I)}{\Gamma \vdash t_1\ t_2 : \sigma}$$

$$\frac{\Gamma, x : \bigwedge_{i\in I} \tau_i \vdash t : \sigma \qquad \tau_i :: \kappa \quad (\text{for all } i \in I)}{\Gamma \vdash \lambda x^{\kappa}.t : (\bigwedge_{i\in I} \tau_i) \to \sigma}$$

Lemma 12. *If $\Delta \vdash t :: \kappa$ and $\Gamma :: \Delta$ and $\Gamma \vdash t : \tau$, then $\tau :: \kappa$.*

4.2 Representing Derivations by Proof Terms

For notational convenience, we use a Church-style simply-kinded lambda calculus with (indexed) product as a term representation of derivations. The raw terms are defined as follows.

$$M \ ::= \ \mathsf{p}_i(x) \ \mid \ \lambda x^{\bigwedge_{i\in I} \tau_i}.M \ \mid \ M_1\, M_2 \ \mid \ \prod_{i\in I} M_i$$

where I is a finite indexing set. We omit I and simply write $\lambda x^{\bigwedge_i \tau_i}$ and so on if I is clear from the context or unimportant. We say a term M is *well-formed* just if for every application subterm $M_1\, M_2$ of M, M_2 has the form $\prod_{i\in I} N_i$. We consider only well-formed terms. By abuse of notation, we write $\top$ for $\prod \emptyset$.

We give a type system for terms of the calculus, which ressemble the intersection type system, but is syntax directed, i.e., a term completely determines the structure of a derivation.

$$\overline{\Gamma, x : \bigwedge_{i\in I} \tau_i \Vdash \mathsf{p}_i(x) : \tau_i}$$

$$\frac{\Gamma, x : \bigwedge_{i\in I} \tau \Vdash M : \sigma}{\Gamma \Vdash \lambda x^{\bigwedge_{i\in I} \tau_i}.M : (\bigwedge_{i\in I} \tau_i) \to \sigma}$$

$$\frac{\Gamma \Vdash M_1 : (\bigwedge_i \tau_i) \to \sigma \qquad \Gamma \Vdash M_2 : \bigwedge_i \tau_i}{\Gamma \Vdash M_1\ M_2 : \sigma}$$

$$\frac{\Gamma \Vdash M_i : \tau_i \qquad \tau_i :: \kappa \qquad (\text{for all } i)}{\Gamma \Vdash \prod_i M_i : \bigwedge_i \tau_i}$$

We call a term-in-context $\Gamma \Vdash M : \tau$ a *proof term.* Observe that a proof term is essentially a typed lambda term with (indexed) product. Here an intersection type $\tau_1 \wedge \cdots \wedge \tau_n$ is interpreted as a product type $\tau_1 \times \cdots \times \tau_n$ and a proof term

$M_1 \sqcap \cdots \sqcap M_n$ is a tuple $\langle M_1, \ldots, M_n \rangle$. Then all variables are bound to tuples and a proof term $\mathsf{p}_i(x)$ is a projection into the ith element.

Unfortunately, not all the proof terms correspond to a derivation of the intersection type system. For example, $\lambda f^{(q_1 \wedge q_2) \to p}.\lambda x^{q_1}.\lambda y^{q_2}.f(\mathsf{p}(x) \sqcap \mathsf{p}(y))$ is a proof term of the type $((q_1 \wedge q_2) \to p) \to q_1 \to q_2 \to p$, but there is no inhabitant of that type. In the intersection type system, $t : \tau \wedge \sigma$ only if $t : \tau$ and $t : \sigma$ for the same term t, but the proof term $\mathsf{p}(x) \sqcap \mathsf{p}(y)$ violates the requirement.

We introduce a judgement $M :: t$ that means the structure of M coincides with the structure of t.

$$\begin{array}{rll} \mathsf{p}_i(x) :: x & & \\ \lambda x^{\bigwedge_i \tau_i}.M :: \lambda x^{\kappa}.t & \text{iff} & M :: t \text{ and } \tau_i :: \kappa \text{ for all } i \\ M_1\ M_2 :: t_1\ t_2 & \text{iff} & M_1 :: t_1 \text{ and } M_2 :: t_2 \\ \bigsqcap_i M_i :: t & \text{iff} & M_i :: t \text{ for all } i \end{array}$$

By definition, $\top :: t$ for every term t.

Lemma 13 (Coincidence). *(i) For every derivations $\mathcal{D}$ whose conclusion is $\Gamma \vdash t : \tau$, there exists a proof term* $\mathrm{Term}(\mathcal{D})$ *such that $\Gamma \Vdash \mathrm{Term}(\mathcal{D}) : \tau$ and* $\mathrm{Term}(\mathcal{D}) :: t$. *(ii) If $\Gamma \Vdash M : \tau$ and $M :: t$, there exists a unique derivation $\mathcal{D}$ of $\Gamma \vdash t : \tau$ such that* $\mathrm{Term}(\mathcal{D}) = M$.

Proof. Easy induction on the structure of $\mathcal{D}$ and of M, respectively.

We write $[\Gamma :: \Delta] \vdash [M :: t] : [\tau :: \kappa]$ just if $\Gamma :: \Delta$, $M :: t$, $\tau :: \kappa$, $\Delta \vdash t : \kappa$ and $\Gamma \Vdash M : \tau$. Let t be a term such that $\Delta \vdash t :: \kappa$. The previous lemma says that there is a one-one correspondence between a derivation of $\Gamma \vdash t : \tau$ and a proof term M such that $[\Gamma :: \Delta] \vdash [M :: t] : [\tau :: \kappa]$.

Lemma 14. *It is decidable, given $\Gamma \Vdash M : \tau$, whether $M :: t$ for some t. Hence, thanks to Lemma 13, it is decidable whether a proof term represents a derivation.*

Proof. The definition of $M :: t$ itself gives a simple decision procedure.

We define an operational semantics for terms and proof terms. The reduction relation is the least congruence defined by the following β-reduction and η-expansion redex rules:

$$\begin{array}{l} (\lambda x^{\kappa}.s)\ t \longrightarrow_{\beta} [t/x]\, s \\ \quad t^{\kappa_1 \to \kappa_2} \longrightarrow_{\eta} \lambda x^{\kappa_1}.(t^{\kappa_1 \to \kappa_2}\ x^{\kappa_1})^{\kappa_2} \qquad (x \text{ is fresh}) \end{array}$$

Here $[t'/x]$ is the standard capture-avoiding substitution of t' for x. We write $\longrightarrow$ for $\longrightarrow_{\beta} \cup \longrightarrow_{\eta}$, $\longrightarrow^*$ for reflexive and transitive closure of $\longrightarrow$, and $=_{\beta\eta}$ for reflexive, transitive and symmetric closure of $\longrightarrow$.

The reduction relation of proof terms is defined similarly:

$$\begin{array}{l} (\lambda x^{\bigwedge_i \tau_i}.M)\ (\bigsqcap_i N_i) \longrightarrow_{\beta} [\bigsqcap_i N_i/x]\ M \\ \qquad\qquad M^{\bigwedge_i \tau_i \to \sigma} \longrightarrow_{\eta} \lambda x^{\bigwedge_i \tau_i}.M\ (\bigsqcap_i \mathsf{p}_i(x)) \quad (x \text{ is fresh}) \end{array}$$

where (the base case of) the substitution is given by

$$[\prod_i N_i/x]\ (\mathsf{p}_i(x)) = N_i$$
$$[\prod_i N_i/x]\ (\mathsf{p}_i(y)) = \mathsf{p}_i(y) \quad (\text{if } x \neq y).$$

We write $[M :: t] \longrightarrow [M' :: t']$ if $t \longrightarrow t'$ and $M \longrightarrow^* M'$. It is easy to see that if $M :: t$ and $t \longrightarrow t'$, then there exists a unique M' such that $M \longrightarrow^* M'$ and $M' :: t'$ ($M \longrightarrow^* M'$ reduces all the redexes at the positions similar to the redex of $t \longrightarrow t'$).

4.3 Game Semantics of Intersection Types

A two-level arena represents a proof of well-kindedness, $\tau :: \kappa$. The interpretation is straightforward since we have arena constructors $\Rightarrow$ and $\wedge$:

$$\begin{aligned}
[\![q :: o]\!] &:= ([\![q]\!], U, [\![o]\!]) \\
[\![(\bigwedge_{i \in I} \tau_i) \to \sigma :: \kappa \to \kappa']\!] &:= (\bigwedge_{i \in I} [\![\tau_i :: \kappa]\!]) \Rightarrow [\![\sigma :: \kappa']\!],
\end{aligned}$$

where $[\![q]\!]$ is a Q-coloured arena with a single move of the colour q, $[\![o]\!]$ is a $\{o\}$-coloured arena with a single move, and U maps the unique move of $[\![q]\!]$ to the unique move of $[\![o]\!]$. Let Γ be a type environment with $\Gamma :: \Delta$. Suppose

$$\begin{aligned}
\Gamma &= x_1 : \textstyle\bigwedge_{i \in I_1} \tau_i^1, \ldots, x_n : \bigwedge_{i \in I_n} \tau_i^n \\
\Delta &= x_1 : \kappa_1, \qquad\quad \ldots, x_n : \kappa_n
\end{aligned}$$

Then $[\![\Gamma :: \Delta]\!] := \prod_{j \leq n} (\bigwedge_{i \in I_j} [\![\tau_i^j :: \kappa_i]\!])$.

A proof $[\Gamma :: \Delta] \vdash [M :: t] : [\tau :: \kappa]$, which is equivalent to a derivation of $\Gamma \vdash t : \tau$ (Lemma 13), is interpreted as a strategy of the two-level arena $[\![\Gamma :: \Delta]\!] \Rightarrow [\![\tau :: \kappa]\!]$, defined by the following rules (for simplicity, we write $[\![M :: t]\!]$ instead of $[\![[\Gamma :: \Delta] \vdash [M :: t] : [\tau :: \kappa]]\!]$):

$$\begin{aligned}
[\![\mathsf{p}_i(x) :: x]\!] &:= \pi_x ; \mathsf{p}_i \\
[\![\prod_i M_i :: t]\!] &:= \prod_i [\![M_i :: t]\!] \\
[\![M_1\ M_2 :: t_1\ t_2]\!] &:= \langle [\![M_1 :: t_1]\!], [\![M_2 :: t_2]\!] \rangle ; \mathbf{eval} \\
[\![\lambda x.M :: \lambda x.t]\!] &:= \Lambda([\![M :: t]\!])
\end{aligned}$$

where π_x is the projection $[\![(\Gamma, x : \bigwedge_i \tau_i) :: (\Delta, x : \kappa)]\!] \longrightarrow [\![\bigwedge_i \tau_i :: \kappa]\!]$ and for strategies $\sigma_i : [\![\Gamma :: \Delta]\!] \longrightarrow [\![\tau_i :: \kappa]\!]$ indexed by i, the strategy $\prod_i \sigma_i : [\![\Gamma :: \Delta]\!] \longrightarrow \bigwedge_i [\![\tau_i :: \kappa]\!]$ is the canonical map of the pullback.

Lemma 15 (Componentwise Interpretation). *Let* $[\Gamma :: \Delta] \vdash [M :: t] : [\tau :: \kappa]$ *be a derivation. Then* $[\![M :: t]\!] = ([\![M]\!], [\![t]\!])$.

Proof. By induction on M.

Lemma 16 (Substitution). *Suppose* $[(\Gamma, x : \bigwedge_i \tau_i) :: (\Delta, x : \kappa)] \vdash [M :: t] : [\sigma :: \kappa']$ *and* $[\Gamma :: \Delta] \vdash [\prod_i N_i :: u] : [\bigwedge_i \tau_i :: \kappa]$*. Then*

$$\langle \mathbf{id}_{[\![\Gamma :: \Delta]\!]}, [\![\prod_i N_i :: u]\!]\rangle ; [\![M :: t]\!] = [\![([\prod_i N_i/x]\ M) :: ([u/x]\ t)]\!].$$

Proof. By Lemma 15 and a well-know result for the standard interpretation [**?**].

Lemma 17. *Suppose* $[\Gamma :: \Delta] \vdash [M :: t] : [\bigwedge_i \tau_i \to \sigma :: \kappa \to \kappa']$*. Then*

$$[\![M :: t]\!] = [\![(\lambda x^{\bigwedge_i \tau_i}.M\ (\prod_i \mathsf{p}_i(x))) :: (\lambda x^{\kappa}.t\ x)]\!].$$

Theorem 4 (Adequacy). *Let* $[\Gamma :: \Delta] \vdash [M_1 :: t_1] : [\tau :: \kappa]$ *and* $[\Gamma :: \Delta] \vdash [M_2 :: t_2] : [\tau :: \kappa]$ *be two proofs such that* $[M_1 :: t_1] =_{\beta\eta} [M_2 :: t_2]$*. Then* $[\![M_1 :: t_1]\!] = [\![M_2 :: t_2]\!]$*.*

Proof. A consequence of the two lemmas above.

Theorem 5 (Definability). *Let* $(\sigma, \bar{\sigma}) : [\![\Gamma :: \Delta]\!] \to [\![\tau :: \kappa]\!]$ *be a winning strategy. There is a derivation* $[\Gamma :: \Delta] \vdash [M :: t] : [\tau :: \kappa]$ *such that* $(\sigma, \bar{\sigma}) = [\![M :: t]\!]$*.*

Proof. We can assume without loss of generality that $\tau = q$ for some q. By induction on size of the domain of the view function f_σ. By the condition on σ, it is at least 1. Suppose $\Gamma = \prod_{j \in J}(\bigwedge_{i \in I_j} \tau_i^j)$.

Assume the domain of f_σ is of size 1. Let m be the unique element of the domain of f_σ. Then m is the initial move of $[\![q :: o]\!]$. Let $m' = f_\sigma(m)$. Then m' is an initial move of $[\![\tau_i^j]\!]$ for some i and j. It is the case that $\tau_i^j = \bigwedge \emptyset \to \cdots \to \bigwedge \emptyset \to q'$ for some q' (if not, $mm'm'' \in \sigma$ for some m'' by contingent completeness and thus $mm'm''m''' \in \sigma$ for some m''' that contradict to the assumption). Since $c(m) = c(m')$, we have $q = q'$. Let $t = x_j \top \dots \top$ and

$$\mathcal{D} = \frac{\Gamma \Vdash x_j : \bigwedge \emptyset \to \cdots \to \bigwedge \emptyset \to q}{\Gamma \Vdash x_j \top \dots \top : q}.$$

It is easy to check that $[\![\mathcal{D}]\!] = \sigma$.

Assume that the domain of f_σ is of size more than 1. Similar to the above argument, $\varepsilon, m, mm' \in \sigma$, where m is the unique initial move of $[\![\Gamma]\!] \Rightarrow [\![q]\!]$ and m' is an initial move of $[\![\tau_i^j]\!]$ for some i and j, and all other legal positions in σ are post-fixes of mm' by determinacy and uniqueness of the initial move m. Let $\tau_j^i = \bigwedge_{l \in L_1} \theta_l^1 \to \cdots \to \bigwedge_{l \in L_n} \theta_l^n \to q$ (recall that $c(m) = c(m')$). There is a bijective correspondence between sets of legal positions

$$\{s \mid mm's \text{ is a legal position of } [\![\Gamma]\!] \Rightarrow [\![q]\!]\}$$

and

$$\{s \mid s \text{ is a legal position of } [\![\Gamma]\!] \Rightarrow \textstyle\prod_k [\![\bigwedge_l \theta_l^k]\!]\}\ .$$

This bijection gives a strategy σ_l^k of $[\![\Gamma]\!] \Rightarrow \theta_l^k$ for each k and l. By induction hypothesis, we have a term t_l^k and a derivation $\mathcal{D}_l^k$ of $\Gamma \Vdash t_l^k : \sigma_l^k$ such that $[\![\mathcal{D}_l^k]\!] = \sigma_l^k$ and $U(\sigma_l^k) = [\![t_l^k]\!]$. If $t_l^k \sim t_{l'}^k$ for all k, l and l', by applying the rule for application as needed, we have a derivation $\mathcal{D}$ of $\Gamma \Vdash x_i^j \, (\bigsqcap_l t_l^1) \ldots (\bigsqcap_l t_l^n) : q$. To show $t_l^k \sim t_{l'}^k$, by Lemma **??**, it suffices to show that $\sigma_l^k \cup \sigma_{l'}^k$ is a strategy. That follows from the fact that σ is a strategy.

It is easy to check that $[\![\mathcal{D}]\!] = \sigma$.

Example 2. Let $Q = \{ q_1, q_2 \}$ and take $\theta \to (q_1 \wedge q_2) \to q_1 :: (o \to o) \to o \to o$ where $\theta = (q_1 \to q_1) \wedge (q_2 \to q_1) \wedge (q_1 \wedge q_2 \to q_1)$ and terminal $f : q_1 \to q_2$. Set $M := \lambda x^\theta \, y^{q_1 \wedge q_2}.\mathsf{p}_2(x)(f^{q_1 \to q_2}(\mathsf{p}_1(x)(\mathsf{p}_3(x)(\mathsf{p}_1(y) \sqcap \mathsf{p}_2(y)))))$. Then we have $M :: \lambda xy.x(f(x(x\,y))$.

We can use Church-style type-annotated terms in β-normal η-long form, called *canonical terms*, to represent winning strategies, which are terms-in-context of the form: $\Gamma \Vdash \mathsf{p}_i(x)\, M_1 \cdots M_n : q$ where $\Gamma = \cdots, x : \bigwedge_i \alpha_i, \cdots$ and $\alpha_i = \tau_1 \to \cdots \to \tau_n \to q$, and for each $k \in \{ 1, \ldots, n \}$,

$$M_k \; = \; \bigsqcap_{j \in J_k} \lambda y_{kj1}^{\tau_{kj1}} \ldots y_{kjr}^{\tau_{kjr}}.N_{kj} \; : \; \bigwedge_{j \in J_k} \beta_{kj} \; = \; \tau_k$$

such that for each $j \in J_k$, $\beta_{kj} = \tau_{kj1} \to \cdots \to \tau_{kjr} \to q_{kj}$ with $r = r_{kj}$ and $\Gamma, y_{kj1} : \tau_{kj1}, \cdots, y_{kjr} : \tau_{kjr} \Vdash N_{kj} : q_{kj}$ is a canonical term. (We assume that canonical terms are proof terms that represent derivations.)

By definition, canonical terms are not λ-abstractions. We call terms-in-context such as t_k above canonical terms in (partially) *curried form*; they have the shape $\Gamma \Vdash \lambda\overline{x}.M : \tau_1 \to \cdots \to \tau_n \to q$. Note that in case $n = 0$, the curried form retains an outermost "dummy lambda" $\Gamma \Vdash \lambda.M : q$. With this syntactic convention, we obtain a tight correspondence between syntax and semantics.

Lemma 18. *Let $\tau :: \kappa$ where $\tau = \tau_1 \to \ldots \to \tau_n \to q$. There is a one-one correspondence between winning strategies over the two-level arena $[\![\tau :: \kappa]\!]$ and canonical terms of the shape $x_1 : \tau_1, \ldots, x_n : \tau_n \Vdash M : q$ (with η-long β-normal simply-typed term t such that $M :: t$).*

Proof. First observe that a two-level arena is a forest; each move of the arena can be represented by the subtree rooted at the move. In other words, moves of $[\![\tau :: o]\!]$ correspond to (and can be named by) the prime subtypes of τ. Consider the abstract syntax trees of these terms, so that the nodes at levels 0, 2, 4, etc. are labelled by lambdas (i.e. $\lambda\overline{x}$), and nodes at levels 1, 3, 5, etc. are labelled by variables. The idea is that a node labelled by a lambda (respectively variable) of prime type θ represents the O-move (respectively P-move) named by θ. It suffices to observe that there is a one-one correspondence between the even-length paths in such a tree, and the even-length P-views in the corresponding winning strategy. (Note that an innocent strategy—*qua* set of legal positions—is determined by its subset of even-length P-views, which is just its view function.) We check that the term representation satisfies the axioms of winning strategy. P/O-alternation holds by construction of the canonical term; pointers to

O-moves correspond to the standard lambda binding, and pointers to P-move correspond to the edges from a lambda node to its parent, which is a variable node. Colour-reflection, totality (leaves of a tree are by construction either a variable or $\top$) and contingent completeness all hold by definition of canonical term.

A strategy $(\sigma, \bar{\sigma})$ of $\mathcal{A} = (A, U, K)$ is *P-full* (respectively *O-full*) just if every P-move (respectively O-move) of A occurs in σ. Suppose $(\sigma, \bar{\sigma})$ is a winning strategy of $[\![\tau :: \kappa]\!]$. Then: (i) If $(\sigma, \bar{\sigma})$ is P-full, then it is also O-full. (ii) There is a subtype $\tau' :: \kappa$ of τ such that $(\sigma, \bar{\sigma})$ is winning and P-full over $[\![\tau' :: \kappa]\!]$.

A derivation $[\Gamma :: \Delta] \vdash [M :: t] : [\tau :: \kappa]$ is *relevant* just if for each abstraction subterm $\lambda x^{\bigwedge_{i \in I} \tau_i}.M'$ of M and $i \in I$, M' has a free occurrence of $\mathsf{p}_i(x)$.

Lemma 19. *$[\Gamma :: \Delta] \vdash [M :: t] : [\tau :: \kappa]$ is relevant iff $[\![M :: t]\!]$ is P-full.*

Proof. The right-to-left direction is shown by a easy modification of the proof of definability. To prove the left-to-right direction, we first normalise $M :: t$ to the canonical form, say $M' :: t'$. It is easy to prove (syntactically) that normalisation preserves relevance of a derivation, so $M' :: t'$ is also relevant. Then by (easy) induction on canonical forms, we prove that $[\![M' :: t']\!]$ is full. By adequacy, $[\![M :: t]\!]$ is also full.

4.4 Subtyping

We can naturally introduce the subtyping relation on the set of intersection types. For $\tau_1, \tau_2 :: \kappa$, the subtyping relation $\tau_1 \preceq_\kappa \tau_2$ is defined by the following rules.

$$\frac{}{q \preceq_o q}$$

The subtyping relation can be characterised by the interpretation as follows.

Theorem 6. *Let $\tau_1, \tau_2 :: \kappa$ be intersection types. $\tau_1 \preceq_\kappa \tau_2$ if and only if $(\sigma, \mathrm{id}_\kappa) \in \mathrm{Hom}([\![\tau :: \kappa]\!], [\![\tau_2 :: \kappa]\!])$ for some σ.*

Proof. It is easy to prove that $\tau_1 \preceq_\kappa \tau_2$ if and only if $\emptyset \vdash t : \tau_1 \to \tau_2$, where t is the η-long normal form of $\lambda x^\kappa.x$. So the left-to-right direction follows from the interpretation and the other direction follows from its adequacy.

5 Applications to HORS Model-Checking

Fix a ranked alphabet Σ and a HORS $G = \langle \Sigma, \mathcal{N}, S, \mathcal{R} \rangle$ (see [1] for a definition of HORS). We begin by giving a game-semantic account of the value tree $[\![G]\!]$ of G. Let $\mathcal{N} = \{ F_1 : \kappa_1, \ldots, F_n : \kappa_n \}$ with $F_1 = S$, and $\Sigma = \{ a_1 : r_1, \ldots, a_m : r_m \}$

where each $r_i = ar(a_i)$, the arity of a_i. Writing $[\![\Sigma]\!] := \prod_{i=1}^{m} [\![o^{r_i} \to o]\!]$ and $[\![\mathcal{N}]\!] := \prod_{i=1}^{n} [\![\kappa_i]\!]$, the *value tree* of G, $[\![G]\!] : [\![\Sigma]\!] \longrightarrow [\![o]\!]$, is the composite

$$[\![\Sigma]\!] \xrightarrow{\Lambda(\mathbf{g})} ([\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!]) \xrightarrow{Y} [\![\mathcal{N}]\!] \xrightarrow{\{\, S::o \,\}} [\![o]\!]$$

in the CCC of o-coloured arenas and innocent strategies, where

- $\mathbf{g} = \langle g_1, \ldots, g_n \rangle : [\![\Sigma]\!] \times [\![\mathcal{N}]\!] \longrightarrow [\![\mathcal{N}]\!]$; each component $g_i = [\![\Sigma \cup \mathcal{N} \vdash \mathcal{R}(F_i) :: \kappa_i]\!]$, and Λ(-) is currying
- Y is the standard fixpoint strategy (see [2, §7.2]), and
- $\{\, S :: o \,\} = \pi_1 : [\![\mathcal{N}]\!] \longrightarrow [\![o]\!]$ is the projection map.

Remark 1. There is a one-one correspondence between maximal P-views p in the map $[\![G]\!]$ and maximal traces t_p in the (concrete) value tree of G. If we write the O-move of the ith component of $[\![\Sigma]\!]$ as a_i, then projecting p to these O-moves gives precisely t_p.

Now fix a trivial automaton $\mathcal{B} = \langle Q, \Sigma, q_I, \delta \rangle$. We extend the game-semantic account to express the run tree of $\mathcal{B}$ over the value tree $[\![G]\!]$ in the category of Q-based two-level arenas and innocent strategies. First set

$$\begin{aligned}[\![\delta :: \Sigma]\!] &:= \textstyle\prod_{a \in \Sigma} \bigwedge_{(q,a,\overline{q}) \in \delta} [\![q_1 \to \ldots \to q_{ar(a)} \to q :: \underbrace{o \to \ldots \to o}_{ar(a)} \to o]\!] \\ &= ([\![\delta]\!], U, [\![\Sigma]\!])\end{aligned}$$

where $[\![\delta]\!]$ is the Q-coloured arena $\prod_{a \in \Sigma} \prod_{(q,a,\overline{q}) \in \delta} [\![q_1 \to \ldots \to q_{ar(a)} \to q]\!]$.

A *run tree* of $\mathcal{B}$ over $[\![G]\!]$ is just an innocent strategy $(\rho, [\![G]\!])$ of the arena $[\![\delta :: \Sigma]\!] \Rightarrow [\![q_I :: o]\!] = ([\![\delta]\!] \Rightarrow [\![q_I]\!], V, [\![\Sigma]\!] \Rightarrow [\![o]\!])$. Every P-view $\bar{p} \in [\![G]\!]$ of the value tree has a unique "colouring" i.e. a P-view $p \in \rho$ such that $V(p) = \bar{p}$. This associates a colour (state) with each node of the value tree, which corresponds to a run tree in the concrete presentation.

5.1 Characterisation by Complete Type Environment

Using G and $\mathcal{B}$ as before, Kobayashi [3] showed that $[\![G]\!]$ is accepted by $\mathcal{B}$ if, and only if, there is a *complete type environment* Γ, meaning that (i) $S : q_I \in \Gamma$, (ii) $\Gamma \vdash \mathcal{R}(F) : \theta$ for each $F : \theta \in \Gamma$. As a first application of two-level arena games, we give a semantic counterpart of the characterisation. Let $\Gamma = \{\, F_1 : \bigwedge_{j \in I_1} \tau_{1j} :: \kappa_1, \ldots, F_n : \bigwedge_{j \in I_n} \tau_{nj} :: \kappa_n \,\}$ be a type environment of G. Set $[\![\Gamma :: \mathcal{N}]\!] := \prod_{i=1}^{n} \bigwedge_{j \in I_i} [\![\tau_{ij} :: \kappa_i]\!] = ([\![\Gamma]\!], U_1, [\![\mathcal{N}]\!])$ where $[\![\Gamma]\!] := \prod_{i=1}^{n} \prod_{j \in I_i} [\![\tau_{ij}]\!]$.

Theorem 7. *Using Σ, G and $\mathcal{B}$ as before, $[\![G]\!]$ is accepted by $\mathcal{B}$ if, and only if, there exists Γ such that*

(i) $S : q_I \in \Gamma$, and
(ii) there exists a strategy σ (say) of the Q-coloured arena $[\![\delta]\!] \times [\![\Gamma]\!] \Rightarrow [\![\Gamma]\!]$ such that $(\sigma, \mathbf{g})$ defines a winning strategy of the two-level arena

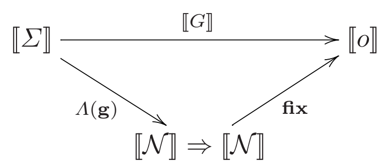
$$([\![\delta :: \Sigma]\!] \times [\![\Gamma :: \mathcal{N}]\!]) \Rightarrow [\![\Gamma :: \mathcal{N}]\!] \;=\; ([\![\delta]\!] \times [\![\Gamma]\!] \Rightarrow [\![\Gamma]\!], V_1, [\![\Sigma]\!] \times [\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!])$$

Proof. Suppose we have Γ and σ that satisfy the conditions. The following composite map in the category of two-level arenas and innocent strategies

$$[\![\delta :: \Sigma]\!] \xrightarrow{\Lambda(\sigma,\mathbf{g})} ([\![\Gamma :: \mathcal{N}]\!] \Rightarrow [\![\Gamma :: \mathcal{N}]\!]) \xrightarrow{(Y,Y)} [\![\Gamma :: \mathcal{N}]\!] \xrightarrow{\{\, S:q_I \,\}} [\![q_I :: o]\!]$$

gives the strategy $(\rho, [\![G]\!])$ over $([\![\delta]\!] \Rightarrow [\![q_I]\!], V, [\![\Sigma]\!] \Rightarrow [\![o]\!])$. To show $V(\rho) = [\![G]\!]$, it suffices to show that if $V(m_1) \cdots V(m_n) \cdot m \in [\![G]\!]$ and m is an O-move of $[\![\Sigma]\!] \Rightarrow [\![o]\!]$, then $V(m') = m$ for some O-move m' of $[\![\delta]\!] \Rightarrow [\![q_I]\!]$. But an O-move m of $[\![\Sigma]\!] \Rightarrow [\![o]\!]$ is either the unique move of $[\![o]\!]$ or a move corresponding to an argument of a tree constructor in Σ (i.e. a move corresponding to o_i for some $a :: o_1 \to \cdots \to o_n \to o \in \Sigma$). By definition of δ and q_I, there exists a corresponding move m' in $[\![\delta]\!] \Rightarrow [\![q_I]\!]$ for each O-move m of $[\![\Sigma]\!] \Rightarrow [\![o]\!]$. The desired property then follows from the contingent completeness of ρ.

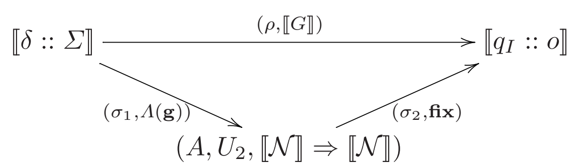
To prove the converse, suppose that $[\![G]\!]$ is accepted by $\mathcal{B}$. I.e. we have a run-tree given by a strategy $(\rho, [\![G]\!])$ over the two-level arena $([\![\delta]\!] \Rightarrow [\![q_I]\!], U_1, [\![\Sigma]\!] \Rightarrow [\![o]\!])$ such that $U_1(\rho) = [\![G]\!]$. By definition of $[\![G]\!]$, the following diagram (in the category of base arenas) commutes:



where **fix** is the composite

$$([\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!])]\!] \xrightarrow{Y} [\![\mathcal{N}]\!] \xrightarrow{\{\, S:o \,\}} [\![o]\!].$$

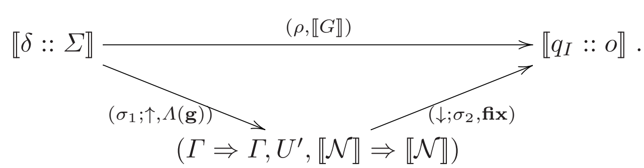
By Subject Expansion (Theorem 3), there exist a two-level arena $\mathcal{A} = (A, U_2, [\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!])$ and strategies σ_1 and σ_2 of Q-coloured arenas that make the diagram



commutes.

By analysis of σ_2, there exist a Q-coloured arena Γ (say) and hence a two-level arena $(\Gamma \Rightarrow \Gamma, U', [\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!])$, a map $(\uparrow, \mathrm{id}_{[\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!]})$ from $(A, U_2, [\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!])$ to $(\Gamma \Rightarrow \Gamma, U', [\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!])$, and a map $(\downarrow, \mathrm{id}_{[\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!]})$ in the opposite direction, such that $\uparrow ; \downarrow = \mathrm{id}_{\mathcal{A}}$.

Thus we have the following commutative diagram:



It follows from $U_1(\rho) = [\![G]\!]$ that $U(\sigma_1;\uparrow) = \Lambda(\mathbf{g})$ where U is the relevant forgetful function. Hence, by uncurrying $(\sigma_1;\uparrow, \Lambda(\mathbf{g}))$, we obtain a total and hence winning strategy of $([\![\delta :: \Sigma]\!] \times [\![\Gamma :: \mathcal{N}]\!]) \Rightarrow [\![\Gamma :: \mathcal{N}]\!]$ as desired. □

5.2 Minimality of Traversals-induced Typing

Using the same notation as before, interaction sequences from $\mathbf{Int}(\Lambda(\mathbf{g}), \mathbf{fix}) \subseteq \mathrm{Int}([\![\Sigma]\!], [\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!], [\![o]\!])$ form a tree, which is (in essence) the *traversal tree* in the sense of Ong [1].

Prime types, which are intersection types of the form $\theta = \bigwedge_{i \in I_1} \theta_{1i} \to \cdots \to \bigwedge_{i \in I_n} \theta_{ni} \to q$, are equivalent to *variable profiles* (or simply *profiles*) [1]. Precisely θ corresponds to profile $\widehat{\theta} := (\{\, \widehat{\theta_{1i}} \mid i \in I_1 \,\}, \cdots, \{\, \widehat{\theta_{ni}} \mid i \in I_n \,\}, q)$. We write profiles of ground kind as q, rather than (q). Henceforth, we shall use prime types and profiles interchangeably.

Tsukada and Kobayashi [8] introduced (a kind-indexed family of) binary relations $\leq_\kappa$ between profiles of kind κ, and between sets of profiles of kind κ, by induction over the following rules.

(i) If for all $\theta \in A$ there exists $\theta' \in A'$ such that $\theta \leq_\kappa \theta'$ then $A \leq_\kappa A'$.
(ii) If $A_i \leq_{\kappa_i} A'_i$ for each i then $(A_1, \ldots, A_n; q) \leq_{\kappa_1 \to \cdots \to \kappa_n \to o} (A'_1, \ldots, A'_n, q)$.

A *profile annotation* (or simply *annotation*) of the traversal tree $\mathbf{Int}(\Lambda(\mathbf{g}), \mathbf{fix})$ is a map of the nodes (which are move-occurrences of $M_{[\![\Sigma]\!]} + M_{[\![\mathcal{N}]\!] \Rightarrow [\![\mathcal{N}]\!]} + M_{[\![o]\!]}$) of the tree to profiles. We say that an annotation of the traversal tree is *consistent* just if whenever a move m, of kind $\kappa_1 \to \cdots \to \kappa_n \to o$ and simulates q, is annotated with profile $(A_1, \cdots, A_n, q')$, then (i) $q' = q$, (ii) for each i, A_i is a set of profiles of kind κ_i, (iii) if m' is annotated with θ and i-points to m, then $\theta \in A_i$. Now consider *annotated moves*, which are moves paired with their annotations, written (m, θ). We say that a profile annotation is *innocent* just if whenever $u_1 \cdot (m_1, \theta_1)$ and $u_2 \cdot (m_2, \theta_2)$ are even-length paths in the annotated traversal tree such that $\ulcorner u_1 \urcorner = \ulcorner u_2 \urcorner$, then $m_1 = m_2$ and $\theta_1 = \theta_2$.

Every consistent (and innocent) annotation α of an (accepting) traversal tree gives rise to a typing environment, written Γ_α, which is the set of bindings $F_i : \theta$ where $i \in \{\, 1, \ldots, n \,\}$ and θ is the profile that annotates an occurrence of an initial move of $[\![\kappa_i]\!]$. Note that Γ_α is finite because there are only finitely many types of a given kind. We define a relation between annotations: $\alpha_1 \leq \alpha_2$ just if for each occurrence m of a move of kind κ in the traversal tree, $\alpha_1(m) \leq_\kappa \alpha_2(m)$.

Theorem 8. *(i) Let α be a consistent and innocent annotation of a traversal tree. Then Γ_α is a complete type environment.*

(ii) There is $\leq$-minimal consistent and innocent annotation, written $\alpha_{\min}$. Then $\Gamma_{\alpha_{\min}} \leq \Gamma_\alpha$ meaning that for all $F : \theta \in \Gamma_{\alpha_{\min}}$ there exists $F : \theta' \in \Gamma_\alpha$ such that $\theta \leq \theta'$.

(iii) Every complete type environment Γ determines a consistent and innocent annotation α_Γ of the traversal tree.

5.3 Game-Semantic Proof of Completeness of GTRecS [9]

GTRecS [9] is a higher-order model checker proposed by Kobayashi. Although GTRecS is inspired by game-semantics, the formal development of the algorithm is purely type-theoretical and no concrete relationship to game semantics is known. Here we give a game-semantic proof of completeness of GTRecS based on two-level arena games.

The novelty of GTRecS lies in a function on type bindings, named **Expand**. For a set Γ of nonterminal-type bindings, $\mathbf{Expand}(\Gamma)$ is defined as

$$\Gamma \cup \bigcup \{\Gamma' \cup \{F_i : \tau'\} \mid \Gamma \preceq_P \Gamma' \wedge \Gamma' \vdash \mathcal{R}(F_i) : \tau' \wedge \Gamma \preceq_O \{F_i : \tau'\}\ \},$$

where $\Gamma' \vdash \mathcal{R}(F_i) : \tau'$ is relevant. Here for types τ_1 and τ_2, $\tau_1 \preceq_P \tau_2$ if the arena $[\![\tau_2]\!]$ is obtained by adding only proponent moves to $[\![\tau_1]\!]$. For example, $(\bigwedge \emptyset) \to q \preceq_P ((\bigwedge \emptyset) \to q') \to q$ but $(\bigwedge \emptyset) \to q \not\preceq_P (q'' \to q') \to q$, since q' is at the proponent position and q'' at the opponent position. $\Gamma \preceq_P \Gamma'$ is defined as $\forall F : \tau' \in \Gamma'.\ \exists F : \tau \in \Gamma.\ \tau \preceq_P \tau'$. Similarly, $\tau \preceq_O \tau'$ and $\Gamma \preceq_O \Gamma'$ are defined.

Fix a type environment Γ and a winning strategy $\sigma : [\![\delta]\!] \longrightarrow (\Gamma^1 \Rightarrow \Gamma^2)$ (here we use superscripts to distinguish occurrences of Γ) that is induced from the derivation of $\vdash G : \Gamma$. The strategy $\mathbf{fix} : ([\![\Gamma^1]\!] \Rightarrow [\![\Gamma^2]\!]) \longrightarrow [\![q_I]\!]$ is defined as the composite of $([\![\Gamma]\!]^1 \Rightarrow [\![\Gamma]\!]^2) \xrightarrow{Y} [\![\Gamma]\!] \xrightarrow{\{S:q_I\}} [\![q_I]\!]$. For a natural number n, the *nth approximation of* **fix** is defined by $\lfloor \mathbf{fix} \rfloor_n = \{s \in \mathbf{fix} \mid |s| \leq 2n+1\}$. For the notational convenience, we define $\lfloor \mathbf{fix} \rfloor_\infty = \mathbf{fix}$.

Our goal is to show the concrete relationship between $\lfloor \mathbf{fix} \rfloor_n$ and $\mathbf{Expand}(\{S : q_I\})$ (Lemma 23). Completeness of GTRecS is an easy consequence of Lemma 23.

Let $n \in \{0, 1, 2, \ldots, \infty\}$. The nth approximation of **fix** induces approximation of arenas and strategies. An arena $\lfloor \Gamma^1 \Rightarrow \Gamma^2 \rfloor_n$ is defined as the restriction of $\Gamma^1 \Rightarrow \Gamma^2$ that consists of only moves appearing at $\mathbf{Int}(\sigma, \lfloor \mathbf{fix} \rfloor_n)$, and a strategy $\lfloor \sigma \rfloor_n : [\![\delta]\!] \longrightarrow \lfloor \Gamma^1 \Rightarrow \Gamma^2 \rfloor_n$ is the restriction of σ to the arena.

Lemma 20. $\lfloor \Gamma^1 \rfloor_\infty = \lfloor \Gamma^2 \rfloor_\infty$ *and* $S : q_0 \in \lfloor \Gamma^2 \rfloor_\infty$.

Proof. Easy. □

Remark 2. It is not necessarily the case that $\lfloor \Gamma^1 \rfloor_\infty = \Gamma^1$ or $\lfloor \Gamma^2 \rfloor_\infty = \Gamma^2$.

Lemma 21. *For* $n \in \{0, 1, \ldots, \infty\}$, $\lfloor \sigma \rfloor_n$ *is a full and winning strategy.*

Proof. All properties other than totality come from the fact that $\lfloor \sigma \rfloor_n$ is a restriction of σ. To prove totality, a key observation is that every maximal sequence $s \in \lfloor \mathbf{fix} \rfloor_n$ ends with a O-move. Thus every maximal interaction sequence $s \in \mathbf{Int}(\sigma, \lfloor \mathbf{fix} \rfloor_n)$ ends with a O-move of $(\Gamma^1 \Rightarrow \Gamma^2) \to \langle q_0 \rangle$, since σ is contingent complete and total. Therefore if $sm \in \lfloor \sigma \rfloor_n$ is maximal, then m is a *P-move* of $(C \times \Gamma^1 \Rightarrow \Gamma^2)$. $\lfloor \sigma \rfloor_n$ is full by definition. □

If $\Gamma = \{F_i : \bigwedge_j \tau_{i,j} \mid F_i \in \mathcal{N}\}$, then the arena $[\![\delta]\!] \Rightarrow (\Gamma^1 \Rightarrow \Gamma^2)$ can be decomposed as $\prod_{i,j}([\![\delta]\!] \Rightarrow (\Gamma^1 \Rightarrow \tau_{i,j}))$. By the same way, the arena $[\![\delta]\!] \Rightarrow \lfloor \Gamma^1 \Rightarrow \Gamma^2 \rfloor_n$ is decomposed as $\prod_{i,j}([\![\delta]\!] \Rightarrow (\lfloor \Gamma^1 \rfloor_{n,i,j} \Rightarrow \lfloor \tau_{i,j} \rfloor_n))$.

Let $\lfloor \Gamma^1 \rfloor_n$ be the union of variable-type bindings corresponding to $\bigcup_{i,j} \lfloor \Gamma^1 \rfloor_{n,i,j}$ and $\lfloor \Gamma^2 \rfloor_n$ be the set of type bindings $\{F_i : \lfloor \tau_{i,j} \rfloor_n\}_{i,j}$.

Lemma 22. *For all $n \in \{0, 1, \ldots, \infty\}$, we have $(\lfloor \Gamma^1 \rfloor_n \cup \lfloor \Gamma^2 \rfloor_n) \preceq_O \lfloor \Gamma^1 \rfloor_{n+1}$ and $(\lfloor \Gamma^1 \rfloor_n \cup \lfloor \Gamma^2 \rfloor_n) \preceq_P \lfloor \Gamma^2 \rfloor_{n+1}$ (here $\infty + 1 = \infty$).*

Proof. Assume that $\lfloor \Gamma^1 \rfloor_n \cup \lfloor \Gamma^2 \rfloor_n \subsetneq \lfloor \Gamma^1 \rfloor_{n+1} \cup \lfloor \Gamma^2 \rfloor_{n+2}$. Let m be an element of their difference. By definition of $\lfloor \Gamma^1 \rfloor_{n+1}$ and $\lfloor \Gamma^2 \rfloor_{n+1}$, there is a sequence $sm \in \mathbf{Int}(\Lambda(\sigma), \lfloor \mathbf{fix} \rfloor_{n+1}$ ending with m. Let $s'm_0$ be the maximal prefix of sm such that $s'm_0 \in \mathbf{Int}(\Lambda(\sigma), \lfloor \mathbf{fix} \rfloor_n)$. Then m_0 is a O-move of Γ^1 or a P-move of Γ^2. (Otherwise m_0 must be a move of C, that implies $s'm_0$ is also maximal in $\mathbf{Int}(\Lambda(\sigma), \mathbf{fix})$, but this contradict to existence of its extension $sm \in \mathbf{Int}(\Lambda(\sigma), \lfloor \mathbf{fix} \rfloor_{n+1})$.) By the definition of $\lfloor \mathbf{fix} \rfloor_{n+1}$, $sm = s'm_0m_0's''m$ for some s'', where m_0' corresponds to m_0. This interaction sequence is maximal. Therefore m is an O-move of Γ^1 or a P-move of Γ^2. Moreover m is justified by a move of the view of $s'm_0m_0's''$, i.e., a move of $\lfloor \Gamma^1 \rfloor_n \cup \lfloor \Gamma^2 \rfloor_n$ or their counterpart. So the proposition holds. □

By Lemma 19 and Lemma 21, we have a relevant derivation of $\delta \cup \lfloor \Gamma^1 \rfloor_{n,i,j} \vdash \mathcal{R}(F_i) : \tau_{i,j}$. Combination of these derivations and Lemma 22 leads to the next lemma.

Lemma 23. $\lfloor \Gamma^1 \rfloor_n \cup \lfloor \Gamma^2 \rfloor_n \subseteq \mathbf{Expand}^n(\{S : q_0\})$.

Proof. By induction on n. The case $n = 0$ is trivial since $\lfloor \Gamma^1 \rfloor_0 = \emptyset$ and $\lfloor \Gamma^2 \rfloor_0 = \{S : q_0\}$.

Suppose $\lfloor \Gamma^1 \rfloor_n \cup \lfloor \Gamma^2 \rfloor_n \subseteq \mathbf{Expand}^n(\{S : q_0\})$. We use the following proposition (see [9, Appendix C].)

> If $\Gamma \preceq_O \Gamma'$ and $\theta \preceq_P \theta'$ and $\Gamma' \vdash \mathcal{R}(F) : \theta'$, then $\Gamma' \cup \{F : \theta'\} \subseteq \mathbf{Expand}(\Gamma \cup \{F : \theta\})$.

The lemma follows from the proposition and previous lemmas. □

Conclusions and Further Directions Two-level arena games are an accurate model of intersection types. Thanks to Subject Expansion, they are a useful semantic framework for reasoning about higher-order model checking.

For future work, we aim to (i) consider properties that are closed under disjunction and quantifications, and (ii) study a call-by-value version of intersection games. In orthogonal directions, it would be interesting to (iii) construct an intersection game model for untyped recursion schemes [8], and (iv) build a CCC of intersection games parameterised by an alternating parity tree automaton, thus extending our semantic framework to mu-calculus properties.

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