Monads for behaviour
(extended version, v0.9)

Maciej Piróg and Jeremy Gibbons

Department of Computer Science
University of Oxford

Abstract

The monads used to model effectful computations traditionally concentrate on the 'destination'—the final results of the program. However, sometimes we are also interested in the 'journey'—the intermediate course of a computation—especially when reasoning about non-terminating interactive systems. In this article we claim that a necessary property of a monad for it to be able to describe the behaviour of a program is complete iterativity. We show how an ordinary monad can be modified to disclose more about its internal computational behaviour, by applying an associated transformer to a completely iterative monad. To illustrate this, we introduce two new constructions: a coinductive cousin of Cenciarelli and Moggi's generalised resumption transformer, and States—a State-like monad that accumulates the intermediate states.

1 Introduction

In this article we are concerned with semantics of programs like the following Haskell fragment:

```
echo :: IO ()
echo = do { x <- getChar ; putChar x ; echo }
```

More precisely, we are interested in programs that (1) have side-effects, and (2) depend on a (not necessarily terminating) recursion—or a corecursion, if you will. In the example, `echo` performs observable actions and then calls itself, 'unfolding' the whole infinite series of events.

Since Moggi's groundbreaking work [20], monads have become the standard model for computational effects. A popular choice for I/O is to employ the State monad $A \mapsto (A \times S)^S$, model the outside world as an object S, and see the program semantics as a function transforming an initial state into a final state [7, 15]. Alternatively, we could consider side-effects as communication with the environment, so no assumption about semantics of effects needs to be made at this point: the program semantics is a free structure generated by the 'effectful' constructs (`getChar` and `putChar`), which is then interpreted by an external handler [13, 25, 28].

The situation becomes much more complicated in the context of (2). For example, the State monad does not build the final state incrementally, so in case of non-terminating programs, such as `echo`, it is useless. The free structure, on the other hand, sometimes needs to be infinite, so in general the free monad Σ^* (for an endofunctor Σ representing the signature) is 'too small'. Evidently, to encompass these examples we need monads that capture not only the final results of the program, but rather its behaviour, for example in the form of execution traces. In this article we identify this property as *complete iterativity*. A monad is completely iterative ('is a cim') if it is equipped with a certain corecursion scheme that is coherent with its monadic structure (for the full definition see Section 2). In particular, the free cim Σ^∞ generated by an endofunctor Σ captures both finite and infinite Σ-terms.

Nevertheless, we should not discard the 'usual' monads too hastily. For example, if we program a divergent computation in the State monad, the intermediate states are physically 'put' and 'gotten' somewhere in the memory of the computer, so the internal behaviour of the computation is, in a sense, accurate. The point is to reify it as a mathematical model. An interesting fact is that the `IO` monad in the Haskell Glasgow Compiler (GHC) is implemented as the State monad [17], so whatever its mathematical model, the two have to be related.

Our idea is to use transformers associated with the 'usual' monads to trace computations. For a cim T and an adjunction $F \dashv U$ that gives rise to a monad M (that is $UF = M$), we use the monad UTF to trace computations in M. Clearly, UTF supports M-computations (via the canonical monad morphism $M \to UTF$), but it can also store some observations about the course of the computation in the inner cim. The choice of the monad T and the adjunction reveals different aspects of computations in M. As our main technical result, we prove that UTF is completely iterative.

As an example, we use the currying adjunction to derive what we call the States monad, which behaves like State, but it also gathers the intermediate states in a stream. This way, the result of the computation is not a single, final state, but rather a possibly infinite trace consisting of intermediate states.

Then we introduce the Coinductive Generalised Resumption transformer $M(\Sigma M)^\infty$, where Σ is an endofunctor. It is a coalgebraic cousin of Cenciarelli and Moggi's Generalised Resumption transformer $M(\Sigma M)^*$ [9]. We characterise it as the composition of a free cim in the category of free Eilenberg-Moore M-algebras with the standard free-underlying adjunction, which yields that it is itself a cim.

2 Completely iterative monads

2.1 Initial assumptions and notations

For the entire article, we assume that we are working in a base category $\mathcal{B}$ with coproducts and all the necessary final coalgebras. We denote the composition of a natural transformation with a functor by a subscript; for example, for functors H and J, if $\xi : F \to G$ is natural, then $\xi_H : FH \to GH$. If ξ is natural in two variables, by $\xi_{H,J}$ we mean a natural transformation $\zeta_A = \xi_{HA,JA}$. We define $\mathsf{gr}_A = [\mathsf{id}_A, \mathsf{id}_A] : A + A \to A$.

Working with infinite computations means working also with infinite data structures. To set the notation, we recall a few standard definitions. For an endofunctor F, an F-coalgebra is a pair $\langle A, f : A \to FA \rangle$. We call A the *carrier* of the coalgebra. A morphism $h : A \to B$ is an F-coalgebra *homomorphism*, denoted as $h : \langle A, f\rangle \to \langle B, g\rangle$, if $g \cdot h = Fh \cdot f$. An F-coalgebra $\langle \nu F, \beta\rangle$ is *final* if for every F-coalgebra $\langle A, f\rangle$ there exists precisely one homomorphism $\langle A, f\rangle \to \langle \nu F, \beta\rangle$, called an *anamorphism* and denoted as $[\![f]\!]$.

2.2 Ideal and idealised monads

In this article we deal with monads that support corecursion: infinite computations are described by single steps. However, a step might not produce any observable behaviour, for example if it is a pure computation constructed with the unit, or we want to be more selective about which monadic actions are observable. To formalise productive computations, we need the notion of (right) ideals of monads. These are analogous to ideals in a ring or a semigroup—subsets closed under the operations. (All the definitions in this section are as given by Adámek, Milius, and Velebil [2].)

Definition 1. *For a monad $\langle M, \eta, \mu\rangle$, let $\overline{M}$ together with a natural transformation $\sigma : \overline{M} \to M$ with monomorphic components be a subfunctor of M. We call $\overline{M}$ an ideal of M if there exists a natural transformation $\overline{\mu} : \overline{M}M \to \overline{M}$ such that the following diagram commutes.*

$$\begin{array}{ccc} \overline{M}M & \xrightarrow{\sigma_M} & M^2 \\ \downarrow{\scriptstyle \overline{\mu}} & & \downarrow{\scriptstyle \mu} \\ \overline{M} & \xrightarrow{\sigma} & M \end{array}$$

We call a pair of a monad and its ideal an idealised monad. An idealised monad M is called an ideal monad if $M = \mathsf{Id} + \overline{M}$ with $\eta = \mathsf{inl}_{\mathsf{Id},\overline{M}}$ and $\sigma = \mathsf{inr}_{\mathsf{Id},\overline{M}}$.

Examples of ideal monads include: free monads, exceptions, interactive output, and nonempty lists.

We also need morphisms that respect the internal structure of idealised monads. If Σ is an endofunctor, then a natural transformation $\xi : \Sigma \to M$ is ideal if its codomain contains only productive computations. Intuitively, this means that an interpretation of a symbol from the signature should never yield a pure computation. An ideal monad morphism $r : M \to N$ always maps productive computations in M to productive computations in N. Formally:

Definition 2. *Let $\langle M, \sigma^M\rangle$ and $\langle N, \sigma^N\rangle$ be idealised monads. A natural transformation $\xi : \Sigma \to M$ is ideal if it factors through σ^M. A monad morphism $r : M \to N$ is idealised if it preserves the ideals, that is there exists $\overline{r}$ such that $r \cdot \sigma^M = \sigma^N \cdot \overline{r}$, for a natural transformation $\overline{r} : \overline{M} \to \overline{N}$.*

2.3 Cims defined

For an idealised monad M, we describe a step of a computation by a morphism of type $e : X \to M(A + X)$, called an *equation morphism*. The object X represents (a set of) *variables*—the seeds of the corecursion. The object A represents (a set of) *parameters*, which are final values of the computation.

An equation morphism is *guarded* if it always produces effects (in the sense of idealised monads) or a final value, but not a variable:

Definition 3. *A morphism $e : X \to M(A+Y)$ is guarded if it factors through the morphism $[\sigma_{A+Y}, \eta_{A+Y} \cdot \mathsf{inl}_{A,Y}]$, that is there exists a morphism j such that the following diagram commutes.*

$$\begin{tikzcd} X \arrow[rr, "e"] \arrow[dr, dashed, "j"] & & M(A+Y) \\ & \overline{M}(A+Y)+A \arrow[ur, "{[\sigma_{A+Y},\ \eta_{A+Y} \cdot \mathsf{inl}_{A,Y}]}"'] & \end{tikzcd}$$

If $X = Y$, we call e a guarded equation morphism.

We use a guarded equation morphism e to unfold a computation $e^\dagger$, called a solution. Intuitively, a solution is an infinite iteration of parameter-preserving Kleisli-compositions of e. A monad is a cim if such a composition always exists and is unique. Formally:

Definition 4. *Let $e : X \to M(A+X)$ be a morphism. We call a morphism $e^\dagger : X \to MA$ a solution of e if the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"] & MA \\ M(A+X) \arrow[r, "{M[\eta_A, e^\dagger]}"] & M^2A \arrow[u, "\mu_A"'] \end{tikzcd}$$

An idealised monad M is completely iterative if every guarded equation morphism has a unique solution.

Cims make it possible to separate the corecursion guarded by invocation of effects from a recursive structure of the base category, like order or metric enrichment. This separation is important conceptually. Consider a server dealing with some requests: though it is non-terminating, it probably does not require unbounded recursion in between handling two requests.

Conversely, in a language with unbounded recursion, M-computations consisting of guarded steps are necessarily solutions: An infinite computation can be seen as the colimit of the ω-chain consisting of single steps. Consider an ω-chain $\{f_i : X_i \to MX_{i+1}\}_{i\in\mathbb{N}}$ of Kleisli morphisms that factor through $\sigma^M_{X_{i+1}}$. In a category with countable coproducts, we define a guarded equation morphism $e = [(id_0 + M\mathsf{in}_{i+1}) \cdot f_i]_{i\in\mathbb{N}} : \coprod_{i\in\mathbb{N}} X_i \to M(0 + \coprod_{i\in\mathbb{N}} X_i)$. One can show that the family of morphisms $\{e^\dagger \cdot \mathsf{in}_i : X_i \to M0\}_{i\in\mathbb{N}}$ is the colimit of the chain in the Kleisli category of M.

2.4 The free cim

An example of a cim is a generalisation of the infinite term monad generated by an endofunctor (intuitively, a signature) Σ. Its functorial part is given by a family of final coalgebras $\Sigma^\infty A = \nu X.A + \Sigma X$. Below we define the unit, η^∞, and a natural transformation $\mathsf{emb} : \Sigma \to \Sigma^\infty$ that embeds Σ in Σ^∞. For an explicit definition of the multiplication μ^∞ refer to Section **??**, and put Id for M in the definition of μ^K.

$$\begin{tikzcd} \mathsf{Id} \arrow[d, "\eta^\infty = \mathsf{inl}_{\mathsf{Id},\Sigma\Sigma^\infty}"] \\ \mathsf{Id} + \Sigma\Sigma^\infty \cong \Sigma^\infty \end{tikzcd} \qquad \begin{tikzcd} \Sigma \arrow[d, "\mathsf{emb} = \mathsf{inr}_{\mathsf{Id},\Sigma\Sigma^\infty} \cdot \Sigma\eta^\infty"] \\ \mathsf{Id} + \Sigma\Sigma^\infty \cong \Sigma^\infty \end{tikzcd}$$

As discussed by Aczel *et al.* [1], Σ^∞ is the free cim generated by Σ. Intuitively, this means that every interpretation of Σ in a cim M extends in a unique way to an interpretation of the entire (possibly infinite) term Σ^∞ in M. Formally, for an ideal natural transformation $\xi : \Sigma \to M$, there exists a unique monad morphism $\iota(\xi) : \Sigma^\infty \to M$ such that the following diagram commutes.

$$\begin{tikzcd} \Sigma \arrow[r, "\mathsf{emb}"] \arrow[dr, "\xi"'] & \Sigma^\infty \arrow[d, "\iota(\xi)"] \\ & M \end{tikzcd}$$

The monad morphism $\iota(\xi)$ is given by $[\eta^M, \xi^\dagger_{\Sigma^\infty}]$. Diagrammatically:

$$\begin{tikzcd} \Sigma\Sigma^\infty \arrow[d, "\xi_{\Sigma^\infty}"] \\ M\Sigma^\infty \cong M(\mathsf{Id} + \Sigma\Sigma^\infty) \end{tikzcd} \qquad \begin{tikzcd} \Sigma\Sigma^\infty \arrow[d, "\xi^\dagger_{\Sigma^\infty}"] \\ M\mathsf{Id} = M \end{tikzcd} \qquad \begin{tikzcd} \Sigma^\infty \cong \mathsf{Id} + \Sigma\Sigma^\infty \arrow[d, "\iota(\xi) = {[\eta^M, \xi^\dagger_{\Sigma^\infty}]}"] \\ M \end{tikzcd}$$

Apart from the free cim and the Exception monad $A \mapsto A + E$, there are hardly any examples of cims commonly used in programming or semantics. This paper aims to fill this void in a rather generic fashion.

3 Cims and adjunctions

Let M be a monad, and let $\langle F, U, \eta, \varepsilon \rangle : \mathcal{B} \to \mathcal{C}$ be a factorization of M as an adjunction, that is $M = \langle UF, \eta, U\varepsilon_F \rangle$. Let $\langle T, \eta^T, \mu^T, \sigma^T \rangle$ be a cim with solutions -†. It is standard that UTF is a monad with $\eta^{UTF} = U\eta^T_F \cdot \eta$ and $\mu^{UTF} = U\mu^T_F \cdot UT\varepsilon_{TF}$, and that $\mathsf{lift} = U\eta^T_F : UF \to UTF$ is a monad morphism. We prove that UTF inherits complete iterativity from T.

Theorem 5. *The functor $U\overline{T}F$ together with the natural transformation $U\sigma^T_F : U\overline{T}F \to UTF$ form an ideal. The monad UTF is completely iterative with respect to this ideal.*

Proof. Right adjoints preserve monomorphisms, hence the components of natural transformation $U\sigma^T_F$ are monic, and so $U\overline{T}F$ is a subfunctor. We define $\overline{\mu}$ to be $U\overline{\mu^T_F} \cdot U\overline{T}\varepsilon_{TF}$. It satisfies the condition for ideals:

$$
\begin{tikzcd}
U\overline{T}FUTF \arrow[r, "U\sigma^T_{FUTF}"] \arrow[d, "U\overline{T}\varepsilon_{TF}"] \arrow[dd, bend right=60, "\overline{\mu}"'] & UTFUTF \arrow[d, "UT\varepsilon_{TF}"] \arrow[dd, bend left=60, "\mu"] \\
U\overline{T}TF \arrow[r, "U\sigma^T_{TF}"] \arrow[d, "U\overline{\mu^T_F}"] & UT^2F \arrow[d, "U\mu^T_F"] \\
U\overline{T}F \arrow[r, "U\sigma^T_F"] & UTF
\end{tikzcd}
$$

The top square is the naturality of σ^T. The bottom square is the U-image of the ideal condition for T.

Let $e : X \to UTF(A+X)$ be a $U\sigma^T_F$-guarded equation morphism. By $\lfloor\text{-}\rfloor : \mathcal{C}[FA, B] \cong \mathcal{B}[A, UB] : \lceil\text{-}\rceil$ we denote the natural isomorphism associated with the adjunction. Recall that left adjoints preserve coproducts, that is $F(A+B) \cong FA + FB$. We calculate:

$$e = [\sigma^{UTF}_{A+X}, \eta^{UTF}_{A+X} \cdot \mathsf{inl}_{A,X}] \cdot j$$

$\Leftrightarrow$ $\{$ definitions $\}$

$$e = [U\sigma^T_{F(A+X)}, U\eta^T_{F(A+X)} \cdot \eta_{A+X} \cdot \mathsf{inl}_{A,X}] \cdot j$$

$\Leftrightarrow$ $\{$ F-image and F preserve coproducts $\}$

$$Fe = [FU\sigma^T_{F(A+X)}, FU\eta^T_{F(A+X)} \cdot F\eta_{A+X} \cdot F\mathsf{inl}_{A,X}] \cdot Fj$$

$\Rightarrow$ $\{$ coposing both sides with ε $\}$

$$\varepsilon_{TF(A+X)} \cdot Fe = [\varepsilon_{TF(A+X)} \cdot FU\sigma^T_{F(A+X)},$$
$$\varepsilon_{TF(A+X)} \cdot FU\eta^T_{F(A+X)} \cdot F\eta_{A+X} \cdot F\mathsf{inl}_{A,X}] \cdot Fj$$

$\Leftrightarrow$ $\{$ right adjunct and naturality $\}$

$$\lceil e \rceil = [\sigma^T_{F(A+X)} \cdot \varepsilon_{\overline{T}F(A+X)}, \eta^T_{F(A+X)} \cdot \varepsilon_{F(A+X)} \cdot F\eta_{A+X} \cdot F\mathsf{inl}_{A,X}] \cdot Fj$$

$\Leftrightarrow$ $\{$ zig-zag equalities $\}$

$$\lceil e \rceil = [\sigma^T_{F(A+X)} \cdot \varepsilon_{\overline{T}F(A+X)}, \eta^T_{F(A+X)} \cdot F\mathsf{inl}_{A,X}] \cdot Fj$$

$\Leftrightarrow$ $\{$ coproducts $\}$

$$\lceil e \rceil = [\sigma^T_{F(A+X)}, \eta^T_{F(A+X)} \cdot F\mathsf{inl}_{A,X}] \cdot (\varepsilon_{\overline{T}F(A+X)} + \mathsf{id}_A) \cdot Fj$$

$\Leftrightarrow$ $\{$ F preserves coproducts $\}$

$$\lceil e \rceil \cong [\sigma^T_{(FA+FX)}, \eta^T_{(FA+FX)} \cdot \mathsf{inl}_{(FA,FX)}] \cdot (\varepsilon_{\overline{T}F(A+X)} + \mathsf{id}_A) \cdot Fj$$

This means that $\lceil e \rceil : FX \to TF(A+X) \cong T(FA+FX)$ is a guarded equation morphism in T with a unique solution $\lceil e \rceil^\dagger : FX \to TFA$.

We define the solution of e as $\lfloor\lceil e\rceil^\dagger\rfloor$. The following diagram commutes:

$$\begin{tikzcd}
X \arrow[drr, bend left, "\lfloor\lceil e\rceil^\dagger\rfloor = U\lceil e\rceil^\dagger \cdot \eta_X"] \arrow[dr, "\eta_X"] \arrow[ddr, bend right, "e"'] & & & \\
& UFX \arrow[r, "U\lceil e\rceil^\dagger"] \arrow[d, "U\lceil e\rceil"] & UTFA & \\
& \begin{array}{c} UTF(A+X) \\ \cong UT(FA+FX) \end{array} \arrow[r, "UT[\eta^T_{FA}, \lceil e\rceil^\dagger]"] \arrow[drr, bend right, "{UTF[\eta^{UTF}_A, U\lceil e\rceil^\dagger \cdot \eta_X]}"'] & UT^2FA \arrow[u, "U\mu^T_{FA}"] & \\
& & & (UTF)^2A \arrow[ul, "UT\varepsilon_{TFA}"] \arrow[uul, bend right, "\mu^{UTF}_A"']
\end{tikzcd}$$

The inner square is the U-image of the solution diagram for $\lceil e\rceil^\dagger$. The outer triangles commute due to properties of adjunctions and the definition of μ^{UTF}.

For uniqueness, let $g : X \to UTFA$ be a solution of e. Substitute $\lceil g\rceil$ for $\lceil e\rceil^\dagger$ in the above diagram. The outer square commutes, because $\lfloor\lceil g\rceil\rfloor = g$ is a solution, and the triangles commute, because of properties of adjunctions, hence the inner square precomposed with η_X also commutes. For all morphisms $f, f' : FB \to C$, if $Uf \cdot \eta_B = Uf' \cdot \eta_B$ then $f = f'$. Therefore, $\lceil g\rceil$ is a solution of $\lceil e\rceil$, so $\lceil g\rceil = \lceil e\rceil^\dagger$, hence $g = \lfloor\lceil g\rceil\rfloor = \lfloor\lceil e\rceil^\dagger\rfloor$. □

3.1 Tracing

Intuitively, T collects observations about a computation in M. Thus, we need a new operation that allows us to actually observe the current state of the computation, for example the current state in the State monad (this example is elaborated in the next section). It could be given as a natural transformation $\mathsf{olift} : M \to UTF$ with components that factor through $U\sigma^T_F$. It will not in general be a monad morphism; on the contrary, performing two actions and then observing the effect differs in general from observing the effect of each action individually. More formally, let $f \circ g$ be a computation in the Kleisli category of M. We can decorate it with observers in two different ways: $\mathsf{olift} \cdot (f \circ g)$ or $(\mathsf{olift} \cdot f) \circ (\mathsf{olift} \cdot g)$. For example, when tracing a computation in State, we may want to observe only 'set' operations, as long as we are certain that there are only finitely many invocations of 'get' in between every two invocations of 'set'. In the rest of the paper we always define olift as $U\mathsf{obs}$ for a natural transformation $\mathsf{obs} : F \to TF$. For convenience, we define a 'save the current state of computation' operation $\mathsf{save} = \mathsf{olift} \cdot \eta : \mathsf{Id} \to UTF$.

Though we do not use this property directly in the rest of the article, observations should not modify the computation. This could be captured by the following cancellation property: for all morphisms $f, f' : A \to MB$ and $g, g' : B \to MC$, if $(\mathsf{lift} \cdot g) \circ \mathsf{save}_B \circ (\mathsf{lift} \cdot f) = (\mathsf{lift} \cdot g') \circ \mathsf{save}_B \circ (\mathsf{lift} \cdot f')$ then $g \circ f = g' \circ f'$.

4 The States monad

Our first example is a monad we call States. Consider the currying adjunction $- \times S \dashv -^S$ that gives rise to the State monad on cartesian closed categories. We choose $(- \times S)^\infty$, for which we write $\overrightarrow{S}$, to be the inner cim, and the result is the monad $A \mapsto (\overrightarrow{S}(A \times S))^S$. Intuitively, $\overrightarrow{S}$ is a possibly infinite stream of states of type S. The 'base' of the exponential is the trace of the computation: a stream that, if finite, is terminated with an answer A and a current state S. The latter is used only to compose two computations and is not stored in the stream.

We define 'put' and 'get' operations as standard liftings of 'put' and 'get' for State. The natural transformation obs duplicates the current state and puts it in the stream as follows.

$$A \times S \xrightarrow{\langle\langle \mathsf{outl}, \mathsf{outr}\rangle, \mathsf{outr}\rangle} (A \times S) \times S \xrightarrow{\mathsf{emb}_{A\times S}} \overrightarrow{S}(A \times S)$$

For example, consider the following computation in States on **Set** for $S = \mathbb{N}$ (using Haskell syntax):

```
let f = do {put 2; save; put 3; save; put 5}
    g = do {x <- get; put (x+1); save; g}
in do {f; g}
```

For any initial state, `f` evaluates to the trace $(2, 3, \langle\star, 5\rangle)$, while the whole computation evaluates to $(2, 3, 6, 7, 8, 9, \ldots)$.

4.1 Example: Control structures for While

Consider a generalised While language, as given by Rutten [26]:

$$P, Q \ ::= \ A \mid P; Q \mid \textbf{if } b \textbf{ then } P \textbf{ else } Q \mid \textbf{while } b \textbf{ do } P$$

For a monad M, the symbol A represents a set of actions (denoted as $\underline{a}$), that is morphisms of type $1 \to M1$. The symbol b represents a set B of Boolean expressions, that is a set of morphisms of type $1 \to M(1+1)$. We parametrise the semantics with a 'guard' operation $\gamma : 1 \to M1$, which allows the addition of behaviour on every choice point of a control structure. The denotation of a program P is given by $[\![P]\!] : 1 \to M1$, defined as follows, where $\circ$ is Kleisli composition.

$$\begin{aligned}
[\![\underline{a}]\!] &= \underline{a} \\
[\![P; Q]\!] &= [\![Q]\!] \circ [\![P]\!] \\
[\![\textbf{if } b \textbf{ then } P \textbf{ else } Q]\!] &= [[\![P]\!], [\![Q]\!]] \circ b \circ \gamma \\
[\![\textbf{while } b \textbf{ do } P]\!] &= ([M\mathsf{inr}_{1,1} \cdot [\![P]\!],\ M\mathsf{inl}_{1,1} \cdot \eta_1^M] \circ b \circ \gamma)^\dagger
\end{aligned}$$

Actions denote themselves, and compositions of programs are just Kleisli compositions of morphisms. The denotation of **if** statements first performs the guard γ, then b, and then the appropriate branch is chosen (we use the left component of $1 + 1$ to represent 'true'). The denotation of **while** first builds an equation morphism by composing the guard, the condition, and the choice between returning the left component of the coproduct (a constant, which means 'stop the iteration'), or performing the body, and right-injecting the result (which makes it a 'continue the iteration' variable). The denotation of the entire **while** expression is a solution to that morphism. The solution might not exist, or might not be unique; hence, depending on the choice of M, A, B, and γ, the denotation might not be well-defined. This semantics specialises to a couple of known cases:

If we choose the regular State monad on **Dcppo** (the category of pointed directed-complete partial orders and continuous functions) for M and its unit on 1 for γ, the solution diagram simplifies to the familiar equation for denotation of While [23, Chapter 4]. So, if we assume $-^{\dagger}$ to be the least fixed point, we yield the standard denotational semantics.

If we instantiate M with a cim, we can ensure that unique solutions always exist by an appropriate γ-guarding of **while** loops. (Note that it is not sufficient to ask for the A actions to be guarded, since **while true do while false do** $\underline{a}$ diverges without invoking an action.) In case of the States monad, this means that every iteration stores its initial state in the stream, that is $\gamma = \mathsf{save}$. Additionally, if we assume that 'put' operations are always guarded and 'get' are not, we obtain a semantics trace-equivalent to Nakata and Uustalu's trace operational semantics [22].

5 Resumptions

5.1 Monadic structure

Let $\langle M, \eta^M, \mu^M \rangle$ be a monad, and Σ be an endofunctor on the base category $\mathcal{B}$. In this section we give a monadic structure to $M(\Sigma M)^{\infty}$ and examine its basic properties. We proceed by first giving a monadic structure to the endofunctor

$$KA = \nu X.M(A + \Sigma X),$$

which is isomorphic to $M(\Sigma M)^{\infty}$ through the coalgebraic version of the rolling rule [5]:

Lemma 6. *Let F, G be endofunctors. Then $\nu FG \cong F\nu GF$.*

Proof. Let $\langle \nu FG, \beta : \nu FG \to FG\nu FG \rangle$ be the final FG-coalgebra, and $\langle \nu GF, \gamma : \nu GF \to GF\nu GF \rangle$ be the final GF-coalgebra. We define the following morphisms (subscripts for the lens brackets indicate the functor for the final coalgebra):

$$\begin{aligned} r &: \nu FG \to F\nu GF \\ r &= F[\![G\beta]\!]_{GF} \cdot \beta \end{aligned}$$

$$\begin{aligned} r^{-1} &: F\nu GF \to \nu FG \\ r^{-1} &= [\![F\gamma]\!]_{FG} \end{aligned}$$

To prove that r is an isomorphism, with r^{-1} being its inverse, we show that $\beta \cdot r^{-1} \cdot r = FGr^{-1} \cdot FGr \cdot \beta$, which means that $r^{-1} \cdot r$ is an FG-coalgebra homomorphism, $r^{-1} \cdot r : \langle \nu FG, \beta \rangle \to \langle \nu FG, \beta \rangle$. By uniqueness and reflexion, $r^{-1} \cdot r = [\![\beta]\!] = id_{\nu FG}$.

$$
\begin{aligned}
& \beta \cdot r^{-1} \cdot r = \beta \cdot [\![F\gamma]\!]_{FG} \cdot F[\![G\beta]\!]_{GF} \cdot \beta \\
= \quad & \{ \text{ computation } \} \\
& \beta \cdot \beta^{-1} \cdot FG[\![F\gamma]\!]_{FG} \cdot F\gamma \cdot F\gamma^{-1} \cdot FGF[\![G\beta]\!]_{GF} \cdot FG\beta \cdot \beta \\
= \quad & \{ \text{ cancellation of } \gamma \text{ and } \gamma^{-1}, \text{ and } \beta \text{ and } \beta^{-1} \} \\
& FG[\![F\gamma]\!]_{FG} \cdot FGF[\![G\beta]\!]_{GF} \cdot FG\beta \cdot \beta \\
= \quad & \{ \text{ definitions of } r \text{ and } r^{-1} \} \\
& FGr^{-1} \cdot FGFr \cdot \beta
\end{aligned}
$$

To prove the converse, we notice the following.

$$
\begin{aligned}
& r \cdot r^{-1} = F[\![G\beta]\!]_{GF} \cdot \beta \cdot [\![F\gamma]\!]_{FG} \\
= \quad & \{ \text{ computation } \} \\
& F[\![G\beta]\!]_{GF} \cdot \beta \cdot \beta^{-1} \cdot FG[\![F\gamma]\!]_{FG} \cdot F\gamma \\
= \quad & \{ \text{ cancellation of } \beta \text{ and } \beta^{-1} \} \\
& F[\![G\beta]\!]_{GF} \cdot FG[\![F\gamma]\!]_{FG} \cdot F\gamma \\
= \quad & \{ \text{ (see below) } \} \\
& F\mathsf{id}_{\nu GF} = id_{F\nu GF}
\end{aligned}
$$

The left-hand side of the penultimate equation is an F-image of $r^{-1} \cdot r$, but with F and G swapped. Thus, we can prove similarly to the previous case that it is equal to the identity on νGF. $\square$

For convenience, we define two auxiliary natural transformations. The first one, $\mathsf{flat}_{A,B} : M(MA+B) \to M(A+B)$, flattens a computation that may return a value or a new computation.

$$
\begin{aligned}
\mathsf{flat}_{A,B} = \quad & M(MA+B) \\
& \downarrow M(\mathsf{id}_{MA} + \eta^M_B) \\
& M(MA+MB) \\
& \downarrow M[M\mathsf{inl}_{A,B}, M\mathsf{inr}_{A,B}] \\
& M^2(A+B) \\
& \downarrow \mu^M_{A+B} \\
& M(A+B)
\end{aligned}
$$

The second one, unf, unfolds and flattens two levels of structure of K. In the following, α is the final coalgebra map $\alpha : K \to M(\mathsf{Id} + \Sigma K)$.

$$\mathsf{unf} = \begin{array}{c} K^2 \\ \downarrow \alpha_K \\ M(K + \Sigma K^2) \\ \downarrow M(\alpha + \mathsf{id}_{\Sigma K^2}) \\ M(M(\mathsf{Id} + \Sigma K) + \Sigma K^2) \\ \downarrow \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\ M(\mathsf{Id} + \Sigma K + \Sigma K^2) \end{array}$$

The unit (return) of the monad K, η^K, is given below. The multiplication (join) is defined as a anamorphism $\mu^K = [\![m]\!]$ of the following transformation m.

$$\eta^K = \begin{array}{c} \mathsf{Id} \\ \downarrow \mathsf{inl}_{\mathsf{Id},\Sigma K} \\ \mathsf{Id} + \Sigma K \\ \downarrow \eta^M_{\mathsf{Id}+\Sigma K} \\ M(\mathsf{Id} + \Sigma K) \\ \downarrow \alpha^{-1} \\ K \end{array} \qquad m = \begin{array}{c} K^2 \\ \downarrow \mathsf{unf} \\ M(\mathsf{Id} + \Sigma K + \Sigma K^2) \\ \downarrow M(\mathsf{id} + \Sigma \eta^K_K + \mathsf{id}_{\Sigma K^2}) \\ M(\mathsf{Id} + \Sigma K^2 + \Sigma K^2) \\ \downarrow M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \\ M(\mathsf{Id} + \Sigma K^2) \end{array}$$

Also, for any natural transformation $u : K^2 \to K$, we define the transformation $\widetilde{u}$ to be the following composition.

$$\begin{array}{c} K^2 \\ \downarrow \mathsf{unf} \\ M(\mathsf{Id} + \Sigma K + \Sigma K^2) \\ \downarrow M(\mathsf{id} + [\mathsf{id}_{\Sigma K}, \Sigma u]) \\ M(\mathsf{Id} + \Sigma K) \\ \downarrow \alpha^{-1} \\ K \end{array}$$

Theorem 7. *The following hold:*

1. *The triple $\langle K, \eta^K, \mu^K \rangle$ is a monad.*

2. *For a transformation $u : K^2 \to K$ that cancels η^K_K on the left (that is $u \cdot \eta^K_K = \mathsf{id}_K$), the following universal property holds: $u = \widetilde{u}$ if and only if $u = \mu^K$.*

Proof. **Right unit.** The equations below prove that $m \cdot K\eta^K = M(\mathsf{id} + \Sigma K \eta^K) \cdot \alpha$, which means that $K\eta^K$ is a coalgebra homomorphism $K\eta^K : \langle K, \alpha \rangle \to \langle K^2, m \rangle$, and so is $[\![m]\!] \cdot K\eta^K : \langle K, \alpha \rangle \to \langle K, \alpha \rangle$, since composition of two homomorphisms is a homomorphism. By uniqueness, $\mu^K \cdot K\eta^K = \llbracket m \rrbracket \cdot K\eta^K = \llbracket \alpha \rrbracket = \mathsf{id}_K$. Diagrammatically:

$$\begin{tikzcd}
\langle K, \alpha \rangle \arrow[dd, "\mathsf{id}_K = \llbracket \alpha \rrbracket"'] \arrow[dr, "K\eta^K"] & \\
& \langle K^2, m \rangle \arrow[dl, "\mu = \llbracket m \rrbracket"] \\
\langle K, \alpha \rangle &
\end{tikzcd}$$

$$\begin{aligned}
& m \cdot K\eta^K \\
= \quad & \{ \text{ definitions } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \cdot K\alpha^{-1} \cdot K\eta^M_{\mathsf{Id}+\Sigma K} \cdot K\mathsf{inl}_{\mathsf{Id},\Sigma K} \\
= \quad & \{ \text{ naturality of } \alpha \text{ } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot M(\alpha^{-1} + \Sigma K\alpha^{-1}) \\
& \quad \cdot M(\eta^M_{\mathsf{Id}+\Sigma K} + \Sigma K\eta^M_{\mathsf{Id}+\Sigma K}) \cdot M(\mathsf{inl}_{\mathsf{Id},\Sigma K} + \Sigma K\mathsf{inl}_{\mathsf{Id},\Sigma K}) \cdot \alpha \\
= \quad & \{ \text{ cancellation of } \alpha \text{ and } \alpha^{-1} \text{ } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot M(\mathsf{id}_{M(\mathsf{Id}+\Sigma K)} + \Sigma K\alpha^{-1}) \cdot M(\eta^M_{\mathsf{Id}+\Sigma K} + \Sigma K\eta^M_{\mathsf{Id}+\Sigma K}) \\
& \quad \cdot M(\mathsf{inl}_{\mathsf{Id},\Sigma K} + \Sigma K\mathsf{inl}_{\mathsf{Id},\Sigma K}) \cdot \alpha \\
= \quad & \{ \text{ naturality of } \mathsf{flat} \text{ } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K} + \Sigma K\alpha^{-1}) \\
& \quad \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K M(\mathsf{Id}+\Sigma K)} \cdot M(\eta^M_{\mathsf{Id}+\Sigma K} + \Sigma K\eta^M_{\mathsf{Id}+\Sigma K}) \\
& \quad \cdot M(\mathsf{inl}_{\mathsf{Id},\Sigma K} + \Sigma K\mathsf{inl}_{\mathsf{Id},\Sigma K}) \cdot \alpha \\
= \quad & \{ \text{ monad laws } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K} + \Sigma K\alpha^{-1}) \\
& \quad \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K} + \Sigma K\eta^M_{\mathsf{Id}+\Sigma K}) \cdot M(\mathsf{inl}_{\mathsf{Id},\Sigma K} + \Sigma K\mathsf{inl}_{\mathsf{Id},\Sigma K}) \cdot \alpha \\
= \quad & \{ \text{ istributivity of composition over coproduct } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{inl}_{\mathsf{Id},\Sigma K} + \Sigma K\alpha^{-1}) \\
& \quad \cdot M(\mathsf{id} + \Sigma K\eta^M_{\mathsf{Id}+\Sigma K}) \cdot M(\mathsf{id} + \Sigma K\mathsf{inl}_{\mathsf{Id},\Sigma K}) \cdot \alpha \\
= \quad & \{ \text{ associativity of coproducts } \}
\end{aligned}$$

$$M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \mathsf{inr}_{\Sigma K^2, \Sigma K^2}) \cdot M(\mathsf{id} + \Sigma K \alpha^{-1}) \\ \cdot M(\mathsf{id} + \Sigma K \eta^M_{\mathsf{Id}+\Sigma K}) \cdot M(\mathsf{id} + \Sigma K \mathsf{inl}_{\mathsf{Id}, \Sigma K}) \cdot \alpha$$

$= \quad \{$ properties of coproducts $\}$

$$M(\mathsf{id} + \Sigma K \alpha^{-1}) \cdot M(\mathsf{id} + \Sigma K \eta^M_{\mathsf{Id}+\Sigma K}) \cdot M(\mathsf{id} + \Sigma K \mathsf{inl}_{\mathsf{Id}, \Sigma K}) \cdot \alpha$$

$= \quad \{$ definition of η^K $\}$

$$M(\mathsf{id} + \Sigma K \eta^K) \cdot \alpha$$

Left unit. Similarily to the previous case, we prove that $m \cdot \eta^K_K = M(\mathsf{id} + \Sigma \eta^K_K) \cdot \alpha$, which means that η^K_K is a coalgebra homomorphism $\eta^K_K : \langle K, \alpha \rangle \to \langle K^2, m \rangle$. By uniqueness and reflexion, $\mu^K \cdot \eta^K_K = \llbracket m \rrbracket \cdot \eta^K_K = \llbracket \alpha \rrbracket = \mathsf{id}_K$. Diagrammatically:

$$\begin{tikzcd}
\langle K, \alpha \rangle \arrow[dd, "\mathsf{id}_K = \llbracket \alpha \rrbracket"'] \arrow[dr, "\eta^K_K"] & \\
& \langle K^2, m \rangle \arrow[dl, "\mu = \llbracket m \rrbracket"] \\
\langle K, \alpha \rangle &
\end{tikzcd}$$

$$m \cdot \eta^K_K$$

$= \quad \{$ definitions $\}$

$$M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma \eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\ \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \cdot \alpha^{-1}_K \cdot \eta^M_{K+\Sigma K^2} \cdot \mathsf{inl}_{K, \Sigma K^2}$$

$= \quad \{$ cancellation of α_K nad α^{-1}_K, and naturality of η^M $\}$

$$M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma \eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\ \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot M \mathsf{inl}_{K, \Sigma K^2} \cdot \eta^M_K$$

$= \quad \{$ naturality of inl $\}$

$$M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma \eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\ \cdot M \mathsf{inl}_{M(\mathsf{Id}+\Sigma K), \Sigma K^2} \cdot M \alpha \cdot \eta^M_K$$

$= \quad \{$ naturality of η^M $\}$

$$M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma \eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\ \cdot M \mathsf{inl}_{M(\mathsf{Id}+\Sigma K), \Sigma K^2} \cdot \eta^M_{M(\mathsf{Id}+\Sigma K)} \cdot \alpha$$

$= \quad \{$ monad laws $\}$

$$
\begin{aligned}
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot M\mathsf{inl}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot \mu^M_{\mathsf{Id}+\Sigma K} \cdot \eta^M_{M(\mathsf{Id}+\Sigma K)} \cdot \alpha \\
= \quad & \{ \text{ monad laws } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot M\mathsf{inl}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \cdot \alpha \\
= \quad & \{ \text{ naturality of } \mathsf{inl} \ \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M\mathsf{inl}_{\mathsf{Id}+\Sigma K^2, \Sigma K^2} \cdot M(\mathsf{id} + \Sigma\eta^K_K) \cdot \alpha \\
= \quad & \{ \text{ properties of coproducts } \} \\
& M(\mathsf{id} + \Sigma\eta^K_K) \cdot \alpha
\end{aligned}
$$

Universal property of μ. Let $u : K^2 \to K$ be a natural transformation such that $u = \widetilde{u}$. We prove below that $M(\mathsf{id} + \Sigma u) \cdot m = \alpha \cdot u$, which means that u is a coalgebra homomorphism $u : \langle K^2, m \rangle \to \langle K, \alpha \rangle$. By uniqueness, $u = [\![m]\!] = \mu^K$.

$$
\begin{aligned}
& M(\mathsf{id} + \Sigma u) \cdot m \\
= \quad & \{ \text{ definition } \} \\
& M(\mathsf{id} + \Sigma u) \cdot M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \\
& \quad \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \\
= \quad & \{ \text{ naturality of } \mathsf{gr} \text{ and } \mathsf{flat}, \text{ and } u = \widetilde{u} \ \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K} \cdot M(M(\mathsf{id} + \Sigma\widetilde{u}) + \Sigma u) \\
& \cdot M(M(\mathsf{id} + \Sigma\eta^K_K) + \mathsf{id}_{\Sigma K^2}) \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \\
= \quad & \{ \ \widetilde{u} \cdot \eta^K_K = \mathsf{id}_K \ \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K} \cdot M(\mathsf{id}_{M(\mathsf{Id}+\Sigma K)} + \Sigma u) \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \\
= \quad & \{ \text{ naturality of } \mathsf{flat} \ \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K}) \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K} + \Sigma u) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \\
= \quad & \{ \text{ introduction of } \alpha \cdot \alpha^{-1} \text{ on the left and definition of } u \ \} \\
& \alpha \cdot u
\end{aligned}
$$

Conversely, assume that $u = \mu^K = [\![m]\!]$, which gives us $M(\mathsf{id} + \Sigma u) \cdot m = \alpha \cdot u$, since $[\![m]\!]$ is by definition a coalgebra homomorphism. We prove that $u = \widetilde{u}$ as follows.

$$
\begin{aligned}
& u = \alpha^{-1} \cdot \alpha \cdot u = \alpha^{-1} \cdot M(\mathsf{id} + \Sigma u) \cdot m \\
= \quad & \{ \text{ definition of } m \ \} \\
& \alpha^{-1} \cdot M(\mathsf{id} + \Sigma u) \cdot M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \\
& \quad \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot \alpha_K
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \{\ \mathsf{gr} \text{ is natural }\} \\
& \alpha^{-1} \cdot M(\mathsf{id} + \mathsf{gr}_{\Sigma K}) \cdot M(\mathsf{id} + \Sigma u + \Sigma u) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \\
& \quad \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2} \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \\
= \quad & \{\ u \text{ cancels } \eta^K_K, \text{ definition of } \widetilde{u}\ \} \\
& \widetilde{u}
\end{aligned}
$$

Associativity. To prove that $\mu^K \cdot \mu^K_K = \mu^K \cdot K\mu^K$, we introduce an auxiliary transformation $w : K^3 \to M(\mathsf{Id} + \Sigma K^3)$ defined as the following composition:

$$
\begin{tikzcd}
K^3 \arrow[d, "\alpha_{K^2}"] \\
M(K^2 + \Sigma K^3) \arrow[d, "M(\alpha_K + \mathsf{id}_{\Sigma K^3})"] \\
M(M(K + \Sigma K^2) + \Sigma K^3) \arrow[d, "\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}"] \\
M(K + \Sigma K^2 + \Sigma K^3) \arrow[d, "M(\alpha + id_{\Sigma K^2+\Sigma K^3})"] \\
M(M(\mathsf{Id} + \Sigma K) + \Sigma K^2 + \Sigma K^3) \arrow[d, "\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2+\Sigma K^3}"] \\
M(\mathsf{Id} + \Sigma K + \Sigma K^2 + \Sigma K^3) \arrow[d, "M(\mathsf{id}_{\mathsf{Id}} + \Sigma(\eta^K_{K^2} \cdot \eta^K_K) + \Sigma\eta^K_{K^2} + \Sigma\mathsf{id}_{K^3})"] \\
M(\mathsf{Id} + \Sigma K^3 + \Sigma K^3 + \Sigma K^3) \arrow[d, "M(\mathsf{id}_{\mathsf{Id}+\Sigma K^3} + \mathsf{gr}_{\Sigma K^3})"] \\
M(\mathsf{Id} + \Sigma K^3 + \Sigma K^3) \arrow[d, "M(\mathsf{id} + \mathsf{gr}_{\Sigma K^3})"] \\
M(\mathsf{Id} + \Sigma K^3)
\end{tikzcd}
$$

Below, we prove that $m \cdot \mu^K_K = M(\mathsf{id} + \Sigma\mu^K_K) \cdot w$ and $m \cdot K\mu^K = M(\mathsf{id} + \Sigma K\mu^K) \cdot w$, which means that both μ^K_K and $K\mu^K$ are coalgebra homomorphisms $\mu^K_K, K\mu^K : \langle K^3, w\rangle \to \langle K^2, m\rangle$. By uniqueness, $\mu^K \cdot \mu^K_K = [\![m]\!] \cdot \mu^K_K = [\![w]\!] = [\![m]\!] \cdot K\mu^K = \mu^K \cdot K\mu^K$. Diagrammatically:

$$
\begin{tikzcd}
 & \langle K^3, w\rangle \arrow[dl, "\mu^K_K"'] \arrow[dr, "K\mu^K"] \arrow[dd, "[\![w]\!]"'] & \\
\langle K^2, m\rangle \arrow[dr, "[\![m]\!]"'] & & \langle K^2, m\rangle \arrow[dl, "[\![m]\!]"] \\
 & \langle K, \alpha\rangle &
\end{tikzcd}
$$

$$
\begin{aligned}
& m \cdot \mu^K_K \\
= \quad & \{ \text{ definition of } m, \text{ universal property of } \mu^K \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \cdot \alpha_K^{-1} \cdot M(\mathsf{id}_K + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}_K + \mathsf{id}_{\Sigma K^2} + \Sigma\mu^K_K) \\
& \quad \cdot \mathsf{flat}_{K+\Sigma K^2, \Sigma K^3} \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ cancellation of } \alpha_K \text{ and } \alpha_K^{-1}, \text{ and naturality of } \alpha \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot M(\mathsf{id}_{M(\mathsf{Id}+\Sigma K)} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}_{M(\mathsf{Id}+\Sigma K)} + \mathsf{id}_{\Sigma K^2} + \Sigma\mu^K_K) \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2 + \Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2, \Sigma K^3} \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ naturality of } \mathsf{flat} \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K} + \mathsf{gr}_{\Sigma K^2}) \\
& \quad \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K} + \mathsf{id}_{\Sigma K^2} + \Sigma\mu^K_K) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2 + \Sigma K^3} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2 + \Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2, \Sigma K^3} \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ distributivity of composition over coporduct } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K^2} + \mathsf{gr}_{\Sigma K^2}) \\
& \quad \cdot M(\mathsf{id}_{\mathsf{Id}} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2} + \Sigma\mu^K_K) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2 + \Sigma K^3} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2 + \Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2, \Sigma K^3} \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ naturality of } \mathsf{gr}, \text{ right unit for } K \} \\
& M(\mathsf{id} + \Sigma\mu^K_K) \cdot M(\mathsf{id} + \mathsf{gr}_{\Sigma K^3}) \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K^3} + \mathsf{gr}_{\Sigma K^3}) \\
& \quad \cdot M(\mathsf{id}_{\mathsf{Id}} + \Sigma(\eta^K_{K^2} \cdot \eta^K_K) + \Sigma\eta^K_{K^2} + \Sigma\mathsf{id}_{K^3}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2 + \Sigma K^3} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2 + \Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2, \Sigma K^3} \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ definition of } w \} \\
& M(\mathsf{id} + \Sigma\mu^K_K) \cdot w
\end{aligned}
$$

* * *

$$
\begin{aligned}
& m \cdot K\mu^K \\
= \quad & \{ \text{ definition of } m \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot \alpha_K \cdot K\mu^K \\
= \quad & \{ \text{ naturality of } \alpha \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot M(\mu^K + \Sigma K\mu^K) \cdot \alpha_{K^2}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \{ \text{ properties of coproducts } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_K + \Sigma K \mu^K) \cdot M(\mu^K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ universal property of } \mu^K \text{ } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_K + \Sigma K \mu^K) \cdot M(\alpha^{-1} + \mathsf{id}_{\Sigma K^3}) \\
& \quad \cdot M(M(\mathsf{id} + \mathsf{gr}_{\Sigma K}) + \mathsf{id}_{\Sigma K^3}) \cdot M(M(\mathsf{id} + \mathsf{id}_{\Sigma K} + \Sigma \mu^K) + \mathsf{id}_{\Sigma K^3}) \\
& \quad \cdot M(\mathsf{flat}_{\mathsf{Id}+\Sigma K^2, \Sigma K^2} + \mathsf{id}_{\Sigma K^3}) \cdot M(M(\alpha + \mathsf{id}_{\Sigma K^2}) + \mathsf{id}_{\Sigma K^3}) \\
& \quad \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ cancellation of } \alpha \text{ and } \alpha^{-1} \text{ } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2} \\
& \quad \cdot M(\mathsf{id}_{M(\mathsf{Id}+\Sigma K)} + \Sigma K \mu^K) \\
& \quad \cdot M(M(\mathsf{id} + \mathsf{gr}_{\Sigma K}) + \mathsf{id}_{\Sigma K^3}) \cdot M(M(\mathsf{id} + \mathsf{id}_{\Sigma K} + \Sigma \mu^K) + \mathsf{id}_{\Sigma K^3}) \\
& \quad \cdot M(\mathsf{flat}_{\mathsf{Id}+\Sigma K^2, \Sigma K^2} + \mathsf{id}_{\Sigma K^3}) \cdot M(M(\alpha + \mathsf{id}_{\Sigma K^2}) + \mathsf{id}_{\Sigma K^3}) \\
& \quad \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ naturality of } \mathsf{flat} \text{ } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K} + \Sigma K \mu^K) \\
& \quad \cdot M(\mathsf{id} + \mathsf{gr}_{\Sigma K} + \mathsf{id}_{\Sigma K^3}) \cdot M(\mathsf{id} + \mathsf{id}_{\Sigma K} + \Sigma \mu^K + \mathsf{id}_{\Sigma K^3}) \\
& \quad \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K^2+\Sigma K^2, \Sigma K^3} \cdot M(\mathsf{flat}_{\mathsf{Id}+\Sigma K^2, \Sigma K^2} + \mathsf{id}_{\Sigma K^3}) \\
& \quad \cdot M(M(\alpha + \mathsf{id}_{\Sigma K^2}) + \mathsf{id}_{\Sigma K^3}) \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ monad laws } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K} + \Sigma K \mu^K) \\
& \quad \cdot M(\mathsf{id} + \mathsf{gr}_{\Sigma K} + \mathsf{id}_{\Sigma K^3}) \cdot M(\mathsf{id} + \mathsf{id}_{\Sigma K} + \Sigma \mu^K + \mathsf{id}_{\Sigma K^3}) \\
& \quad \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2+\Sigma K^3} \cdot M(\alpha + \mathsf{id}_{\Sigma K^2} + \mathsf{id}_{\Sigma K^3}) \\
& \quad \cdot \mathsf{flat}_{K+\Sigma K^2, \Sigma K^3} \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ naturality of } \mathsf{gr} \text{ } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2} + \mathsf{id}_{\Sigma K^2}) \\
& \quad \cdot M(\mathsf{id} + \Sigma\eta^K_K + \Sigma\eta^K_K + \mathsf{id}_{\Sigma K^2}) \cdot M(\mathsf{id}_{\mathsf{Id}+\Sigma K+\Sigma K} + \Sigma K \mu^K) \\
& \quad \cdot M(\mathsf{id} + \mathsf{id}_{\Sigma K} + \Sigma \mu^K + \mathsf{id}_{\Sigma K^3}) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2+\Sigma K^3} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2} + \mathsf{id}_{\Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2, \Sigma K^3} \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2} \\
= \quad & \{ \text{ properties of coproducts } \} \\
& M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2}) \cdot M(\mathsf{id} + \mathsf{gr}_{\Sigma K^2} + \mathsf{id}_{\Sigma K^2}) \\
& \quad \cdot M(\mathsf{id} + \Sigma\eta^K_K + \Sigma(\eta^K_K \cdot \mu^K) + \Sigma K \mu^K) \cdot \mathsf{flat}_{\mathsf{Id}+\Sigma K, \Sigma K^2+\Sigma K^3} \\
& \quad \cdot M(\alpha + \mathsf{id}_{\Sigma K^2} + \mathsf{id}_{\Sigma K^3}) \cdot \mathsf{flat}_{K+\Sigma K^2, \Sigma K^3} \cdot M(\alpha_K + \mathsf{id}_{\Sigma K^3}) \cdot \alpha_{K^2}
\end{aligned}
$$

$=\qquad \{$ naturality of η^K $\}$

$$M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2})\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^2})$$
$$\cdot M(\mathsf{id}+\Sigma\eta^K_K+\Sigma(K\mu^K\cdot\eta^K_{K^2})+\Sigma K\mu^K)\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2+\Sigma K^3}$$
$$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3})\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$$

$=\qquad \{$ right unit for K, naturality of gr $\}$

$$M(\mathsf{id}+\Sigma K\mu^K)\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^3})\cdot M(\mathsf{id}+\mathsf{gr}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3})$$
$$\cdot M(\mathsf{id}+\Sigma(\eta^K_{K^2}\cdot\eta^K_K)+\Sigma\eta^K_{K^2}+\mathsf{id}_{\Sigma K^3})\cdot\mathsf{flat}_{\mathsf{Id}+\Sigma K,\Sigma K^2+\Sigma K^3}$$
$$\cdot M(\alpha+\mathsf{id}_{\Sigma K^2}+\mathsf{id}_{\Sigma K^3})\cdot\mathsf{flat}_{K+\Sigma K^2,\Sigma K^3}\cdot M(\alpha_K+\mathsf{id}_{\Sigma K^3})\cdot\alpha_{K^2}$$

$=\qquad \{$ definition of w $\}$

$$M(\mathsf{id}+\Sigma K\mu^K)\cdot w$$

□

The fact that $\mu^K=\widetilde{\mu^K}$ serves as an efficient definition of μ^K in a programming language: μ^K can be defined as the greatest fixed point of the operation $\widetilde{-}$. The middle component of the coproduct in the codomain of unf in a step of computation of μ^K is already the final result of the computation, and is always preserved. That is why Vene and Uustalu's apomorphism may seem to provide a more accurate level of abstraction for a definition of μ^K, since they allow to give a short-cut answer in a single step. In this case μ^K can be defined as $〖M(\mathsf{id}+[\Sigma\mathsf{inr}_{K,K^2},\Sigma\mathsf{inl}_{K,K^2}])\cdot\mathsf{unf}〗$. Nevertheless, we stick to the definition with an anamorphism, since we have found that it makes the present proofs much simpler—instead of pushing the short-cut answers through coproducts, we can push them with η^K, which gives us, in this case, more flexibility, especially in the proof of the associativity of μ^K, where we need to deal with ΣK^2, which is neither the final structure (K), nor the seed for unfolding (K^3).

5.2 Distributive law

Theorem 8 (Distributive law)**.** *There exists a monad distributive law $\lambda:(\Sigma M)^\infty M\to M(\Sigma M)^\infty$, given by $\lambda=\mu^K\cdot\eta^M_{(\Sigma M)^\infty M(\Sigma M)^\infty}\cdot(\Sigma M)^\infty M\eta^\infty$.*

Proof. It is sufficient to check that the monads M and $(\Sigma M)^\infty$ are compatible (see Barr and Wells' book [6, Chapter 6]) with $K\cong M(\Sigma M)^\infty$. In this case the compatibility conditions specialise to:

1. $\eta^K=\eta^M_{(\Sigma M)^\infty}\cdot\eta^\infty=M\eta^\infty\cdot\eta^M$
2. $M\mu^\infty=\mu^K\cdot M(\Sigma M)^\infty\eta^M_{(\Sigma M)^\infty}$
3. $\mu^M_{(\Sigma M)^\infty}=\mu^K\cdot M\eta^\infty_{M(\Sigma M)^\infty}$

(Note that Barr and Wells give five conditions for compatibility, but the last two are redundant, and follow from the first three. See [10] for discussion.)

Equalities (1) and (2) are trivial. To prove (3) we introduce a morphism $w:M(\Sigma M)^\infty(\Sigma M)^\infty\to M(\mathsf{Id}+\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty)$ defined as:

$$\begin{tikzcd}
M(\Sigma M)^\infty(\Sigma M)^\infty \arrow[d, "M\alpha^\infty_{(\Sigma M)^\infty}"] \\
M((\Sigma M)^\infty + \Sigma M(\Sigma M)^\infty(\Sigma M)^\infty) \arrow[d, "M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty})"] \\
M(\mathsf{Id} + \Sigma M(\Sigma M)^\infty + \Sigma M(\Sigma M)^\infty(\Sigma M)^\infty) \arrow[d, "M(\mathsf{id} + [\Sigma M\eta^\infty_{(\Sigma M)^\infty} + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty}])"] \\
M(\mathsf{Id} + \Sigma M(\Sigma M)^\infty(\Sigma M)^\infty)
\end{tikzcd}$$

Below we prove that $\alpha\cdot\mu^K\cdot M(\Sigma M)^\infty\eta^M_{(\Sigma M)^\infty} = M(\mathsf{id}+\Sigma(\mu^K\cdot M(\Sigma M)^\infty\eta^M_{(\Sigma M)^\infty}))\cdot w$, which means that $\mu^K \cdot M(\Sigma M)^\infty\eta^M$ is a coalgebra homomorphism $\langle M(\Sigma M)^\infty(\Sigma M)^\infty, w\rangle \to \langle M(\Sigma M)^\infty, \alpha\rangle$, and that $\alpha\cdot M\mu^\infty = M(\mathsf{id}+\Sigma M\mu^\infty)\cdot w$, which means that $M\mu^\infty$ is a coalgebra homomorphism $\langle M(\Sigma M)^\infty(\Sigma M)^\infty, w\rangle \to \langle M(\Sigma M)^\infty, \alpha\rangle$. By uniqueness, $M\mu^\infty = [\![w]\!] = \mu^K \cdot M(\Sigma M)^\infty\eta^M_{(\Sigma M)^\infty}$. Diagrammatically:

$$\begin{tikzcd}[row sep=huge]
\langle M(\Sigma M)^\infty(\Sigma M)^\infty, w\rangle \arrow[d, "[\![w]\!]"'] \arrow[d, bend right=70, "\mu^K \cdot M(\Sigma M)^\infty\eta^M_{(\Sigma M)^\infty}"'] \arrow[d, bend left=70, "M\mu^\infty"] \\
\langle M(\Sigma M)^\infty, \alpha\rangle
\end{tikzcd}$$

$$\begin{aligned}
&\alpha \cdot \mu^K \cdot M(\Sigma M)^\infty\eta^M_{(\Sigma M)^\infty} \\
=\quad &\{ \text{ universal property } \} \\
&\alpha \cdot \alpha^{-1} \cdot M(\mathsf{id} + [\mathsf{id}_{\Sigma K}, \Sigma\mu^K]) \cdot \mathsf{unf} \cdot M(\Sigma M)^\infty\eta^M_{(\Sigma M)^\infty} \\
=\quad &\{ \text{ cancellation, } \mathsf{unf} \} \\
&M(\mathsf{id} + [\mathsf{id}_{\Sigma K}, \Sigma\mu^K]) \cdot M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty M(\Sigma M)^\infty}) \\
&\cdot \mathsf{flat}_{(\Sigma M)^\infty, \Sigma M(\Sigma M)^\infty M(\Sigma M)^\infty} \cdot M\alpha^\infty \cdot M(\Sigma M)^\infty\eta^M_{(\Sigma M)^\infty} \\
=\quad &\{ \text{ monad laws } \} \\
&M(\mathsf{id} + [\mathsf{id}_{\Sigma K}, \Sigma\mu^K]) \cdot M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty M(\Sigma M)^\infty}) \\
&\cdot M(\mathsf{id}_{(\Sigma M)^\infty} + \Sigma M(\Sigma M)^\infty\eta^M_{(\Sigma M)^\infty}) \cdot M\alpha^\infty_{(\Sigma M)^\infty} \\
=\quad &\{ \text{ coproducts } \} \\
&M(\mathsf{id} + [\mathsf{id}_{\Sigma K}, \Sigma\mu^K]) \cdot M(\alpha^\infty + \Sigma M(\Sigma M)^\infty\eta^M_{(\Sigma M)^\infty}) \cdot M\alpha^\infty_{(\Sigma M)^\infty} \\
=\quad &\{ \text{ coproducts } \}
\end{aligned}$$

$$M(\mathsf{id} + [\mathsf{id}_{\Sigma K}, \Sigma(\mu^K \cdot M(\Sigma M)^\infty \eta^M_{(\Sigma M)^\infty})]) \cdot M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty})$$
$$\cdot M\alpha^\infty_{(\Sigma M)^\infty}$$

$=$ { monad laws }

$$M(\mathsf{id} + [\Sigma(\mu^K \cdot M(\Sigma M)^\infty \eta^M_{(\Sigma M)^\infty}) \cdot \Sigma M\eta^\infty_{(\Sigma M)^\infty},$$
$$\Sigma(\mu^K \cdot M(\Sigma M)^\infty \eta^M_{(\Sigma M)^\infty})]) \cdot M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty}) \cdot M\alpha^\infty_{(\Sigma M)^\infty}$$

$=$ { coproducts }

$$M(\mathsf{id} + [\Sigma(\mu^K \cdot M(\Sigma M)^\infty \eta^M_{(\Sigma M)^\infty}), \Sigma(\mu^K \cdot M(\Sigma M)^\infty \eta^M_{(\Sigma M)^\infty})])$$
$$\cdot M(\mathsf{id} + \Sigma M\eta^\infty_{(\Sigma M)^\infty} + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty}) \cdot M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty})$$
$$\cdot M\alpha^\infty_{(\Sigma M)^\infty}$$

$=$ { coproducts }

$$M(\mathsf{id} + \Sigma(\mu^K \cdot M(\Sigma M)^\infty \eta^M_{(\Sigma M)^\infty}))$$
$$\cdot M(\mathsf{id} + [\Sigma M\eta^\infty_{(\Sigma M)^\infty} + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty}]) \cdot M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty})$$
$$\cdot M\alpha^\infty_{(\Sigma M)^\infty}$$

$=$ { definition }

$$M(\mathsf{id} + \Sigma(\mu^K \cdot M(\Sigma M)^\infty \eta^M_{(\Sigma M)^\infty})) \cdot w$$

* * *

$$\alpha \cdot M\mu^\infty$$

$=$ { universal property }

$$\alpha \cdot M\alpha^{\infty-1} \cdot M(\mathsf{id} + [\mathsf{id}_{\Sigma M(\Sigma M)^\infty}, \Sigma M\mu^\infty]) \cdot M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty})$$
$$\cdot M\alpha^\infty_{(\Sigma M)^\infty}$$

$=$ { cancellation }

$$M(\mathsf{id} + [\mathsf{id}_{\Sigma M(\Sigma M)^\infty}, \Sigma M\mu^\infty]) \cdot M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty})$$
$$\cdot M\alpha^\infty_{(\Sigma M)^\infty}$$

$=$ { monad laws }

$$M(\mathsf{id} + [\Sigma M\mu^\infty \cdot \Sigma M\eta^\infty_{(\Sigma M)^\infty}, \Sigma M\mu^\infty]) \cdot M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty})$$
$$\cdot M\alpha^\infty_{(\Sigma M)^\infty}$$

$=$ { coproducts }

$$M(\mathsf{id} + \Sigma M\mu^\infty) \cdot M(\mathsf{id} + [\Sigma M\eta^\infty_{(\Sigma M)^\infty} + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty}])$$
$$\cdot M(\alpha^\infty + \mathsf{id}_{\Sigma M(\Sigma M)^\infty(\Sigma M)^\infty}) \cdot M\alpha^\infty_{(\Sigma M)^\infty}$$

$=$ { definition }

$$M(\mathsf{id} + \Sigma M\mu^\infty) \cdot w$$

$\square$

5.3 Free Eilenberg-Moore M-algebras

Definition 9. *We call M-**Fema** the category of free Eilenberg-Moore algebras, that is algebras $\langle MA,\ \mu_A^M : M^2A \to MA\rangle$, where A is an object in the base category. We identify $\langle MA, \mu_A^M\rangle$ with MA.*

*A morphism in M-**Fema** (called a homomorphism) is a morphism $f : MA \to MB$ in $\mathcal{B}$ such that the following diagram commutes.*

$$\begin{tikzcd} M^2A \arrow[r, "\mu_A^M"] \arrow[d, "f"] & MA \arrow[d, "Mf"] \\ M^2B \arrow[r, "\mu_B^M"] & MB \end{tikzcd}$$

The category M-**Fema** is equivalent to the more widely used Kleisli category of M. We prefer to use M-**Fema** over Kleisli for two reasons: it is a subcategory of the base category $\mathcal{B}$, which makes a commuting diagram in M-**Fema** a commuting diagram in $\mathcal{B}$, and it makes a clearer connection with the monad K, as can be seen in the next section. We use the following properties of morphisms and functors:

Lemma 10. *The following hold:*

- *For an object A in the base category, $\mu_A^M : M^2A \to MA$ is a homomorphism.*
- *For a morphism $f : A \to B$ in the base category, Mf is a homomorphism.*
- *For an endofunctor F over the base category, the composition MF is an endofunctor over M-**Fema**.*

The inclusion functor $U : M\text{-}\mathbf{Fema} \to \mathcal{B}$ has a left adjoint F defined as $FA = MA$ and $Ff = Mf$, with the unit and counit defined as $\eta = \eta^M$ and $\varepsilon = \mu^M$ respectively. It means that M-**Fema** has coproducts, which we denote as $MA \oplus MB$, given as $M(A + B)$, with $M\mathsf{inl}_{A,B}$ and $M\mathsf{inr}_{A,B}$ as the left and right injections respectively. For two homomorphisms $f : MA \to MC$ and $g : MB \to MC$, their mediator, denoted by $[[f, g]]$, is given by the following composition:

$$\begin{tikzcd} MA \oplus MB = M(A + B) \arrow[d, "{M[f \cdot \eta^M, g \cdot \eta^M]}"] \\ M^2C \arrow[d, "\mu^M"] \\ MC \end{tikzcd}$$

Lemma 11. *For morphisms $f : A \to C$ and $g : B \to C$, the mediator of their M-images $[[Mf, Mg]] : MA \oplus MB \to MC$ is equal to $M[f, g]$.*

As discussed by Mulry [21], liftings of an endofunctor T on $\mathcal{B}$ to M-**Fema** are in one-to-one correspondence with distributive laws $TM \to MT$. Moreover, a simple calculation shows that if T has a monadic structure and the distributive law respects this structure, the corresponding lifting $\langle T \rangle$ is also a monad.

More precisely, let $\lambda : TM \to MT$ be a monad distributive law. We define an endofunctor $\langle T \rangle$ on M-**Fema** as:

$$\begin{aligned}
&\langle T \rangle MA = MTA \\
&\langle T \rangle (f : MA \to MB) = \mu^M_{TB} \cdot M\lambda_B \cdot MTf \cdot MT\eta^M_A : MTA \to MTB
\end{aligned}$$

We define a monadic structure for $\langle T \rangle$ as:

$$\begin{aligned}
&\eta^{\langle T \rangle}_{MA} : MA \to \langle T \rangle MA \quad (= MA \to MTA) \\
&\eta^{\langle T \rangle}_{MA} = M\eta^T_A
\end{aligned}$$

$$\begin{aligned}
&\mu^{\langle T \rangle}_{MA} : \langle T \rangle^2 MA \to \langle T \rangle MA \quad (= MTTA \to MTA) \\
&\mu^{\langle T \rangle}_{MA} = M\mu^T_A
\end{aligned}$$

It is an easy calculation to show that $\langle T \rangle$ is an endofunctor and that $\eta^{\langle T \rangle}$ and $\mu^{\langle T \rangle}$ are natural transformations and that monad laws hold.

Lemma 12. *The monad induced by the distributive law (that is,* $\langle MT,\ \eta^M_T \cdot \eta^T,\ M\mu^T \cdot \mu^M_{T^2} \cdot M\lambda_T \rangle$*) is equal to* $U\langle T \rangle F$*.*

Proof. Since $\langle T \rangle$ is a lifting, that is $\langle T \rangle F = FT$, we reason that $U\langle T \rangle F = UFT = UMT = MT$, so they are equal as functors.

Unit:

$$\begin{aligned}
&\eta^{U\langle T \rangle F}_A = U\eta^{\langle T \rangle}_{FA} \cdot \eta_A = U\eta^{\langle T \rangle}_{MA} \cdot \eta_A \\
= \quad & \{ \text{ definition of } \eta^{\langle T \rangle} \text{ and } \eta \ \} \\
&UM\eta^T_A \cdot \eta^M_A \\
= \quad & \{ \text{ definition of } U \ \} \\
&M\eta^T_A \cdot \eta^M_A \\
= \quad & \{ \text{ naturality of } \eta^M \ \} \\
&\eta^M_{TA} \cdot \eta^T_A = \eta^{MT}_A
\end{aligned}$$

Multiplication:

$$\begin{aligned}
&\mu^{U\langle T \rangle F}_A = U\mu^{\langle T \rangle}_{FA} \cdot U\langle T \rangle \varepsilon_{\langle T \rangle FA} \\
= \quad & \{ \text{ definitions of } U,\ F, \text{ and } \langle T \rangle \ \} \\
&\mu^{\langle T \rangle}_{MA} \cdot \langle T \rangle \varepsilon_{MTA} \\
= \quad & \{ \text{ definition of } \varepsilon \ \} \\
&\mu^{\langle T \rangle}_{MA} \cdot \langle T \rangle \mu^M_{TA}
\end{aligned}$$

$$\begin{aligned}
= \quad & \{ \text{ definition of } \langle T \rangle \ \} \\
& \mu^{\langle T \rangle}_{MA} \cdot \mu^M_{TTA} \cdot M\lambda_{TA} \cdot MT\mu^M_{TA} \cdot MT\eta^M_{MTA} \\
= \quad & \{ \text{ monad laws } \} \\
& \mu^{\langle T \rangle}_{MA} \cdot \mu^M_{TTA} \cdot M\lambda_{TA} \\
= \quad & \{ \text{ definition of } \mu^{\langle T \rangle} \ \} \\
& M\mu^T_A \cdot \mu^M_{TTA} \cdot M\lambda_{TA} = \mu^{MT}_A
\end{aligned}$$

□

5.4 Complete iterativity of K

Consider the monad $(\Sigma M)^\infty$. The monad distributive law λ from Theorem 8 gives rise to a lifting $\langle (\Sigma M)^\infty \rangle$, defined on objects as

$$\langle (\Sigma M)^\infty \rangle MA = M(\Sigma M)^\infty A \cong KA.$$

The following theorem states that the lifting is also a free cim. Note that $M\Sigma$ is an endofunctor also over M-**Fema** (Lemma 10):

Theorem 13. *The monad $\langle (\Sigma M)^\infty \rangle$ is the free cim generated by $M\Sigma$ in M-***Fema***. We denote it as $\widehat{(M\Sigma)^\infty}$.*

Proof. For an M-**Fema** homomorphism $f : MX \to M(A + \Sigma MX)$, consider the following diagram in the base category. It commutes, because KA is the carrier of the final $M(A + \Sigma -)$-coalgebra.

$$\begin{tikzcd}
KA \arrow[r, "\alpha_A"] & M(A + \Sigma KA) \\
MX \arrow[u, "\llbracket f \rrbracket"] \arrow[r, "f"] & M(A + \Sigma MX) \arrow[u, "M(\mathsf{id}_A + \Sigma\llbracket f \rrbracket)"']
\end{tikzcd}$$

It is easy to see that α_A and $M(A + \Sigma\llbracket f \rrbracket)$ are also homomorphisms, and $\llbracket f \rrbracket$ is a homomorphism as a composition of homomorphisms via the computation law: $\llbracket f \rrbracket = \alpha_A^{-1} \cdot M(A + \Sigma\llbracket f \rrbracket) \cdot f$. This means that this diagram commutes also in M-**Fema**. Expanding the definition of coproducts we get the following commutative diagram in M-**Fema**:

$$\begin{tikzcd}
\langle (\Sigma M)^\infty \rangle MA \arrow[r, "\alpha_A"] & M(A + \Sigma \langle (\Sigma M)^\infty \rangle MA) = MA \oplus M\Sigma \langle (\Sigma M)^\infty \rangle MA \\
MX \arrow[u, "\llbracket f \rrbracket"] \arrow[r, "f"] & M(A + \Sigma MX) = MA \oplus M\Sigma MX \arrow[u, "M(\mathsf{id}_A + \Sigma\llbracket f \rrbracket) = \mathsf{id}_{MA} \oplus M\Sigma\llbracket f \rrbracket"']
\end{tikzcd}$$

Note that in M-**Fema**, $M(A + \Sigma -) = MA \oplus M\Sigma -$ is a functor (Lemma 10), hence $\langle (\Sigma M)^\infty \rangle MA$ is the carrier of the final $(MA \oplus M\Sigma -)$-coalgebra, and so, according to [18, Corollary 6.3], $\langle(\Sigma M)^\infty\rangle$ is the functorial part of the free cim generated by $M\Sigma$, understood as a functor in M-**Fema**. We denote free cims in M-**Fema** as $\text{-}^{\widehat{\infty}}$, so $\langle(\Sigma M)^\infty\rangle \cong (M\Sigma)^{\widehat{\infty}}$ as functors.

It is left to see that $\langle(\Sigma M)^\infty\rangle \cong (M\Sigma)^{\widehat{\infty}}$ also as monads. It is easy to see that the units are equivalent. For multiplications, compare the universal properties from Theorem 7 (for the identity monad and ΣM and $M\Sigma$ as respective functors). In case of $\langle(\Sigma M)^\infty\rangle$ the multiplication is equal to $M\mu^\infty$. The M-image of the universal property of μ^∞ simplifies to:

$$\begin{array}{c}
\langle(\Sigma M)^\infty\rangle^2 M \cong M(\Sigma M)^\infty(\Sigma M)^\infty \\
\cong M(\mathsf{Id} + \Sigma M(\Sigma M)^\infty + \Sigma M(\Sigma M)^\infty(\Sigma M)^\infty) \\
\Big\downarrow M(\mathsf{id} + [\mathsf{id}_{\Sigma M(\Sigma M)^\infty}, \Sigma M\mu^\infty]) \\
M(\mathsf{Id} + \Sigma M(\Sigma M)^\infty) \cong M(\Sigma M)^\infty
\end{array}$$

In case of $(M\Sigma)^{\widehat{\infty}}$, the universal property can be simplified using Lemma 11 as follows:

$$\begin{array}{c}
(M\Sigma)^{\widehat{\infty}}(M\Sigma)^{\widehat{\infty}}M \cong (M\Sigma)^{\widehat{\infty}}M \oplus M\Sigma(M\Sigma)^{\widehat{\infty}}(M\Sigma)^{\widehat{\infty}}M \\
\cong M \oplus M\Sigma(M\Sigma)^{\widehat{\infty}}M \oplus M\Sigma(M\Sigma)^{\widehat{\infty}}(M\Sigma)^{\widehat{\infty}}M \\
= M(\mathsf{Id} + \Sigma(M\Sigma)^{\widehat{\infty}}M + \Sigma(M\Sigma)^{\widehat{\infty}}(M\Sigma)^{\widehat{\infty}}M) \\
\Big\downarrow \begin{array}{l} \mathsf{id} \oplus [[\mathsf{id}_{M\Sigma(M\Sigma)^{\widehat{\infty}}}, M\Sigma\mu^{\widehat{\infty}}]] \\ = M(\mathsf{id} + [\mathsf{id}_{\Sigma(M\Sigma)^{\widehat{\infty}}M}, \Sigma\mu^{\widehat{\infty}}]) \end{array} \\
M \oplus M\Sigma(M\Sigma)^{\widehat{\infty}}M \cong (M\Sigma)^{\widehat{\infty}}M \\
= M(\mathsf{Id} + \Sigma(M\Sigma)^{\widehat{\infty}}M) \cong M(\Sigma M)^\infty
\end{array}$$

It means that the multiplication $M\mu^\infty$ satisfies the universal property of $\mu^{\widehat{\infty}}$. □

Theorem 5 and the above characterisation yields that K is completely iterative. The guardedness specialises as:

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[dr, dashed, "j"'] & & K(A+X) \\
& M\Sigma K(A+X) + A \arrow[ur, "{[\alpha^{-1}_{A+X} \cdot M\mathsf{inr}_{A+X,\Sigma K(A+X)},\ \eta^K_{A+X} \cdot \mathsf{inl}_{A,X}]}"'] &
\end{tikzcd}$$

5.5 A more robust cim-like property of K

We can define a more general notion of iterativity for K than one provided by Theorem 5. It states that an equation morphism $X \to K(A+X)$ has a unique solution as long as every 'value' of the outer monad M is a parameter or at least one level of the Σ-guarded recursive structure, but not a variable. (The guardedness of Theorem 5 needs the result of an equation morphism to be either a parameter or an M where all the 'values' are Σ-guarded.)

Theorem 14. *Let $e : X \to K(A + X)$ be a morphism such that it factorizes through $\alpha^{-1} \cdot M(\mathsf{inl}_{A,X} + \mathsf{id}_{\Sigma K(A+X)})$, that is there exists a morphism j such that t.f.d.c.*

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[d, dashed, "j"] & & K(A+X) \\
M(A + \Sigma K(A+X)) \arrow[rr, "M(\mathsf{inl}_{A,X} + \mathsf{id}_{\Sigma K(A+X)})"] & & M((A+X) + \Sigma K(A+X)) \arrow[u, "\alpha^{-1}"']
\end{tikzcd}$$

There exists a unique morphism $e^{\ddagger}$ for which t.f.d.c.

$$\begin{tikzcd}
X \arrow[r, "e^{\ddagger}"] \arrow[d, "e"] & KA \\
K(A+X) \arrow[r, "K[\eta^K, e^{\ddagger}]"] & K^2 A \arrow[u, "\mu^K"']
\end{tikzcd}$$

Lemma 15. *1. Every morphism $f : X \to MY$ factors as $f' \cdot \eta^M_X$ for some homomorphism $f' : MX \to MX$.*

2. The natural transformation η^M is epic for homomorphisms, that is, for two homomorphisms $f, g : MA \to MB$, if $f \cdot \eta^M_A = g \cdot \eta^M_A$ then $f = g$.

Proof. (i) It is easy to check that $f = \mu^M_X \cdot Mf \cdot \eta^M_X$. By Lemma 10, the morphism $\mu^M_X \cdot Mf$ is a homomorphism. For (ii) we calculate:

$$\begin{aligned}
& f \\
= \quad & \{ \text{ monad laws } \} \\
& f \cdot \mu^M_A \cdot M\eta^M_A \\
= \quad & \{ \ f \text{ is a homomorphism } \} \\
& \mu^M_C \cdot Mf \cdot M\eta^M_A \\
= \quad & \{ \text{ assumprion } \} \\
& \mu^M_C \cdot Mh \cdot M\eta^M_A \\
= \quad & \{ \ h \text{ is a homomorphism } \} \\
& h \cdot \mu^M_A \cdot M\eta^M_A \\
= \quad & \{ \text{ monad laws } \} \\
& h
\end{aligned}$$

□

Proof of Theorem 14. Let $f : MX \to \widehat{(M\Sigma)^{\infty}}(MA \oplus MX)$ be a guarded equation morphism in M-**Fema**. Then there exists a unique homomorphism $f^{\dagger}$ such that t.f.d.c.

$$
\begin{tikzcd}
MX \arrow[r, "f^\dagger"] \arrow[d, "f"] \arrow[dr, phantom, "\text{(A)}"] & \begin{array}{c}(M\Sigma)^{\widehat{\infty}}MA\\ = M(\Sigma M)^\infty A\end{array} & \\
\begin{array}{c}(M\Sigma)^{\widehat{\infty}}(MA\oplus MX)\\ = \langle(\Sigma M)^\infty\rangle(MA\oplus MX)\\ = \langle(\Sigma M)^\infty\rangle M(A+X)\\ = M(\Sigma M)^\infty(A+X)\end{array} \arrow[r, "(M\Sigma)^{\widehat{\infty}}[[\eta,f^\dagger]]", "\langle(\Sigma M)^\infty\rangle[[\eta,f^\dagger]]"'] \arrow[dd, "M(\Sigma M)^\infty\eta^M"] \arrow[ddr, "M(\Sigma M)^\infty[\eta^K,f^\dagger\cdot\eta^M]" description] \arrow[dr, phantom, "\text{(B)}"] \arrow[dd, phantom, bend left, "\text{(C)}"] & \begin{array}{c}(M\Sigma)^{\widehat{\infty}}(M\Sigma)^{\widehat{\infty}}MA\\ = \langle(\Sigma M)^\infty\rangle\ (M\Sigma)^{\widehat{\infty}}MA\\ = \langle(\Sigma M)^\infty\rangle\ \langle(\Sigma M)^\infty\rangle MA\\ = M(\Sigma M)^\infty(\Sigma M)^\infty A\end{array} \arrow[u, "\mu^{(M\Sigma)^{\widehat{\infty}}} = M\mu^{(\Sigma M)^\infty}"] & \\
 & M^2(\Sigma M)^\infty(\Sigma M)^\infty A \arrow[u, "\mu^M"] \arrow[r, phantom, "\text{(E)}"] & {} \\
M(\Sigma M)^\infty M(A+X) \arrow[r, "M(\Sigma M)^\infty[[\eta,f^\dagger]]"] \arrow[d, "M(\Sigma M)^\infty M2\eta^M"] \arrow[dr, phantom, "\text{(D)}"] & M(\Sigma M)^\infty M(\Sigma M)^\infty A \arrow[u, "M\lambda"] \arrow[uuu, bend right=60, "\mu^K"'] & \\
M(\Sigma M)^\infty M(MA+MX) \arrow[r, "M(\Sigma M)^\infty M[\eta,f^\dagger]"] & M(\Sigma M)^\infty M^2(\Sigma M)^\infty A \arrow[u, "M(\Sigma M)^\infty\mu^M"] &
\end{tikzcd}
$$

(A) solution property for $(M\Sigma)^{\widehat{\infty}}$, (B+C) definition of lifting to M-**Fema** for morphisms, (D) definition of mediator for coproducts in M-**Fema**, (E) definition of μ^K via the distributive law. The 'diagonal' $M(\Sigma M)^\infty[\eta^K, f^\dagger\cdot\eta^M]$ is a simplification of the outer edges of (C+D).

Note that the longer path of the outer edges of $(\mathsf{A}+\mathsf{B}+\mathsf{E})$ is a composition of homomorphisms, so $f^\dagger$ is unique also among all morphisms in $\mathcal{B}$.

The following diagram also commutes. Reading the outer edges it states that the morphism $\mu^M\cdot Me$ is a guarded equation morphism in the monad $(M\Sigma)^{\widehat{\infty}}$ in M-**Fema**.

$$
\begin{tikzcd}
MX \arrow[d, "Mj"] \arrow[ddddd, bend right, "Me"'] & & \\
M^2(A + \Sigma M(\Sigma M)^\infty(A + X)) \arrow[rr, "\mu^M"] \arrow[dd, "M^2(\mathsf{inl} + \mathsf{id})"] & & \begin{array}{c} MA \oplus M\Sigma(M\Sigma)^{\widehat{\infty}}(MA \oplus MB) \\ = M(A + \Sigma M(\Sigma M)^\infty(A + X)) \end{array} \arrow[d, "M2\eta^M"] \arrow[ddl, "M(\mathsf{inl} + \mathsf{id})"'] \arrow[dddd, bend left=60, "{[[\eta^{(M\Sigma)^{\widehat{\infty}}} \cdot M\mathsf{inl}, \boldsymbol{\sigma}]]}"] \\
 & & M(MA + M\Sigma M(\Sigma M)^\infty(A + X)) \arrow[dd, "{\begin{array}{l} M[\eta^{(M\Sigma)^{\widehat{\infty}}} \cdot M\mathsf{inl}, M\sigma] = \\ M[M\eta^{(\Sigma M)^\infty} \cdot M\mathsf{inl}, M\sigma] \end{array}}"] \\
M^2(A + X + \Sigma M(\Sigma M)^\infty(A + X)) \arrow[dd, "M\alpha^{-1}"] & M(A + X + \Sigma M(\Sigma M)^\infty(A + X)) \arrow[ddr, "\alpha^{-1}"'] & \\
 & & M^2(\Sigma M)^\infty(A + X) \arrow[d, "\mu^M"] \\
M^2(\Sigma M)^\infty(A + X) \arrow[rr, "\mu^M"] & & \begin{array}{c} M(\Sigma M)^\infty(A + X) \\ = (M\Sigma)^{\widehat{\infty}}(MA \oplus MX) \end{array}
\end{tikzcd}
$$

Define $e^\ddagger$ to be $(\mu^M \cdot Me)^\dagger \cdot \eta^M$. The following diagram also commutes:

$$\begin{tikzcd}
X \arrow[d, "\eta^M"] \arrow[dr, "e^\ddagger = (\mu^M \cdot Me)^\dagger \cdot \eta^M"] \arrow[ddd, bend right, "e"'] & \\
MX \arrow[r, "(\mu^M \cdot Me)^\dagger"] \arrow[d, "Me"] & M(\Sigma M)^\infty A \\
MM(\Sigma M)^\infty(A+X) \arrow[d, "\mu^M"] & \\
M(\Sigma M)^\infty(A+X) \arrow[r, "M(\Sigma M)^\infty[\eta^K, (\mu^M \cdot Me)^\dagger \cdot \eta^M]"] & M(\Sigma M)^\infty M(\Sigma M)^\infty A \arrow[uu, "\mu^K"']
\end{tikzcd}$$

The bottom pentagon is the (A+B+E) square from the diagram above for $f = \mu^M_{(\Sigma M)^\infty(A+X)} \cdot Me$.

For uniqueness, assume that $g : X \to M(\Sigma M)^\infty A$ substituted for $e^\ddagger$ also commutes the diagram. By Lemma 15, g factors as $g' \cdot \eta^M_X$ for a homomorphism g'. Substituting g' for $(\mu^M \cdot Me)^\dagger$ in the diagram yields that the bottom pentagon precomposed with η^M_X commutes. Since η^M is epic for homomorphisms (Lemma 15), the pentagon commutes, so g' is a solution of $\mu^M \cdot Me$. This means that $g' = (\mu^M \cdot Me)^\dagger$, hence $g = g' \cdot \eta^M_X = (\mu^M \cdot Me)^\dagger \cdot \eta^M_X = e^\ddagger$. $\square$

6 Related and future work

Cims were introduced by Elgot [11], and recently brought to attention by Aczel *et al.* [1, 18]. Milius and Moss [19] consider recursive program schemes in terms of solutions in Elgot algebras [3] (that is, Eilenberg-Moore algebras for free cims).

Cenciarelli and Moggi [9] introduced the Generalised Resumption transformer $M(\Sigma M)^*$, which decomposes a monadic computation into a series of steps (layers of free structure). Hyland, Plotkin, and Power [16] proved it to be the coproduct $M + \Sigma^*$ in the category of monads. The monad $M(\Sigma M)^\infty$ captures also potentially infinite computations. In some categories—and so programming languages like Haskell—the limit-colimit coincidence [27] identifies $M(\Sigma M)^*$ and $M(\Sigma M)^\infty$, but the explicit use of the free cim is significant in **Set** and in type theories with guarded (co)recursion. Interleaving data and monadic actions is a powerful abstraction studied recently also by Filinski and Støvring [12], Atkey *et al.* [4], and the present authors [24]. The monad $M(\Sigma M)^\infty$ is also a categorical model for datatypes built around resumptions, like Haskell pipes (for $\Sigma A = A^I + A \times O$). The fact that we use the free cim is crucial, since programming patterns for pipes rely heavily on infinite computations.

Since the free cim is a final coalgebra [18], we can see $(M\Sigma)^\infty$ in M-**Fema** from Theorem 13 as an example of Hasuo, Jacobs, and Sokolova's generic trace semantics [14], which models state-based systems as F-coalgebras in a Kleisli category (or, equivalently, a FEMA). The coalgebra represents transitions (for example, with $\Sigma A = A \times O$ for labelled transitions), and the monad represents the underlying effect (like the Powerset monad for nondeterminism or the Probability Distribution monad for probabilistic systems).

In this paper we concentrate on the monads and tracing, and we only sketch potential applications in defining semantics and reasoning about programs. The natural next step is to formalise a language like Moggi's computational λ-calculus [20] with recursion provided by a background cim. It is also an interesting question whether the presented theory could be used to develop a practical framework for reasoning about effectful programs in type theories, like those implemented by the Coq or Agda proof systems. So far, Capretta [8] represented general recursion by the free cim generated by the identity functor.

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