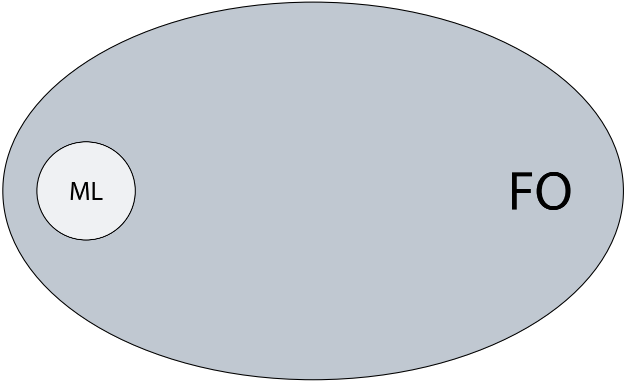
# Effective interpolation and preservation in guarded logics

Michael Benedikt[1], Balder ten Cate[2], **Michael Vanden Boom**[1]

[1]University of Oxford [2]LogicBlox and UC Santa Cruz

CSL-LICS 2014
Vienna, Austria

# Some decidable fragments of first order logic

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| tree-like model property | ✓ |
| Craig interpolation | ✓ |
| Łoś-Tarski preservation | ✓ |

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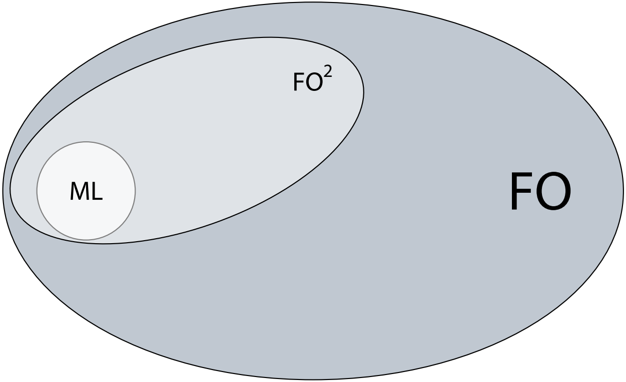


constrain
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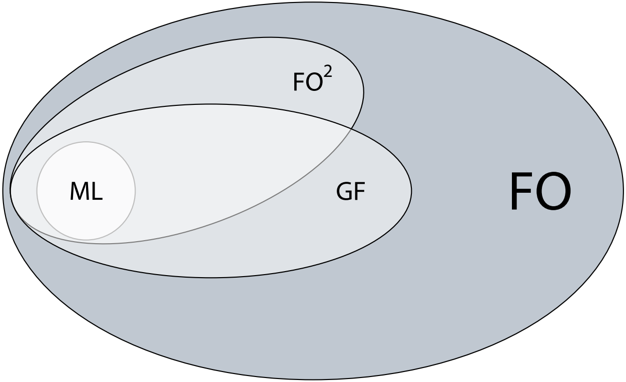


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$\exists \boldsymbol{x}.\alpha(\boldsymbol{xy}) \wedge \psi(\boldsymbol{xy})$
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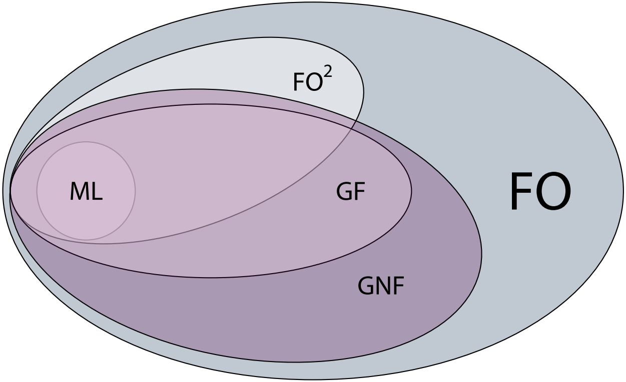


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# Interpolation

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only uses
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# Interpolation example

$$\exists xyz(Txyz \wedge Rxy \wedge Ryz \wedge Rzx) \quad \models \quad \exists xy(Rxy \wedge ((Sx \wedge Sy) \vee (\neg Sx \wedge \neg Sy)))$$

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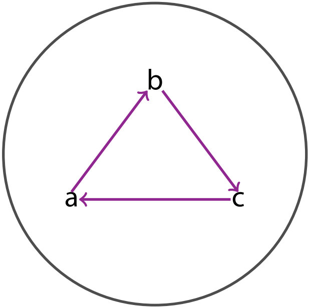

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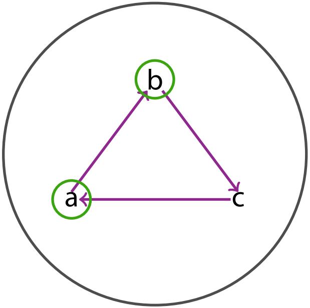

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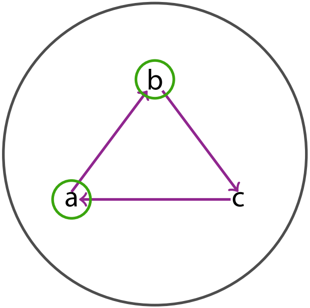


interpolant $\chi := \exists xyz(Rxy \wedge Ryz \wedge Rzx)$

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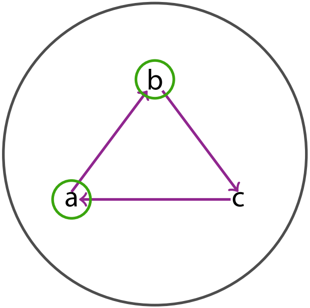


GNF interpolant $\chi := \exists xyz(Rxy \wedge Ryz \wedge Rzx)$

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**Theorem** (Barany+Benedikt+ten Cate '13)

Given GNF formulas $\varphi$ and $\psi$ such that $\varphi \models \psi$, there is a GNF interpolant $\chi$ (but model theoretic proof implies no bound on size of $\chi$).

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Even when input is in GF, no idea how to **compute** interpolants (or other rewritings related to interpolation and preservation).

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Given GNF formulas $\varphi$ and $\psi$ such that $\varphi \models \psi$, there is a GNF interpolant $\chi$ (but model theoretic proof implies no bound on size of $\chi$).

**Theorem** (Constructive interpolation for GNF)

Given GNF formulas $\varphi$ and $\psi$ such that $\varphi \models \psi$, we can construct a GNF interpolant $\chi$ of doubly exponential DAG-size (in size of input).

# Mosaics

A **mosaic** $\tau(\boldsymbol{a})$ for $\varphi$ is a collection of subformulas of $\varphi$ over some guarded set $\boldsymbol{a}$ of parameters.

| $\tau_1(ab)$ | $\tau_2(bc)$ | $\tau_3(d)$ |
|---|---|---|
| $Raa$ | $Sb$ | $Sd$ |
| $\neg Sa$ | $\neg Rbb$ | $\neg Sd$ |
| $\exists z(Rbz \wedge Sz)$ | $Rbc \wedge Sc$ | $\exists yz(Ryz \wedge Sz)$ |
| $Sb$ | $Rcb$ | $\forall z(Rdz)$ |
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| $Raa$ | $Sb$ | **Internally inconsistent** |
| $\neg Sa$ | $\neg Rbb$ | |
| $\exists z(Rbz \wedge Sz)$ | $Rbc \wedge Sc$ | (e.g. $S$ d & $\neg S$ d) |
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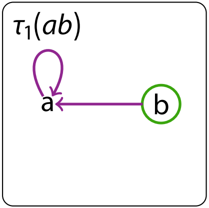
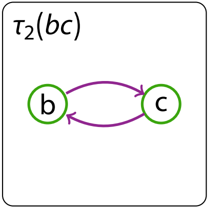



$\tau_3(d)$

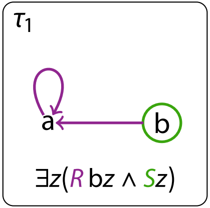
**Internally inconsistent**

(e.g. $S\,d$ & $\neg S\,d$)

Internally consistent mosaics are windows into a (guarded) piece of a structure.

# Linking mosaics

Mosaics can be **linked** together to fulfill an existential requirement if they agree on all formulas that use only shared parameters.



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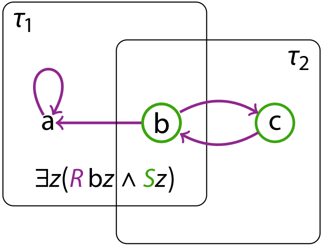

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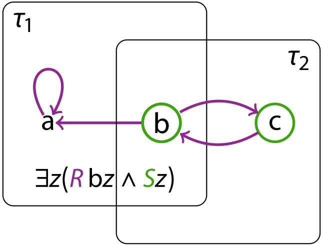


We say a set $S$ of mosaics is **saturated** if every existential requirement in a mosaic $\tau \in S$ is fulfilled in $\tau$ or in some linked $\tau' \in S$.

# Mosaics

Fix some set $P$ of size $2 \cdot \text{width}(\varphi)$ and let $\mathscr{M}_\varphi$ be the set of mosaics for $\varphi$ over parameters $P$. The size of $\mathscr{M}_\varphi$ is doubly exponential in the size of $\varphi$.

**Theorem**

$\varphi$ is satisfiable iff there is a saturated set $S$ of internally consistent mosaics from $\mathscr{M}_\varphi$ that contains some $\tau$ with $\varphi \in \tau$.

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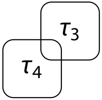


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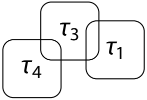


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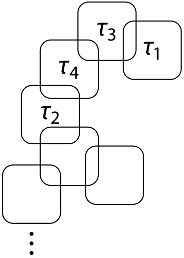
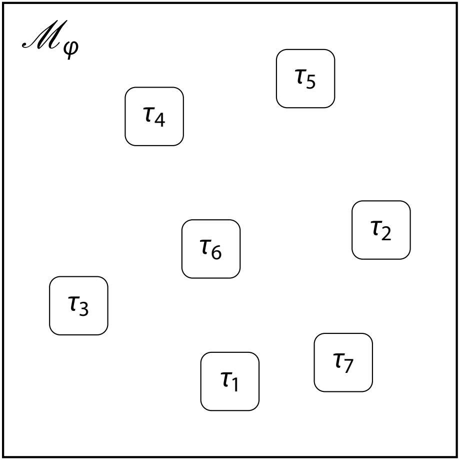


# Mosaic elimination algorithm for satisfiability testing



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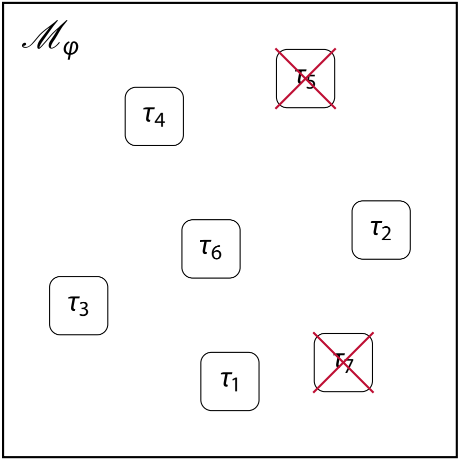
Eliminate mosaics with internal inconsistencies.



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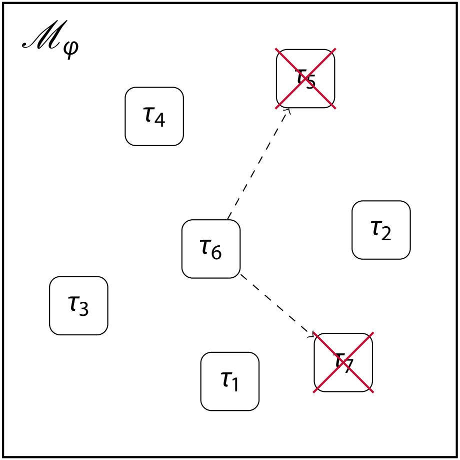
**Stage $i+1$.**
Eliminate mosaics with existential requirements that can only be fulfilled using mosaics eliminated in earlier stages.



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**Stage 1.**
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**Stage** $i + 1$**.**
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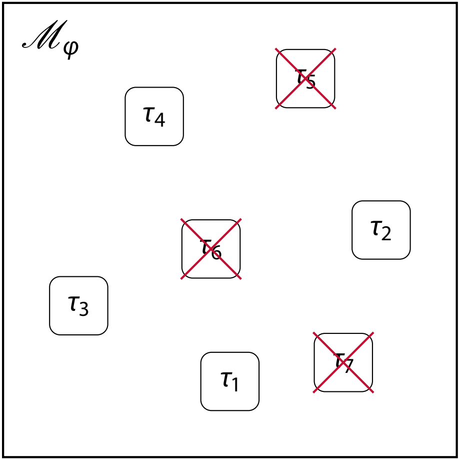
Continue until fixpoint $\mathscr{M}'$ reached. The set $\mathscr{M}'$ is a saturated set of internally consistent mosaics.




**Theorem**

$\varphi$ is satisfiable iff there is some mosaic $\tau \in \mathscr{M}'$ with $\varphi \in \tau$.

# Mosaics for interpolation

Assume $\varphi_{\mathtt{L}} \models \varphi_{\mathtt{R}}$.

**Idea:** Construct interpolant from proof that $\varphi_{\mathtt{L}} \wedge \neg\varphi_{\mathtt{R}}$ is unsatisfiable.

# Mosaics for interpolation

Assume $\varphi_{\mathtt{L}} \models \varphi_{\mathtt{R}}$.

**Idea:** Construct interpolant from proof that $\varphi_{\mathtt{L}} \land \neg\varphi_{\mathtt{R}}$ is unsatisfiable.

Consider mosaics for $\varphi_{\mathtt{L}} \land \neg\varphi_{\mathtt{R}}$.
Annotate each mosaic and each formula with a **provenance** $\mathtt{L}$ or $\mathtt{R}$.

$\mathtt{L} : \tau_1(ab)$

$\mathtt{L} : Raa$
$\mathtt{R} : \neg Sa$
$\mathtt{R} : \exists z(Rbz \land Sz)$
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$\ldots$

$\mathtt{R} : \tau_2(bc)$

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$\mathtt{L} : \tau_3(d)$

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$\mathtt{R} : \exists yz(Ryz \land Sz)$
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Linking must respect the provenance annotations.

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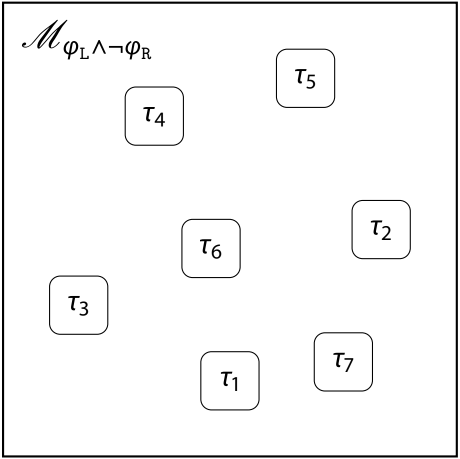

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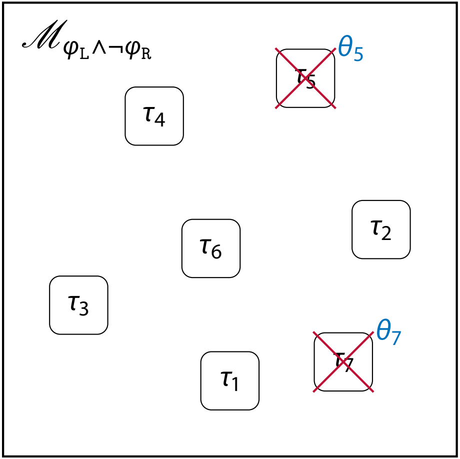
Assign a **mosaic interpolant** $\theta_\tau$ to each eliminated mosaic $\tau$ such that $\tau_{\mathrm{L}} \vDash \theta_\tau$ and $\theta_\tau \vDash \neg\tau_{\mathrm{R}}$.

Mosaic interpolants $\theta_\tau$ describe why the mosaic $\tau$ was eliminated.



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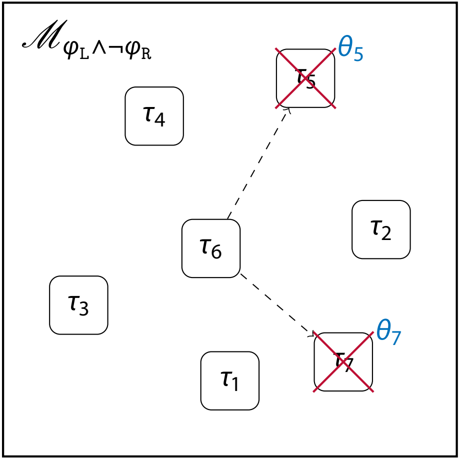

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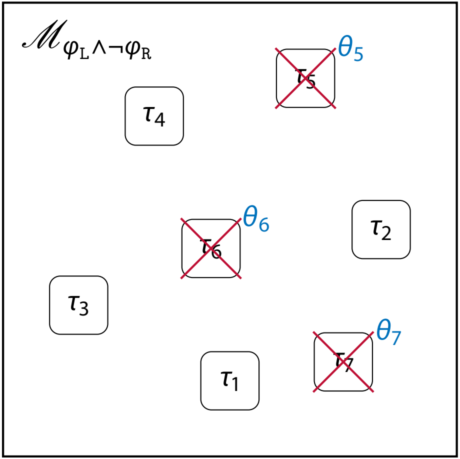

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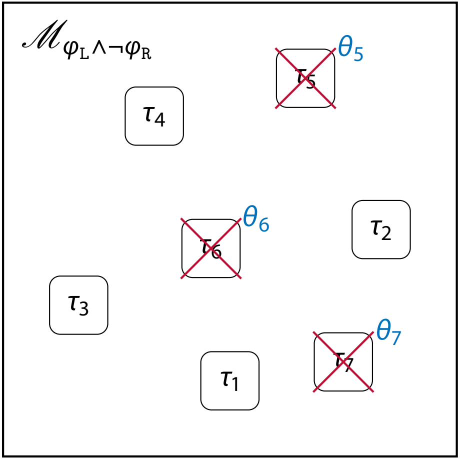


## Theorem

An interpolant $\chi$ for $\varphi_{\mathrm{L}} \models \varphi_{\mathrm{R}}$ of at most doubly exponential DAG-size can be constructed from the mosaic interpolants.

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**Lyndon interpolation:** $\chi$ respects **polarity of relations**

A relation $R$ occurs positively (respectively, negatively) in $\chi$ iff $R$ occurs positively (respectively, negatively) in both $\varphi_{\mathrm{L}}$ and $\varphi_{\mathrm{R}}$.

**Relativized interpolation:** $\chi$ respects **quantification pattern**

If the quantification in $\varphi_{\mathrm{L}}$ and $\varphi_{\mathrm{R}}$ is relativized to a distinguished set of unary predicates $\mathbb{U}$, then $\chi$ is $\mathbb{U}$-relativized.
i.e. quantification is of the form $\exists x\,(Ux \wedge \psi(x\mathbf{y}))$ for $U \in \mathbb{U}$

# Effective preservation theorems

$\varphi$ is **monotone** if $\mathfrak{A} \models \varphi$ implies that $\mathfrak{A}' \models \varphi$ for any $\mathfrak{A}'$ obtained from $\mathfrak{A}$ by adding tuples to the interpretation of some relation.

$\varphi$ is **positive** if every relation appears within the scope of an even number of negations.

**Corollary** (Monotone = Positive)

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$\varphi$ is **preserved under extensions** if $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{B} \models \varphi$.

$\varphi$ is in **existential GNF** if no quantifier is in the scope of a negation.

**Corollary** (Analog of Łoś-Tarski)

If $\varphi$ is preserved under extensions and in GNF, then we can construct an equivalent existential GNF formula $\varphi'$ of doubly exponential DAG-size.

## Summary

GNF is an expressive fragment of FO with good computational and model-theoretic properties.

Proved **constructive interpolation and preservation theorems** for GNF.

Adapted **mosaic method** to prove interpolation.

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More in the paper:

Proved **matching lower bounds** for constructive interpolation results.

Analyzed **special cases** when input is in GF or the unary negation fragment (UNF).

# Shape of interpolants

Mosaic interpolants $\theta_\tau$ satisfy $\tau_{\mathtt{L}} \models \theta_\tau$ and $\theta_\tau \models \neg\tau_{\mathtt{R}}$.
They describe why the mosaic $\tau$ was eliminated

**Stage 1:**

| | | | | |
|---|---|---|---|---|
| | $\mathtt{L} : Rab$ | $\mathtt{L} : \neg Rab$ | $\Rightarrow$ | $\theta_\tau := \bot$ |
| Internal | $\mathtt{R} : Rab$ | $\mathtt{R} : \neg Rab$ | $\Rightarrow$ | $\theta_\tau := \top$ |
| inconsistency | $\mathtt{L} : Rab$ | $\mathtt{R} : \neg Rab$ | $\Rightarrow$ | $\theta_\tau := Rab$ |
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# Shape of interpolants

Mosaic interpolants $\theta_\tau$ satisfy $\tau_{\mathtt{L}} \models \theta_\tau$ and $\theta_\tau \models \neg\tau_{\mathtt{R}}$.
They describe why the mosaic $\tau$ was eliminated

**Stage 1:**

Internal inconsistency

$$\begin{array}{llcl}
\mathtt{L} : Rab & \mathtt{L} : \neg Rab & \Rightarrow & \theta_\tau := \bot \\
\mathtt{R} : Rab & \mathtt{R} : \neg Rab & \Rightarrow & \theta_\tau := \top \\
\mathtt{L} : Rab & \mathtt{R} : \neg Rab & \Rightarrow & \theta_\tau := Rab \\
\mathtt{R} : Rab & \mathtt{L} : \neg Rab & \Rightarrow & \theta_\tau := \neg Rab
\end{array}$$

**Stage** $i+1$**:**

Unfulfilled $\quad \mathtt{L} : \exists \boldsymbol{z}\,[G(\boldsymbol{bz}) \wedge \psi(\boldsymbol{bz})] \quad \Rightarrow \quad \theta_\tau := \bigvee_{\tau'(\boldsymbol{bc})} \exists \boldsymbol{z} \left[ \bigwedge_{\tau'' \supseteq \tau'} \theta_{\tau''}(\boldsymbol{bz}) \right]$

*"there is a mosaic $\tau'$ that can be linked to $\tau$ to fulfil the requirement, but no matter what* R*-formulas are added, the resulting mosaic $\tau''$ has already been eliminated"*