Interpolation with decidable fixpoint logics

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Joint work with
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Fixpoint logics give mechanism to express
dynamic, recursive properties.

Example

binary relation R, unary relation P

“from y, it is possible to R-reach some P-element”

$$[\mathbf{lfp}_{Y,y} \,.\, Py \vee \exists z(Ryz \wedge Yz)](y)$$

Some decidable fixpoint logics

Modal mu-calculus (L_μ)
[Kozen '83]

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extension of modal logic
with fixpoints

describes transition systems
(relations of arity at most 2)

decidable satisfiability
(EXPTIME-complete)

tree model property

Some decidable fixpoint logics

Modal mu-calculus (L_μ) [Kozen '83]	**Unary negation fixpoint logic** (UNFP) [Segoufin, ten Cate '11]
extension of modal logic with fixpoints	fragment of LFP with monadic fixpoints and negation of formulas with at most one free variable
describes transition systems (relations of arity at most 2)	describes relational structures (relations of arbitrary arity)
decidable satisfiability (EXPTIME-complete)	decidable satisfiability (2EXPTIME-complete)
tree model property	tree-like model property (models of bounded tree-width)

UNFP

UNFP is expressive:

- modal logic and L_μ, even with backwards modalities;
- positive existential FO (i.e. unions of conjunctive queries);
- description logics including $\mathcal{ALC}$, $\mathcal{ALCHIO}$, $\mathcal{ELI}$;
- monadic Datalog.

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...what about interpolation?

Interpolation

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interpolant

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$\downarrow$

only uses relations common to φ and ψ

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Craig interpolation: θ depends on φ and ψ

Uniform interpolation: θ depends only on φ and common signature (not on a particular ψ)

Interpolation for L_μ and UNFP

Theorem (D'Agostino, Hollenberg '00)

L_μ has effective uniform interpolation.

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Proof strategy: Bootstrap from modal world, making use of results/ideas of [Grädel, Walukiewicz '99], [Grädel, Hirsch, Otto '00], [D'Agostino, Hollenberg '00].

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Relational structures

Coded structures (tree decompositions of width k)

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$$L_\mu\ \widehat{\theta}$$

over subsignature encoding

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$\downarrow$ [D'Agostino, Hollenberg'00]

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Conclusion

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Can this result about UNFP help us answer any interesting query rewriting problems?

Uniform interpolation example

"S holds at x, and from every position y where S holds,
there is an R-neighbor z where S holds"

$$\begin{aligned}\varphi(x) &:= Sx \wedge \forall y\big(Sy \rightarrow \exists z(Ryz \wedge Sz)\big)\\ &\equiv Sx \wedge \neg\exists y\big(Sy \wedge \neg\exists z(Ryz \wedge Sz)\big)\end{aligned}$$

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Uniform interpolant of φ over subsignature $\{R\}$
"there is an infinite R-path from x"

$$[\mathbf{gfp}_{Y,y} \,.\, \exists z(Ryz \wedge Yz)](x)$$
$$\equiv \neg[\mathbf{lfp}_{Y,y} \,.\, \neg\exists z(Ryz \wedge \neg Yz)](x)$$