Expressive Power of Cost Logics over Infinite Words

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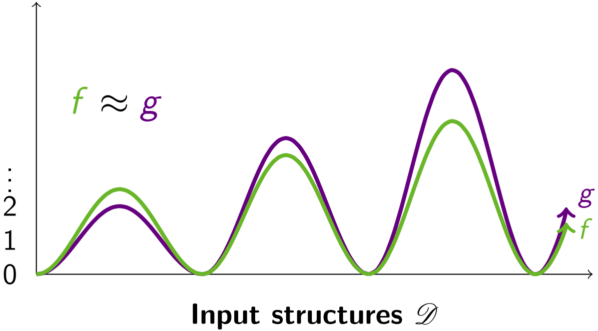
Cost functions

Cost automata and logics define **functions** $f : \mathscr{D} \to \mathbb{N} \cup \{\infty\}$
($\mathscr{D}$ could be words or trees over some fixed finite alphabet)

Only consider functions up to the **boundedness relation** $\approx$
"$f \approx g$": for all $U \subseteq \mathscr{D}$, $f(U)$ bounded iff $g(U)$ bounded



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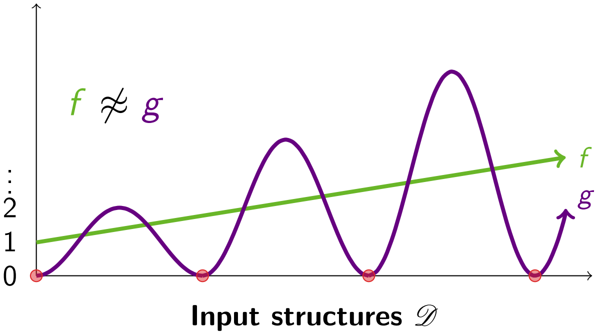

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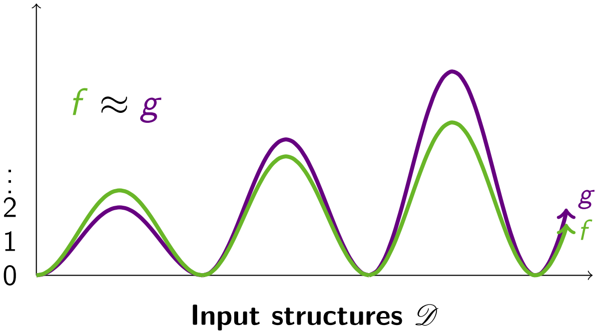

A **cost function** is an equivalence class of $\approx$.

Regular cost functions over finite words [Colcombet'09]

Regular Cost Functions

Cost Automata
Cost MSO
BS Expressions
Stabilization Monoids

$f \approx g$ decidable
[Colcombet'09, Bojańczyk+Colcombet'06]

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Given regular languages K and L, is $K = L$?

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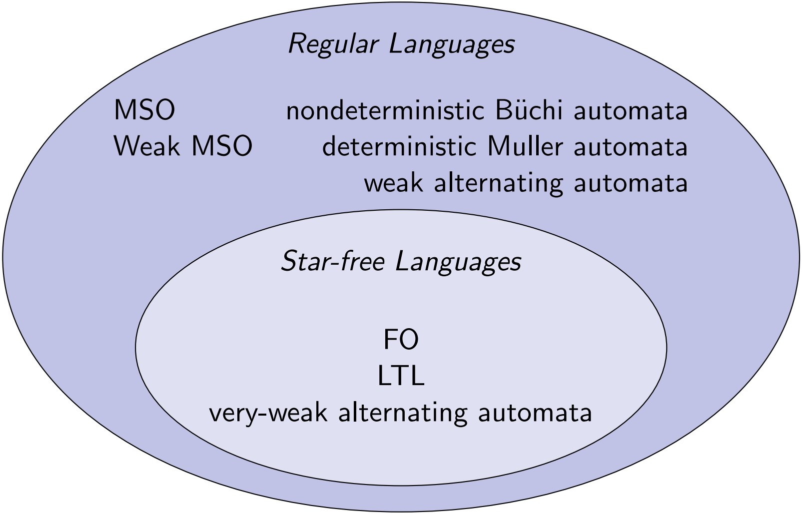
Given regular languages K and L, is $K = L$?

star height problem decidable

[Hashiguchi'88, Kirsten'05]

Given regular language L and $n \in \mathbb{N}$, is there a regular expression for L with at most n nestings of Kleene star?

Classical picture over infinite words



Do these classical results hold for regular cost functions over infinite words?

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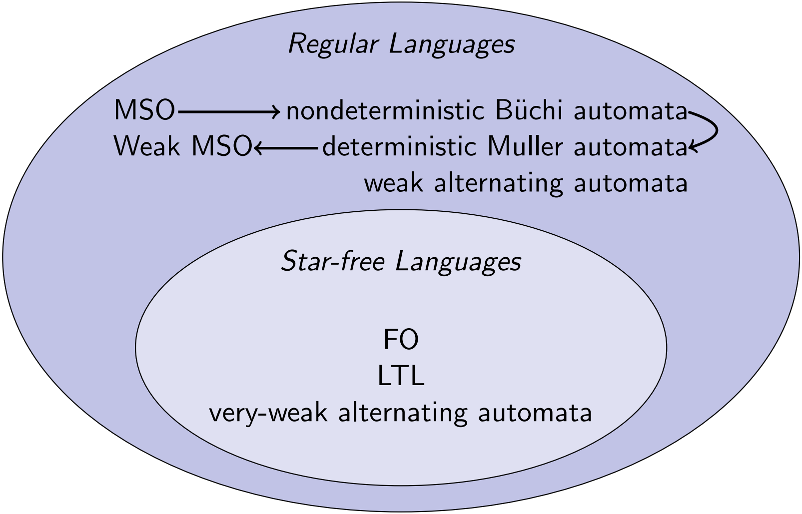

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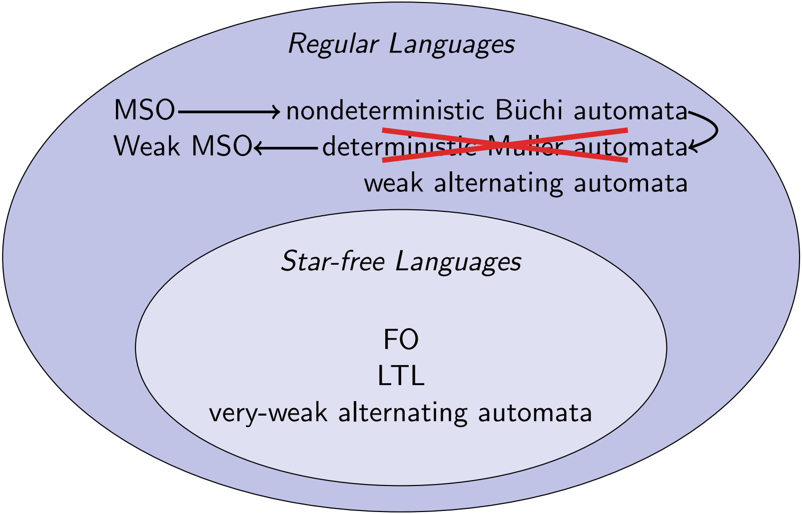

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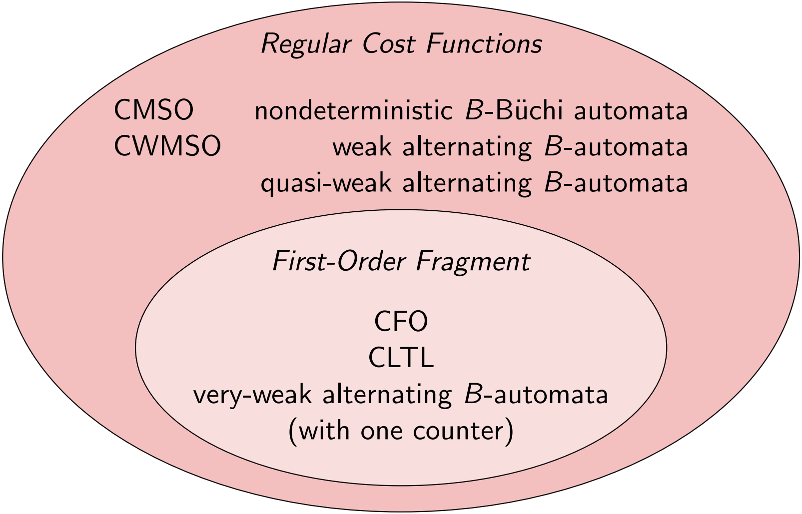
Cost functions over infinite words



B-Büchi automata over infinite words

Nondeterministic finite-state automaton $\mathcal{A}$
+ Büchi acceptance condition (visit accepting state infinitely often)
+ finite set of counters (initialized to 0, values range over $\mathbb{N}$)
+ counter operations (increment I, reset R, no change ε)

B-semantic

$[\![\mathcal{A}]\!] : \mathbb{A}^\omega \to \mathbb{N} \cup \{\infty\}$

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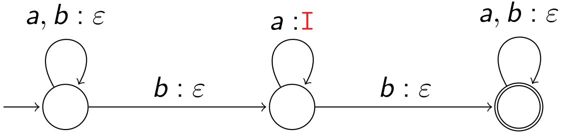
$[\![\mathcal{A}]\!](u) := \inf\{n : \exists$ accepting run with counter values at most $n\}$

Example

$[\![\mathcal{A}]\!](u) =$ min length of block of a's surrounded by b's in u



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Example

If no counter operations used, then

$$[\![\mathcal{A}]\!](u) = \chi_{L(\mathcal{A})}(u) = \begin{cases} 0 & \text{if } u \in L(\mathcal{A}) \\ \infty & \text{otherwise} \end{cases}$$

Cost logics

Cost first-order logic (CFO)

FO + $\forall^{\leq N} x.\psi$ appearing positively

- N is variable representing the error value (ranging over $\mathbb{N}$)
- $(u, n) \models \forall^{\leq N} x.\psi(x)$ iff $\psi(i)$ is false in at most n positions i

Cost function: $[\![\varphi]\!](u) := \inf\{n \in \mathbb{N} : (u, n) \models \varphi\}$

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Cost monadic second-order logic (CMSO)

CFO $+$ second-order quantification over sets

Cost weak monadic second-order logic (CWMSO)

same syntax as CMSO, but second-order quantification interpreted only over finite sets

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Examples

Let $u \in \{a, b\}^\omega$.

- **number of** a **in** u
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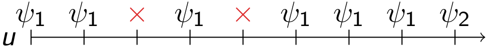
- **number of a in u**
  $= [\![\forall^{\leq N} x.b(x)]\!](u)$

- **min length of block of a (surrounded by b) in u**
  $= [\![\exists x.\exists y.x < y \wedge b(x) \wedge b(y) \wedge \forall^{\leq N} z.(z \leq x \vee z \geq y)]\!](u)$

Cost logics

Cost linear temporal logic (CLTL)
LTL + $\psi_1 \mathbf{U}^{\leq N} \psi_2$ (appearing positively)

$(u, n) \models \psi_1 \mathbf{U}^{\leq N} \psi_2$:



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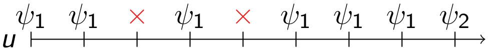

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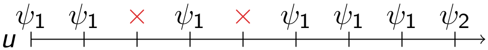

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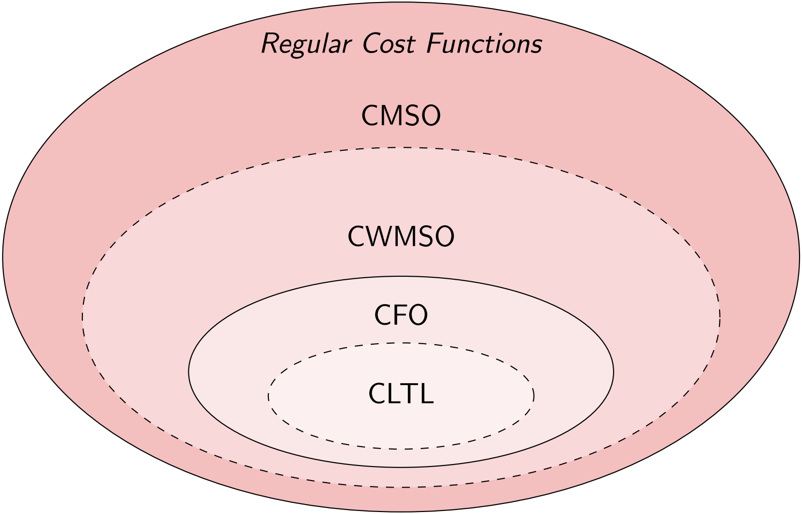
  $= \; [\![b \mathbf{U}^{\leq N} (\mathbf{G} b)]\!](u)$
- **min length of block of** a **(surrounded by** b**) in** u
  $= \; [\![\mathbf{F}(b \wedge \mathbf{X}(\bot \mathbf{U}^{\leq N} b))]\!](u)$

Cost functions over infinite words



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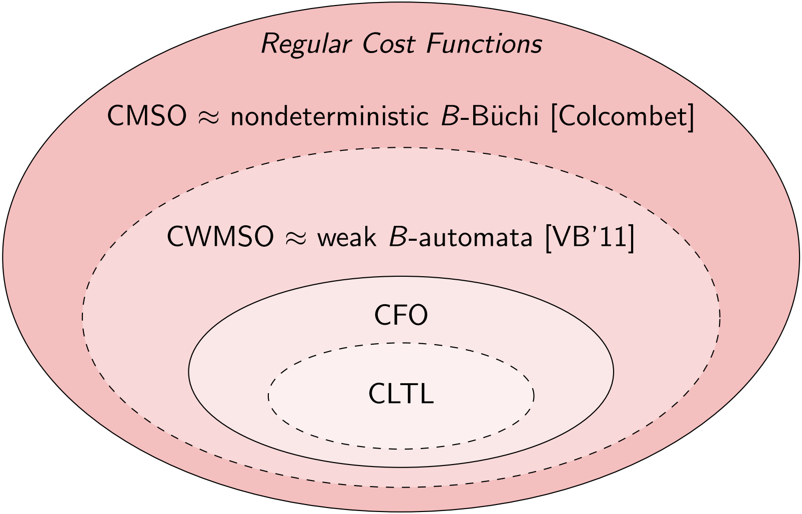

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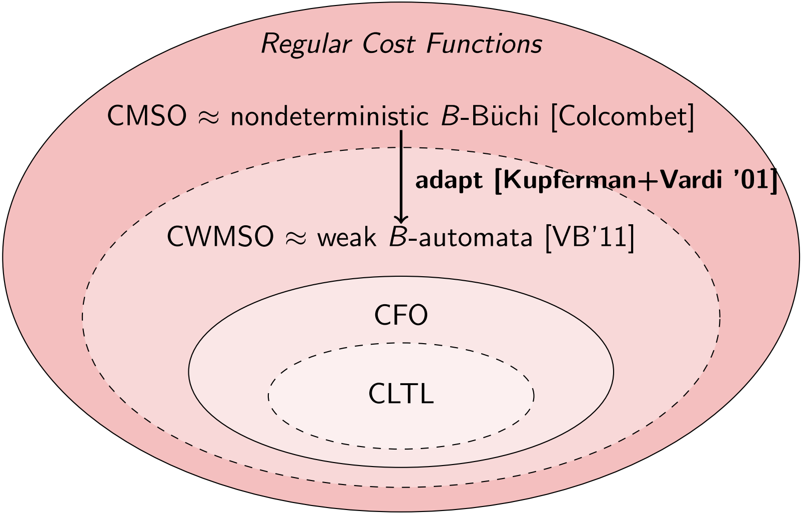

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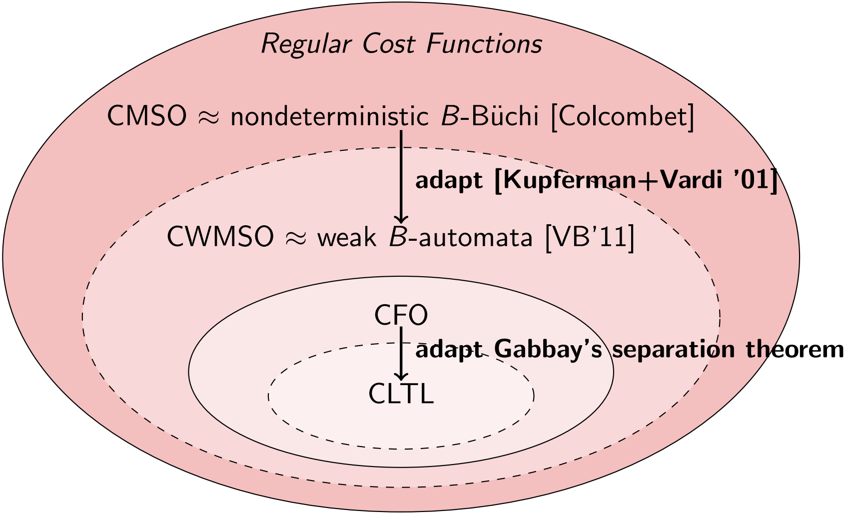
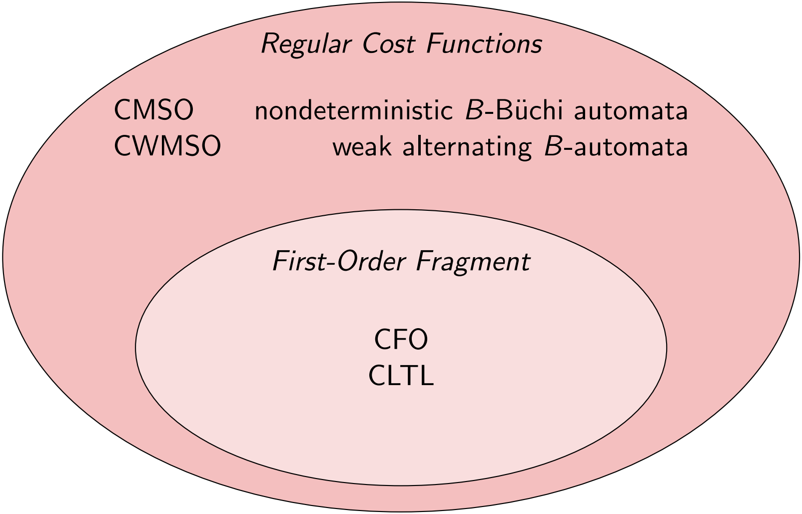


Summary over infinite words



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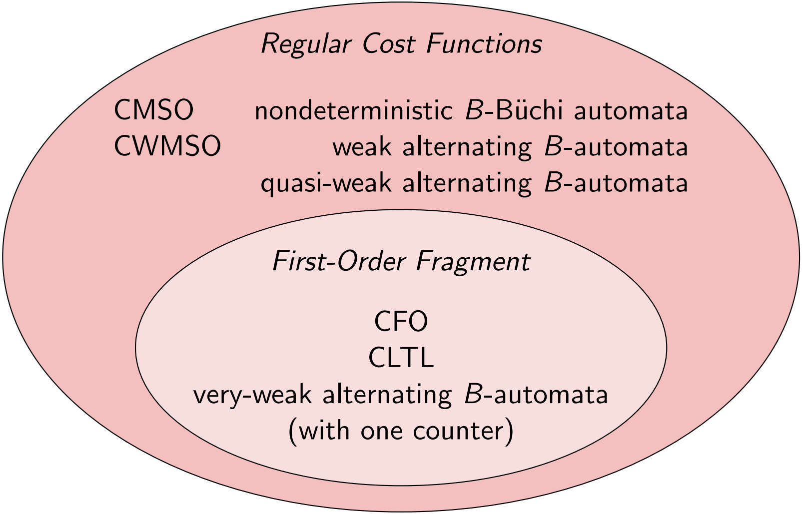

Conclusion

Can classical theorems about regular languages be extended to regular cost functions?

- finite words and trees: **yes** [Colcombet'09,Colcombet+Löding'10]
- infinite words: **yes** [Colcombet] and [Kuperberg+VB]

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- infinite words: **yes** [Colcombet] and [Kuperberg+VB]
- infinite trees: **open** but partial results in [VB'11, Kuperberg+VB'11]

Parity index problem
given regular language L of infinite trees and a set of priorities P, is there a nondeterministic parity automaton using only priorities P which recognizes L?

Weak definability problem
given a regular language L of infinite trees, is there a weak alternating automaton which recognizes L?

open in general case, but reduced to deciding $\approx$ for regular cost functions over infinite trees [Colcombet+Löding'08]