Fixpoint logics with tree-like models

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Including joint work with
Michael Benedikt, Pierre Bourhis, Balder ten Cate, and Thomas Colcombet

Fixpoint logics give mechanism to express
dynamic, recursive properties.

Example

binary relation R, unary relation P

"from x, it is possible to R-reach some P-element"

$$[\texttt{Reach-}P](x) := [\mathbf{lfp}_{Y,y} \,.\, \exists z(Ryz \wedge (Pz \vee Yz))](x)$$

Least fixpoint

Consider $\psi(\boldsymbol{y}, Y)$ positive in Y (of arity $m = |\boldsymbol{y}|$).

For all structures $\mathfrak{A}$, the formula ψ induces a monotone operation

$$\begin{aligned} \mathcal{P}(A^m) &\longrightarrow \mathcal{P}(A^m) \\ V &\longmapsto \psi_{\mathfrak{A}}(V) := \{\boldsymbol{a} \in A^m : \mathfrak{A}, \boldsymbol{a}, V \vDash \psi\} \end{aligned}$$

$\Rightarrow$ there is a unique least fixpoint $[\mathbf{lfp}_{Y,\boldsymbol{y}}.\psi(\boldsymbol{y}, Y)]_{\mathfrak{A}} := \bigcup_\alpha \psi^\alpha_{\mathfrak{A}}$

$$\begin{aligned} \psi^0_{\mathfrak{A}} &:= \varnothing \\ \psi^{\alpha+1}_{\mathfrak{A}} &:= \psi_{\mathfrak{A}}(\psi^\alpha_{\mathfrak{A}}) \\ \psi^\lambda_{\mathfrak{A}} &:= \bigcup_{\alpha<\lambda} \psi^\alpha_{\mathfrak{A}} \end{aligned}$$

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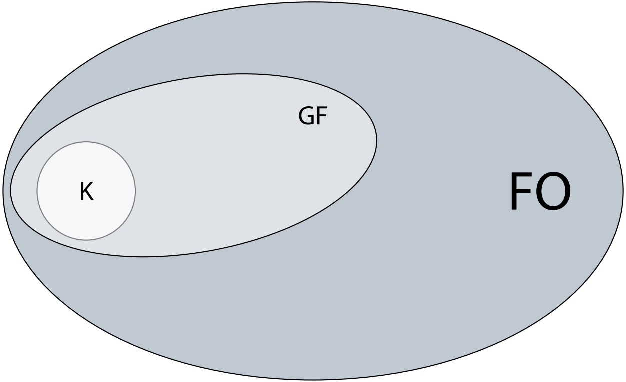
Semantics of fixpoint operator: $\mathfrak{A}, \boldsymbol{a} \models [\mathbf{lfp}_{Y,\boldsymbol{y}}.\psi(\boldsymbol{y}, Y)](\boldsymbol{x})$ iff $\boldsymbol{a} \in \bigcup_\alpha \psi_{\mathfrak{A}}^\alpha$

Some decidable fragments of first-order logic



constrain
quantification

$\exists \boldsymbol{x}(G(\boldsymbol{xy}) \wedge \psi(\boldsymbol{xy}))$
$\forall \boldsymbol{x}(G(\boldsymbol{xy}) \rightarrow \psi(\boldsymbol{xy}))$

[Andréka, van Benthem, Németi '95-'98]

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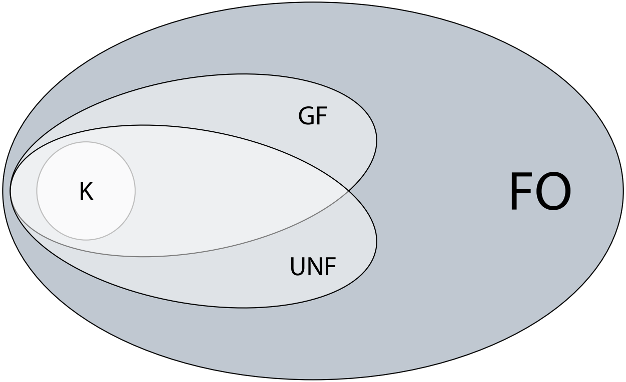

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constrain negation

$\exists \boldsymbol{x}(\psi(\boldsymbol{xy}))$
$\neg\psi(x)$

[ten Cate, Segoufin '11]

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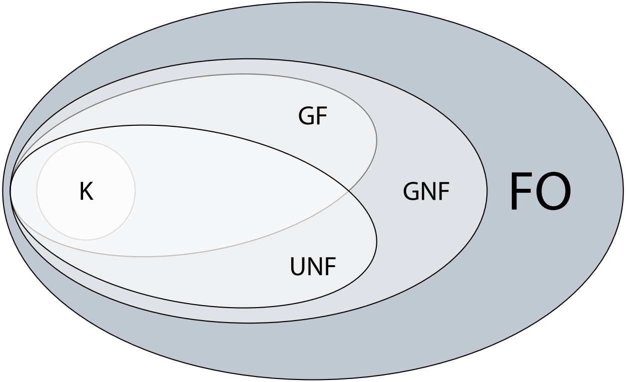

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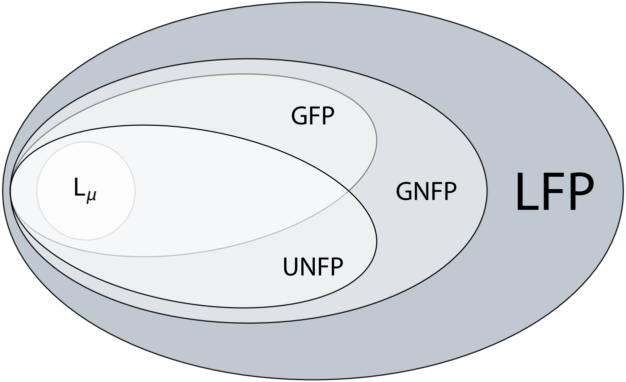
Some decidable fragments of LFP (fixpoint extension of FO)



constrain quantification

$\exists \boldsymbol{x}(G(\boldsymbol{xy}) \land \psi(\boldsymbol{xy}))$
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Guarded fixpoints: tuples in fixpoint are guarded by atom in original signature.
(UNFP has only monadic fixpoints, which are trivially guarded.)

Guarded negation fixpoint logic (GNFP)

Fix some signature σ of relations and constants.

Syntax of GNFP$[\sigma]$

$$\varphi \ ::= \ R\boldsymbol{t} \ \mid \ Y\boldsymbol{t} \ \mid \ \varphi \land \varphi \ \mid \ \varphi \lor \varphi \ \mid \ \exists y(\psi(\boldsymbol{x}y)) \ \mid \ G(\boldsymbol{x}) \land \neg\psi(\boldsymbol{x}) \ \mid$$
$$[\mathbf{lfp}_{Y,\boldsymbol{y}} \,.\, G(\boldsymbol{y}) \land \varphi(\boldsymbol{y}, Y, \boldsymbol{Z})](\boldsymbol{t}) \quad \text{where } Y \text{ only occurs positively in } \varphi$$

where R and G are relations in σ or $=$, and $\boldsymbol{t}$ is a tuple over variables and constants.

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Examples

- unions of conjunctive queries (in GNF)
- frontier-guarded tgds (in GNF):
  $\forall xyz\,((Rxy \wedge Ryz) \rightarrow \exists w(Twyz)) \equiv \neg\exists xyz\,(Rxy \wedge Ryz \wedge \neg\exists w(Twyz))$

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- mu-calculus, even with backwards modalities
- monadic Datalog

GNFP example

$$[\mathbf{lfp}_{Z,xy}\,.\,Sxy \wedge \exists uv(Rxu \wedge Ryv \wedge (Zuv \vee (Pu \wedge Pv)))](xy)$$

$$\begin{tikzcd}
a_1 \arrow[r] \arrow[d] & a_2 \arrow[r] \arrow[d] & a_3 \arrow[r, dotted] \arrow[d] & a_k \arrow[r] \arrow[d] & a_{k+1} \\
b_1 \arrow[r] & b_2 \arrow[r] & b_3 \arrow[r, dotted] & b_k \arrow[r] & b_{k+1}
\end{tikzcd}$$

Some nice computational properties for guarded fixpoint logics

Decidable satisfiability and finite satisfiability
(2EXPTIME in general, EXPTIME for fixed-width formulas in GFP)

[Grädel, Walukiewicz '99 ; Bárány, Segoufin, ten Cate '11; Bárány, Bojańczyk '12]

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Decidable boundedness
(given $\psi(\boldsymbol{y}, Y)$ positive in Y, is there $n \in \mathbb{N}$ such that for all $\mathfrak{A}$, $\psi^n_{\mathfrak{A}} = \psi^{n+1}_{\mathfrak{A}}$?)

[Blumensath, Otto, Weyer '14 ; Bárány, ten Cate, Otto '12 ; Benedikt, ten Cate, Colcombet, VB. '15]

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Constructive interpolation for UNFP

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Why so many nice properties?

These guarded logics all have tree-like models (bounded tree width).

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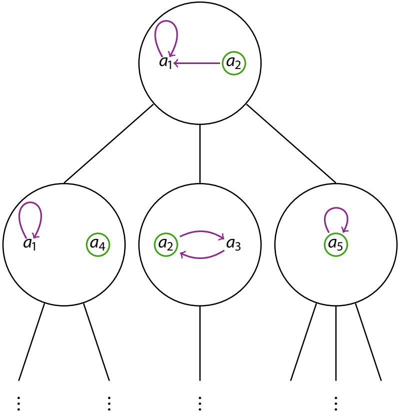
A structure $\mathfrak{A}$ has tree width $k - 1$ if it can be covered by (overlapping) bags of size at most k, arranged in a tree t s.t.

- every fact appears in some bag in t;
- for each element, the set of bags with this element is connected in t.



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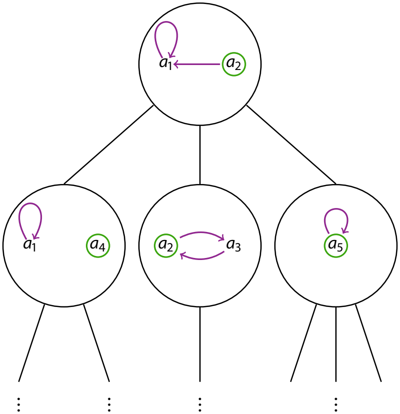

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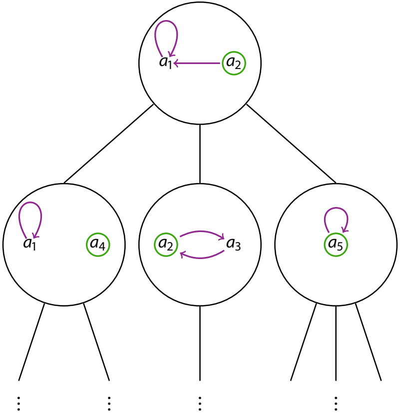

⇒ We can reason about tree encodings rather than relational structures.

Interpolation

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Craig interpolation: θ depends on φ and ψ

Uniform interpolation: θ depends only on φ and common signature (not on a particular ψ)

Uniform interpolation example

"P holds at x, and from every position y where P holds,
there is an R-neighbor z where P holds"

$$
\begin{aligned}
\varphi(x) &:= Px \wedge \forall y\big(Py \to \exists z(Ryz \wedge Pz)\big) \\
&\equiv Px \wedge \neg\exists y\big(Py \wedge \neg\exists z(Ryz \wedge Pz)\big)
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Uniform interpolant of φ over subsignature $\{R\}$
"there is an infinite R-path from x"

$$\begin{aligned} & \neg[\mathbf{lfp}_{Y,y} \,.\, \forall z(Ryz \to Yz)](x) \\ \equiv \;\; & \neg[\mathbf{lfp}_{Y,y} \,.\, \neg\exists z(Ryz \wedge \neg Yz)](x) \end{aligned}$$

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- Interpolation implies several results about going from semantic properties to syntactic properties (e.g., Beth definability, preservation theorems, etc.)

- Interpolation is related to query rewriting over views.

- Interpolation is related to modularity.

- Very little was known about interpolation for fixpoint logics over general relational structures, where relations can have arbitrary arity.

Interpolation for L_μ and UNFP

Theorem (D'Agostino, Hollenberg '00)

L_μ has effective uniform interpolation.

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Proof strategy: Exploit tree-like models and ideas / results from
[Grädel, Walukiewicz '99 ; Grädel, Hirsch, Otto '00 ; D'Agostino, Hollenberg '00].

Uniform interpolation for UNFP^k

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Relational structures

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over subsignature encoding

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\underset{\text{over subsignature}}{\mathrm{UNFP}^k\ \theta} & \underset{\text{over subsignature encoding}}{\mathrm{L}_\mu\ \widehat{\theta}} \arrow[l]
\end{tikzcd}
$$

Summary of interpolation results

UNFP has effective interpolation,
and the construction takes advantage
of its tree-like models.

	K	GF	UNF	GNF	L_μ	GFP	UNFP	GNFP
Craig interpolation	✓	✗	✓	✓	✓	✗	✓	✗

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"from y, it is possible to R-reach x"

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Subsumes

C2RPQs (conjunctive 2-way regular path queries) and
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But still has tree-like models!

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Theorem (Benedikt, Bourhis, VB. unpublished)

Satisfiability is decidable for $\varphi \in \mathrm{GNFP}^{\mathrm{UP}}$ in $(n+2)$-EXPTIME,
where n is "nesting depth of UCQ-shaped formulas with parameters" in φ.

Boundedness is decidable for $\varphi \in \mathrm{GNFP}^{\mathrm{UP}}$.

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Open questions

Does GNFP^{UP} have interpolation?

Is finite satisfiability decidable for GNFP^{UP}?

Conclusion

Guarded fixpoint logics are expressive logics
with nice computational properties
coming from their tree-like models.

Examples

Expressible in GNFP

R is symmetric

$$\begin{aligned} &\forall xy(Rxy \to Ryx) \\ \equiv\ &\neg\exists xy(Rxy \land \neg Ryx) \end{aligned}$$

Every element has an R-successor

$$\begin{aligned} &\forall x(\exists y(Rxy)) \\ \equiv\ &\neg\exists x(\neg\exists y(Rxy)) \end{aligned}$$

Every element is on R-cycle of length 3

$$\begin{aligned} &\forall x\exists yz(Rxy \land Ryz \land Rzx) \\ \equiv\ &\neg\exists x(\neg\exists yz(Rxy \land Ryz \land Rzx)) \end{aligned}$$

Not expressible in GNFP

R is total

$$\begin{aligned} &\forall xy(Rxy \lor Ryx) \\ \equiv\ &\neg\exists xy(\neg Rxy \land \neg Ryx) \end{aligned}$$

Every element has a unique R-successor

$$\begin{aligned} &\forall x\exists y(Rxy \land \forall z(Rxz \to y = z)) \\ \equiv\ &\neg\exists x(\neg\exists y(Rxy \land \neg\exists z(Rxz \land y \neq z))) \end{aligned}$$

Every element is on R-cycle

$$\begin{aligned} &\forall x\,[\mathbf{lfp}^{x}_{Y,y}.\exists z(Ryz \land (z = x \lor Yz))](x) \\ \equiv\ &\neg\exists x\,[\mathbf{lfp}^{x}_{Y,y}.\exists z(Ryz \land (z = x \lor Yz))](x) \end{aligned}$$

Examples, continued

Expressible in GNFP

R is well-founded:

$$\forall yz \Big(Ryz \to [\mathbf{lfp}_{Y,y}.\forall x(Rxy \to Yx)](y) \Big)$$

$$\equiv \neg\exists yz \Big(Ryz \wedge \neg[\mathbf{lfp}_{Y,y}.\neg\exists x(Rxy \wedge \neg Yx)](y) \Big)$$