Hedonic Coalition Nets

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ABSTRACT

In *hedonic games*, players have the opportunity to form coalitions, and have preferences over the coalitions they might join. Such games can be used to model a variety of settings ranging from multi-agent coordination to group formation in social networks. However, the practical application of hedonic games is hindered by the fact that the naive representation for such games is exponential in the number of players. In this paper, we study *hedonic coalition nets*—a succinct, rule-based representation for hedonic games. This formalism is based on marginal contribution nets, which were developed by Ieong and Shoham for representing coalitional games with transferable utility. We show that hedonic coalition nets are universally expressive, yet are at least as succinct as other existing representation schemes for hedonic games. We then investigate the complexity of many natural decision problems for hedonic coalition nets. In particular, we provide a complete characterisation of the computational difficulty of problems related to coalitional stability for hedonic games represented with hedonic nets.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; I.2.4 [**Knowledge representation formalisms and methods**]

General Terms

Theory

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1. INTRODUCTION

In coalitional games, players may form coalitions, and have preferences over the possible resulting coalition structures, i.e., partitions of the players into coalitions. Hedonic games [4, 2] are a subclass of coalitional games in which players are indifferent about coalitions formed by the players outside of their own coalition. In other words, in hedonic games players *only* care about *who* they will join with: the term "hedonic" stems from the idea that the players can be thought of as "enjoying the pleasure of each other's company". Hedonic games can be used to model many multi-agent coordination scenarios. They also provide an interesting approach to representing concepts in social networking services such as Facebook or MySpace. In particular, the framework of hedonic games is useful for capturing the notion of stability in these settings. Intuitively, a coalition structure is stable if no individual player or group of players prefers to deviate from it; this intuition can be formalised in several ways, depending on what deviations are allowed. Stable coalition structures can be seen as feasible outcomes of the game; therefore, identifying such coalition structures allows us to predict the behavior of players.

A key problem in applying concepts from hedonic games in multi-agent settings is, of course, that of representation: naive representations for hedonic games are exponential in the number of players. Accordingly, it is important to develop representations that are capable of succinctly capturing preferences in hedonic games, without being too computationally complex to be practically useful. (Of course, ultimately, the more compact and expressive a representation is, the more complex it is to reason with it: the key is to understand the tradeoffs involved.)

In this paper, we put forward a representation scheme for hedonic games, which we call *hedonic coalition nets*. This representation scheme is based on the marginal contribution nets formalism, which was developed by Ieong and Shoham for representing coalitional games with transferable utility (TU games) [12], and extended to coalitional games with non-transferable utility (NTU games) by Malizia *et al.* [13]. Hedonic coalition nets inherit many of the positive properties of marginal contribution nets: they provide a representation language that is *complete*, i.e., can be used to represent any hedonic game, yet is succinct for many interesting classes of games.

We begin by presenting the technical framework of hedonic games and the key solution concepts for such games. We then introduce hedonic coalition nets, and investigate their relationship to other representation schemes for hedonic games. Next, we study the complexity of natural decision problems for hedonic coalition nets. Specifically, we first consider problems of checking *equivalence* of hedonic nets: when a pair of hedonic nets are equivalent with respect to the actual values of every coalition, and when they are equivalent with respect to the preference relations they induce. We then investigate the computational aspects of stability in hedonic games, and provide a complete characterisation of the computational complexity of stability-related solution concepts in these games. We focus on the core, which is perhaps one of the most important solution concepts in hedonic games, as it represents coalition structures resistant to group deviations. Using the ideas of [12] as a starting point, we identify a class of hedonic games that admits efficient algorithms for checking if a particular outcome is in the core. Our argument demonstrates that TU-games and NTU-games are very different from a computational perspective.

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2. HEDONIC GAMES

We give a self-contained but brief introduction to the technical framework of hedonic games; for more details, including motivations for studying such games, we refer the reader to, e.g., [4]. Let $N = \{1, \ldots, n\}$ be a set of players. A *coalition* in N is a non-empty subset of N; for $i \in N$, let $\mathcal{N}_i$ be the coalitions in N that contain i: $\mathcal{N}_i = \{C \cup \{i\} : C \subseteq N \neq \emptyset\}$. A hedonic game (hereafter referred to simply as a "game") is then a structure

$$G = \langle N, \succeq_1, \ldots, \succeq_n \rangle$$

where $N = \{1, \ldots, n\}$ is the set of players, and $\succeq_i \subseteq 2^{\mathcal{N}_i} \times 2^{\mathcal{N}_i}$ is a complete, reflexive, and transitive preference relation for player $i \in N$, with the intended interpretation that if $C_1 \succeq_i C_2$, then player i prefers coalition C_1 at least as much as coalition C_2. As usual, for each player we can define the indifference relationship $\sim_i$ by setting $C_1 \sim_i C_2$ iff $C_1 \succeq_i C_2$ and $C_2 \succeq_i C_1$ and the strict preference order $\succ_i$ by setting $C_1 \succ_i C_2$ iff $C_1 \succeq_i C_2$ and $C_1 \not\sim_i C_2$. To simplify our presentation, we will often abuse notation by writing $C_1 \succeq_i C_2$ for *arbitrary* coalitions C_1 and C_2, understanding that this is an abbreviation for $C_1 \cup \{i\} \succeq_i C_2 \cup \{i\}$.

2.1 Solution Concepts

The outcome of a game is a *coalition partition*: a partition of the players N into disjoint coalitions. If π is a coalition partition and $i \in N$ then we denote by π_i the coalition in π of which i is a member. We let Π^G denote the possible coalition partitions over G, dropping reference to G where the context is clear. Now, let π be a coalition partition over $G = \langle N, \succeq_1, \ldots, \succeq_n \rangle$; then a number of solution concepts suggest themselves, as follows [4, p.207–208]:

- A coalition $C \subseteq N$ *blocks* π if $C \succ_i \pi_i$ for all $i \in C$. The *core* of a game is the set of coalition partitions that are not blocked by any coalition. We denote the core of game G by $core(G)$.
- π is *individually rational* if $\pi_i \succeq_i \{i\}$ for all $i \in N$. We denote the set of individually rational partitions for G by $ir(G)$. If π is individually rational, then every player does at least as well in π as it would do alone.
- π is *Nash stable* if for all $i \in N$ we have $\pi_i \succeq_i C_k \cup \{i\}$ for all $C_k \in \pi$. Thus Nash stability means that no player would want to join any other coalition in π, assuming the other coalitions did not change. Let $ns(G)$ denote the set of Nash stable solutions for game G.
- π is *individually stable* if there do not exist $i \in N$ and $C \in \pi \cup \{\emptyset\}$ such that $C \cup \{i\} \succ_i \pi_i$ and $C \cup \{i\} \succeq_j C$ for all $j \in C$. Intuitively, individual stability means no player could move to another coalition that it preferred without making some member of the coalition it joined unhappy.
- π is *contractually individually stable* (CIS) if there do not exist $i \in N$ and $C \in \pi \cup \{\emptyset\}$ such that:

  $C \cup \{i\} \succ_i \pi_i$ and $C \cup \{i\} \succeq_j C$ for all $j \in C$; and

  $\pi_i \setminus \{i\} \succeq_j \pi_i$ for all $j \in \pi_i \setminus \{i\}$.

  Intuitively, a CIS partition is one in which no player can move to another coalition that it prefers so that the move is acceptable to both coalitions it joins and leaves.

It is easy to see that, for example, any Nash stable coalition partition or core coalition partition is individually rational (see [4, p.208]).

An obvious issue when considering hedonic games from a computational point of view is that of representation: the naive representation (explicitly listing preference orders $\succeq_i$ for each player i) will be exponential in the number of players. A number of representations for hedonic games have been proposed in the literature; we will discuss these in later sections, where we formally compare them to hedonic coalition nets.

3. HEDONIC COALITION NETWORKS

We now introduce *hedonic coalition nets*, a succinct representation for hedonic games which draws upon the marginal contribution nets formalism for coalitional games [12]. The basic idea behind hedonic coalition nets is to represent a player's preference relation $\succeq_i$ as a collection of *rules* of the form $\varphi \mapsto_i x$, where φ is a predicate over coalitions, expressed as a formula of propositional logic, and x is a real number. To determine the value of a coalition to a player, we take the player's rule set, and sum the values on the right hand side of rules for which the predicate part of the rule is satisfied by the coalition. These values then induce the preference orderings $\succeq_i$ in the obvious way.

We make use of classical propositional logic, and for completeness, we thus begin by recalling the technical framework of this logic. Let Φ be a (finite, fixed, non-empty) vocabulary of Boolean variables, and let $\mathcal{L}_\Phi$ denote the set of (well-formed) formulae of propositional logic over Φ, constructed using the conventional Boolean operators ("$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$", and "$\neg$"), as well as the truth constants "$\top$" (for truth) and "$\bot$" (for falsity). We assume a conventional semantic satisfaction relation "$\models$" for propositional logic. A *valuation*, ξ, is a subset of Φ: a variable $p \in \Phi$ is considered to be true under valuation ξ iff $p \in \xi$. We write $\xi \models \varphi$ to mean that φ is true under, or satisfied by, valuation $\xi \subseteq \Phi$, in which case ξ is a *satisfying assignment* for φ, while if $\xi \not\models \varphi$ then ξ is a *falsifying assignment*.

For hedonic coalition nets, we fix the vocabulary of propositional variables Φ to be the set of players, N, so that we we have a propositional variable for every player. Note that every coalition in N then defines a valuation for $\mathcal{L}_N$. A *rule for player* $i \in N$ is then a pair (φ, β), where $\varphi \in \mathcal{L}_N$ and $\beta \in \mathbb{R}$. We write a rule (φ, β) for player $i \in N$ using the notation $\varphi \mapsto_i \beta$, omitting the index i where it is clear from the context. Let $\mathcal{R}_i$ be the set of possible rules for player i. A *hedonic coalition net* (hereafter simply "net") is a structure

$$H = \langle N, R_1, \ldots, R_n \rangle,$$

where $N = \{1, \ldots, n\}$ is a set of players, and $R_i \subseteq \mathcal{R}_i$ is a set of rules for player i, for each $i \in N$. The *utility* of a coalition $C \in N_i$ for a player i is then:

$$u_i(C) = \sum_{\substack{\varphi \mapsto_i \beta \in R_i: \\ C \models \varphi}} \beta.$$

We say a net is in *simple conjunctive form* if the conditions of all rules are conjunctions of literals, i.e., of the form

$$p_1 \wedge \cdots \wedge p_k \wedge \neg p_{k+1} \wedge \cdots \wedge \neg p_l$$

for Boolean variables $p_1, \ldots, p_k, p_{k+1}, \ldots, p_l$. Similarly, we say a hedonic net is of *simple disjunctive form* if conditions are disjunctions of literals. We say a net is a *unit net* if the only value appearing on the right hand side of rules is 1.

Given a net $H = \langle N, R_1, \ldots, R_n \rangle$, we let $G_H = \langle N, \succeq_1, \ldots, \succeq_n \rangle$ denote the game induced by H, i.e., the game such that for all $i \in N$ and for all $C_1, C_2 \in N_i$, we have that

$$\underbrace{C_1 \succeq_i C_2}_{\text{game}} \text{ iff } \underbrace{u_i(C_1) \geq u_i(C_2)}_{\text{net}}.$$

3.1 Other Representations

The first question to ask is how hedonic nets relate to other existing representations for hedonic games. When we consider the notion of a representation, we are typically thinking of representing elements of a set – in the present paper, we are thinking of representing elements of the set of hedonic games. Given a representation scheme ζ for a set S, we denote by $\zeta(x)$ the shortest representation of x possible using the scheme ζ. When we say that a representation ζ_1 is as compact or succinct as a representation ζ_2, we mean that for every $x \in S$, the size of element x represented using ζ_1 (i.e., $|\zeta_1(x)|$) is at most polynomial in $|\zeta_2(x)|$. We will say that a representation ζ_1 is *strictly more expressive* than ζ_2 if every object that can be represented using ζ_2 can be represented using ζ_1, and, moreover, there exists an object that can be represented using ζ_1, but not ζ_2. In this section, we review several representations that appear in the literature on hedonic games, and for each, show that hedonic nets are at least as compact.

IRCLs: We start by presenting *individually rational coalition lists* (IRCL). This formalism is based on the idea of eliminating redundant information from the naive representation, i.e., eliminating information that can manifestly play no part in the strategic reasoning of players [1]. Specifically, in IRCL, instead of listing the complete preference ordering $\succeq_i$, we only list the ordering for those coalitions that are preferred by i over the singleton coalition $\{i\}$, that is, the individually rational coalitions. It is easy to see that only such individually rational coalitions can form part of a coalition partition satisfying the solution concepts listed above. So, for the IRCL representation we define $\succeq_i$ by explicitly listing individual rational coalitions for i, in order, most preferred first, and indicating whether two consecutive coalitions in this order are equally preferred.

Formally, the preference list of the player i is represented as $C_1 *^1 C_2 *^2 \cdots *^{r-1} C_r$ where $*^j \in \{\succ_i, \sim_i\}$, $C_j \subseteq \mathcal{N}_i$ and $C_r = \{i\}$. Although this representation can sometimes eliminate much redundant information, it is clear that, in many cases, this representation will be no more succinct than the naive representation. For example, if the worst outcome for a player is working alone (which would happen, for example, if the player needed help to achieve its goal) then IRCLs reduce to the naive representation. On the positive side, the IRCL representation *is* complete.

Given an IRCL-representation of a player's preferences, we can transform it into a hedonic net representation as follows: given a preference list $C_1 *^1 C_2 *^2 \cdots *^{r-1} C_r$ of player i, define $x_r = 0$ and for $j = r-1, \ldots, 1$ set $x_j = x_{j+1}$ if $C_j \sim_i C_{j+1}$ and $x_j = x_{j+1} + 1$ if $C_j \succ_i C_{j+1}$. Now, the rule set R_i contains r rules, where the jth rule in R_i is:

$$\left(\bigwedge_{k \in C_j} k\right) \wedge \left(\bigwedge_{l \in N \setminus C_j} \neg l\right) \mapsto_i x_j.$$

It is easy to see that the size of the resulting representation is at most a factor of n larger than that of the original representation (the worst-case blowup corresponds to coalitions of size 1, which are represented as rules with n literals).

We will now review several representation formalisms that are not complete, i.e., strictly less expressive than hedonic nets, and demonstrate that hedonic nets are nevertheless as compact as those representations.

Additively Separable Games: A game $G = \langle N, \succeq_1, \ldots, \succeq_n \rangle$ is said to be *additively separable* if there exists an $|N| \times |N|$ matrix of reals v (the *value matrix*) such that $C_1 \succeq_i C_2$ iff $\sum_{j \in C_1} v[i,j] \geq \sum_{j \in C_2} v[i,j]$. Thus $v[i,j]$ represents the value of player j to player i. If a game is additively separable, then the value matrix v clearly provides a very succinct representation for the game. Of course, not all games are additively separable, and so we cannot use this representation for all games. Additively separable games are straightforward to represent as hedonic nets. Let v be the value matrix for an additively separable game: then for each $i, j \in N$, we create a single rule for i, as follows: $j \mapsto_i v[i,j]$.

In fact, we can assume that the right-hand sides of all rules are integers. Indeed, given a game represented by a non-integer matrix, we can find an equivalent integer matrix representation of this game by solving a system of linear inequalities (one for each pair of coalitions) with 0-1 coefficients. This will produce a rational matrix, which can then be scaled up. However, these integers may have to be quite large, i.e., superpolynomial in n. For example, consider a player i with lexicographic preferences. More formally, define $\succeq_i$ as $C_j \succ_i C_k$ iff $\min(C_j \triangle C_k) \in C_j$, where $\triangle$ denotes the symmetric difference of two sets. These preferences are additively separable: we can set $v[i,j] = 2^{n-j}$. However, as for every pair of coalitions $C_j, C_k \in \mathcal{N}_i$ we have either $C_j \succ C_k$ or $C_k \succ C_j$, the set $\{\sum_{j \in S} v[i,j] \mid S \subseteq \mathcal{N}_i\}$ contains at least 2^{n-1} distinct elements: otherwise, there will be two coalitions that are assigned the same value by this representation. Hence, at least some of the RHSs of the rules in R_i are at least $2^{n-1}/n$.

$\mathcal{B}$- and $\mathcal{W}$-Preferences: Another class of preferences is based on ranking individual players, and ordering coalitions according to the ranks of their worst/best members. In more detail, under $\mathcal{W}$-preferences introduced in [7], each player i has a (reflexive, transitive and complete) preference relation $\succeq'_i$ over the set of of all players, and he prefers a coalition C_1 to a coalition C_2 iff he prefers the worst member of C_1 (according to $\succeq'_i$) to the worst member of C_2. Games with $\mathcal{W}$-preferences are not additively separable; nor can they be compactly represented using individually rational coalition lists. However, they *can* be compactly represented using hedonic coalition nets.

Consider a player i with a preference relation $\succeq'_i$ over all other players. Order all players in $N \setminus \{i\}$ so that $i_1 \succeq'_i \cdots \succeq'_i i_{n-1}$. As with the IRCL translation, set $x_{i_{n-1}} = 1$, and for $j = n-2, \ldots, 1$ set $x_{i_j} = x_{i_{j+1}}$ if $i_j \sim'_i i_{j+1}$ and set $x_{i_j} = x_{i_{j+1}} + 1$ if $i_j \succ'_i i_{j+1}$. We can now represent player i's preferences using the following $n-1$ rules:

$$\begin{array}{rcl} i_{n-1} & \mapsto & x_{i_{n-1}} \\ i_{n-2} \wedge \neg i_{n-1} & \mapsto & x_{i_{n-2}} \\ & \cdots & \\ i_1 \wedge \neg i_2 \wedge \ldots \wedge \neg i_{n-1} & \mapsto & x_{i_1}. \end{array}$$

$\mathcal{B}$-preferences [5] are defined in a similar manner: again, each player i has a preference relation $\succeq'_i$ over individual players, but now we have $C_1 \succeq_i C_2$ if i prefers the best player in C_1 (according to $\succeq'_i$) to the best player in C_2; in addition, the draws are resolved in favor of smaller sets (otherwise, the grand coalition is the best outcome for everybody). They can be represented as hedonic coalition nets using the following set of rules:

$$\begin{array}{rcl} i_1 \mapsto -\delta, \ldots, i_{n-1} & \mapsto & -\delta \\ i_1 & \mapsto & x_{i_1} \\ i_2 \wedge \neg i_1 & \mapsto & x_{i_2} \\ & \ldots & \\ i_{n-1} \wedge \neg i_{n-2} \wedge \ldots \wedge \neg i_1 & \mapsto & x_{i_{n-1}}, \end{array}$$

where δ is sufficiently small (say, $\delta < \frac{1}{n}$) and $i_1, \ldots, i_{n-1}$ and $x_{i_1}, \ldots, x_{i_{n-1}}$ are defined as above.

Anonymous Preferences: Another well-studied class of preferences in hedonic games is that of *anonymous preferences*: each player's preferences solely depend on the sizes of the coalitions, but not on individual players that appear in these coalitions. In other words, each player i is endowed with a preference relation $\succeq'_i$ over $1, \ldots, n$ and his preference relation over coalitions $\succeq_i$ is given by $C_1 \succeq_i C_2$ iff $|C_1| \succeq'_i |C_2|$. Clearly, this class of preferences differs from all other classes considered above. However, it, too, can be compactly represented using hedonic coalition nets. Indeed, it is known [9] that for each $k = 1, \ldots, n$ there is a Boolean formula φ_k over variable $x_1, \ldots, x_n$ of size poly(n) such that for all $\xi \subseteq \{x_1, \ldots, x_n\}$, we have $\xi \models \varphi_k$ iff $|\xi| = k$. Using such formulas, we can express anonymous preferences as hedonic coalition nets; the construction is similar to the one for $\mathcal{W}$-preferences.

Discussion: We summarise the main points above in the following:

THEOREM 1.

1. *Hedonic nets are* just as compact *as all the representation formalisms considered above (the* IRCL*-representation, the matrix representation for additively separable games, and the $\succeq'_i$-representations for $\mathcal{B}$- and $\mathcal{W}$-preferences or anonymous preferences).*

2. *Hedonic nets are* strictly more expressive *than the representations based on additively separable preferences, $\mathcal{B}$- and $\mathcal{W}$-preferences, and anonymous preferences.*

3. *For some games, hedonic nets are* exponentially more compact *than the* IRCL *representation.*

The first point follows from the constructions given in the section above, showing how each representation can be translated to hedonic nets with at most a polynomial blow up in size. The second point follows from the fact that every hedonic game can be represented by a hedonic net, i.e., hedonic nets provide a *complete* description language for hedonic games, while additively separable preferences, $\mathcal{B}$- and $\mathcal{W}$-preferences, and anonymous preferences are not complete. For the third point, it is easy to give examples of games that may be succinctly represented using hedonic nets, but which require space exponential in the number of players using the IRCL representation (e.g., additively separable preferences).

4. EQUIVALENCE PROBLEMS

An obvious first problem is as follows. Suppose we are given hedonic nets H_1, H_2 , and asked whether these are *equivalent* (in this and other similar problems, we assume that the hedonic nets given in the problem instance have exactly the same players). In fact, equivalence can be formulated in several different ways. First, we will say they are *net equivalent* if, assuming $u_i^1(\cdots)$ and $u_i^2(\cdots)$ denote the utility functions for player $i \in N$ induced by H_1 and H_2 respectively, then for all $i \in N$ and for all $C \subseteq N$ we have $u_i^1(C) = u_i^2(C)$. It is trivial to see that, if we allow arbitrary conditions on the left hand side of rules, then checking net equivalence is coNP-complete. A natural question is whether, by constraining the form of conditions in rules, we can obtain tractability. As the following result shows, we have high complexity even with quite strong constraints on rules.

THEOREM 2. NET EQUIVALENCE *is coNP-complete even for simple conjunctive or disjunctive unit nets.*

PROOF. Membership in coNP is obvious, so consider hardness. Proving hardness for simple disjunctive nets is straightforward, so we focus on the conjunctive case. We reduce TAUT, the problem of checking that a formula ψ of propositional logic is true under every valuation. Without loss of generality, we assume that ψ is in 3-Conjunctive Normal Form, i.e., of the form

$$\psi = \bigwedge_{i=1}^{m} \chi_i$$

where each χ_i is a disjunction of three literals:

$$\chi_i = \ell_i^1 \vee \ell_i^2 \vee \ell_i^3.$$

(Recall that a literal is either a Boolean variable or the negation of a Boolean variable.) Given an input instance ψ with m clauses $\chi_1, \ldots, \chi_m$, over the l Boolean variables $p_1, \ldots, p_l$, we create hedonic coalition nets H_1^ψ and H_2^ψ as follows. First, we create $l + m$ players: one player for each clause, and one player for each Boolean variable. For the construction, we need to choose some arbitrary member of N (e.g., the player corresponding to the lexicographically first Boolean variable); we denote this player by d.

The rule set for H_1^ψ is then constructed so that every player gets a utility of 1 for every coalition. More formally, we define the rule sets in H_1^ψ as follows:

$$\begin{array}{rcl} R_1 & = & \{d \mapsto_1 1, \neg d \mapsto_1 1\} \\ & \ldots & \\ R_{l+m} & = & \{d \mapsto_{l+m} 1, \neg d \mapsto_{l+m} 1\} \end{array}$$

The rule set for H_2^ψ is constructed as follows. For each player $1 \leq i \leq m$, corresponding to a clause, we create a rule set R_i with 7 rules, as follows (we assume double negations are eliminated):

$$\begin{array}{rcl} (\neg\ell_i^1) \wedge (\neg\ell_i^2) \wedge \ell_i^3 & \mapsto_i & 1 \\ (\neg\ell_i^1) \wedge \ell_i^2 \wedge (\neg\ell_i^3) & \mapsto_i & 1 \\ (\neg\ell_i^1) \wedge \ell_i^2 \wedge \ell_i^3 & \mapsto_i & 1 \\ \ell_i^1 \wedge (\neg\ell_i^2) \wedge (\neg\ell_i^3) & \mapsto_i & 1 \\ \ell_i^1 \wedge (\neg\ell_i^2) \wedge \ell_i^3 & \mapsto_i & 1 \\ \ell_i^1 \wedge \ell_i^2 \wedge (\neg\ell_i^3) & \mapsto_i & 1 \\ \ell_i^1 \wedge \ell_i^2 \wedge \ell_i^3 & \mapsto_i & 1 \end{array}$$

Finally, for each player $m < i \leq l + m$, we create a rule set

$$R_i = \{d \mapsto_i 1, \neg d \mapsto_i 1\}.$$

Let $u_i^1(\cdots)$ and $u_i^2(\cdots)$ be the utility functions for player i from the nets H_1^ψ and H_2^ψ respectively. By construction, for every $i \in N$ and $C \subseteq N$, we have $u_i^1(C) = 1$, irrespective of the structure of ψ. Where $C \subseteq N$, we denote by ξ_C the valuation for the Boolean variables $p_1, \ldots, p_l$ defined by eliminating from C all players corresponding to clauses, i.e.,

$$\xi_C = C \cap \{p_1, \ldots, p_l\}.$$

Now, H_1^ψ and H_2^ψ are net equivalent iff the input instance ψ is a tautology. Notice that the reduction is polynomial (we use $2(l+m)$

rules in H_1^ψ and $2l+7m$ rules in H_2^ψ), and the form of rules created satisfies the statement of the theorem. □

A related question is whether the two nets induce the same game; we call this GAME EQUIVALENCE. While net equivalence obviously implies game equivalence, the converse does not, of course, hold in general.

THEOREM 3. GAME EQUIVALENCE *is coNP-complete even for simple conjunctive or disjunctive unit nets.*

PROOF. Membership is obvious; for hardness, we use the same reduction as Theorem 3, asking whether $G_{H_1^\psi} = G_{H_2^\psi}$. □

5. SOLUTION CONCEPTS

We now turn to the various solution concepts discussed above. Given a solution concept S, there are several obvious associated decision problems:

- S-MEMBERSHIP: Given a hedonic net H and partition π, is π an instance of the solution S?
- S-NON-EMPTINESS: Given a hedonic net H and partition π, is the set $\{\pi : \pi \text{ is an } S \text{ solution for } H\}$ non-empty?

For the solution concepts that deal with individual deviations, i.e., individual rationality, Nash stability, individual stability, and contractual individual stability, the MEMBERSHIP problems are trivially decidable in polynomial time. Also, every hedonic game has an individually rational and a contractually individually stable solution [1, p.8], so the NON-EMPTINESS problems for these solution concepts are trivially polynomially solvable. Furthermore, it is known that for the IRCL-representation, checking whether there exists a Nash stable partition or an individually stable partition is NP-hard [1]. This immediately implies that NON-EMPTINESS is NP-hard for these solution concepts under the hedonic net representation. Moreover, since, as argued above, under this representation one can check if a given partition is Nash stable or individually stable in polynomial time, checking NON-EMPTINESS for Nash stable or individually rational solutions are NP-complete for hedonic nets. Thus, we have a complete characterisation of the complexity of MEMBERSHIP and NON-EMPTINESS for solution concepts related to individual deviations. We now turn to the solution concept that captures stability under group deviations, i.e., the core.

5.1 The Core

The CORE MEMBERSHIP problem is known to be decidable in polynomial time for the IRCL representation [1, p.10], and is coNP-complete for the additive representation [15, p.157]. While the former result seems attractive, given the fact that the IRCL representation is often not succinct, this result is perhaps not very significant. Moreover, since we can directly encode additive games using hedonic nets, it follows from [15, p.157] that CORE MEMBERSHIP for hedonic nets is coNP-hard, and it is therefore easy to see that CORE MEMBERSHIP is coNP-complete for hedonic nets.

Now consider the CORE NON-EMPTINESS problem. This problem involves checking that

$$\exists \pi \in \Pi : (\forall C \subseteq N : [\forall i \in C : \pi_i \succeq_i C]).$$

Clearly, the problem is in Σ_2^p for hedonic nets, since the inner condition can be checked in polynomial time. Moreover, [13, Theorem 6.3], a general complexity result on non-emptiness of the core for NTU games represented by marginal contribution nets, tells us that the problem is also Σ_2^p-hard, thus giving us a complete picture of the complexity of core-related problems for hedonic coalition nets.

Games with bounded treewidth: An obvious next question is whether restrictions on the form of hedonic nets lead to polynomial time decidability for problems relating to the core. Indeed, Ieong and Shoham [12] give a polynomial-time algorithm for deciding core membership for TU games in the special case where the underlying network has bounded treewidth. The notion of treewidth is applicable in the context of hedonic nets as well. Given a game G_H with a set of players N, and a list of rule sets $(R_1, \ldots, R_n)$ (here H can be either a transferable utility game or a hedonic game), consider the *agent graph* $\mathcal{G}_H$ whose vertices are players, and there is an edge between i and j if there is a rule $(\varphi, \beta) \in R_i$ such that j appears in φ, or there is a rule $(\varphi', \beta') \in R_j$ such that i appears in φ', or for some $k = 1, \ldots, n$ there is a rule $(\varphi'', \beta'') \in R_k$ such that both i and j appear in φ''. Paper [12] then showed that one can decide core membership for TU games in time that is exponential only in the treewidth of $\mathcal{G}_H$, and therefore is polynomial if the treewidth of $\mathcal{G}_H$ is bounded by a constant. It is therefore natural to ask if this result can be extended to hedonic coalition nets. Before we proceed, we provide a review of the basic ideas of tree decomposition and treewidth (this material is based on [3] and [12]).

DEFINITION 1. *A* tree decomposition *of a graph $G = (V, E)$ is a pair $(\mathcal{S}, T)$, where $\mathcal{S} = \{S_1, \ldots, S_K\}$ is a collection of subsets of V and $T = (\mathcal{V}, \mathcal{E})$ is a tree whose vertices are labeled by the elements of $\mathcal{S}$ so that a vertex $v \in \mathcal{V}$ is labeled by $S_v \in \mathcal{S}$ and*

- *each vertex of G is covered by some S_v: $\bigcup_{v\in\mathcal{V}} S_v = V$;*
- *each edge of G is covered by some S_v: for any $(i, j) \in E$ there exists a $v \in \mathcal{V}$ such that $i \in S_v$ and $j \in S_v$.*
- *if $i \in S_v$ and $i \in S_{v'}$ for some $i \in V$ and some $v, v' \in \mathcal{V}$, then $i \in S_{v''}$ for all v'' that appear on the (unique) path from v to v' in T.*

The treewidth of a tree decomposition *$(\mathcal{S}, T)$ is* $\max_{v\in\mathcal{V}} |S_v| - 1$*; the* treewidth of a graph *G, denoted by* $\text{Tr}(G)$*, is the minimum treewidth over all tree decompositions of G.*

Abusing notation, we will refer to the treewidth of $\mathcal{G}_H$ as the treewidth of H. Furthermore, in what follows we refer to the vertices of the agent graph $\mathcal{G}_H$ as agents and the vertices of the tree T as nodes.

A tree decomposition $(\mathcal{S}, T)$ is called *nice* if T is rooted, and each of its nodes is of one of the following four types:

- **Leaf node:** v is a leaf of T and $|S_v| = 1$;
- **Introduce node:** v has one child x and there exists an $i \in N$ such that $S_v = S_x \cup \{i\}$;
- **Forget node:** v has one child x and there exists an $i \in N$ such that $S_v = S_x \setminus \{i\}$.
- **Merge node:** v has two children x and y and $S_v = S_x = S_y$;

It is known that any tree decomposition $(\mathcal{S}, T)$ can be efficiently transformed into a *nice* tree decomposition $(\mathcal{S}', T')$ of the same treewidth and such that $|\mathcal{V}'| = O(|\mathcal{V}|)$. Furthermore, if $\text{Tr}(G)$ is bounded by a constant, one can find a tree decomposition $(\mathcal{S}, T)$ with treewidth $\text{Tr}(G)$ in time polynomial in the size of G; observe that this implies that $|W| = \text{poly}(n)$.

Our first result is that, unlike for marginal contribution nets, for hedonic coalition nets deciding CORE MEMBERSHIP remains hard even if the underlying graph has bounded treewidth.

THEOREM 4. *Checking* CORE MEMBERSHIP *for hedonic coalition nets is coNP-complete, even if the treewidth of the agent graph is at most* 2.

PROOF. We will show that the complementary problem of deciding whether a partition is not in the core is NP-complete.

It is easy to see that this problem is in NP: one can check that a partition π is not in the core by guessing a coalition C and verifying, for each $i \in \pi$, that i prefers C to π_i.

To show NP-hardness, we reduce from PARTITION, a classic NP-complete problem [11]. An instance of PARTITION is given by a list of n integer numbers $a_1, \dots, a_n$ given in binary such that $a_1 + \dots + a_n = 2B$. It is a "yes"-instance if there exists a subset of indices $C \subseteq \{1, \dots, n\}$ such that $\sum_{i \in C} a_i = B$.

Given an instance $\mathcal{I}$ of PARTITION, we construct a hedonic coalition net H as follows. We set $N = \{1, \dots, n, n+1, n+2, n+3\}$. For $i = 1, \dots, n$, we set $R_i = \{n+1 \mapsto 1\}$. Also, we set $R_{n+1} = \{n+3 \mapsto B-1, 1 \wedge n+2 \mapsto a_1, \dots, n \wedge n+2 \mapsto a_n\}$, $R_{n+2} = \{n+1 \mapsto B+1, 1 \mapsto -a_1, \dots, n \mapsto -a_n\}$, $R_{n+3} = \emptyset$. Finally, we set $\pi = \{\{1\}, \dots, \{n\}, \{n+1, n+3\}, \{n+2\}\}$. Observe that we have $\mathrm{Tr}(\mathcal{G}_H) \le 2$; e.g., we can set $\mathcal{V} = \{v_1, \dots, v_{n+1}\}$, $\mathcal{E} = \{(v_i, v_{i+1})\}_{i=1,\dots,n}$ and $S_{v_i} = \{i, n+1, n+2\}$ for $i = 1, \dots, n$, $S_{v_{n+1}} = \{n+1, n+2, n+3\}$.

Now, $n+3$ has no incentive to deviate from π. Players $1, \dots, n$ would like to join a coalition that contains $n+1$. However, $n+1$ is only interested in a coalition with a subset C of $\{1, \dots, n\}$ if $n+2$ joins this coalition as well. Moreover, $n+1$ would prefer such a coalition to its current situation iff $\sum_{i \in C} a_i \ge B$. Now, $n+2$ is happy to form a coalition with $n+1$ and some $C \subseteq \{1, \dots, n\}$ as long as $\sum_{i \in C} a_i \le B$. Together, these observations imply that there is a successful deviation from π if and only if there exists a subset of indices $C \subseteq \{1, \dots, n\}$ such that $\sum_{i \in C} a_i = B$, i.e., if $\mathcal{I}$ is a "yes"-instance of PARTITION. □

A slight modification of the proof shows that CORE NON-EMPTINESS is also hard for hedonic nets with small treewidth.

THEOREM 5. CORE NON-EMPTINESS *for hedonic coalition nets is NP-hard, even if the treewidth of the agent graph is at most 4.*

PROOF. Modify the construction in the proof of Theorem 4 by introducing a player $n+4$ with

$$R_{n+4} = \{ \quad n+1 \wedge \neg(n+3) \wedge \neg(n+2) \mapsto 2, \\ n+3 \wedge \neg(n+1) \wedge \neg(n+2) \mapsto 1 \quad \}$$

and modifying the rule sets R_{n+1}, R_{n+2} and R_{n+3} by setting

$$R_{n+1} = \{ \quad n+3 \wedge \neg(n+4) \wedge \neg(n+2) \mapsto B - 1/2, \\ n+4 \wedge \neg(n+3) \wedge \neg(n+2) \mapsto B - 2/3, \\ 1 \wedge n+2 \mapsto a_1, \dots, n \wedge n+2 \mapsto a_n \quad \},$$

$$R_{n+2} = \{n+1 \mapsto B + 1/2, 1 \mapsto -a_1, \dots, n \mapsto -a_n\},$$

and

$$R_{n+3} = \{ \quad n+4 \wedge \neg(n+3) \wedge \neg(n+2) \mapsto 2, \\ n+1 \wedge \neg(n+4) \wedge \neg(n+2) \mapsto 1 \quad \},$$

(clearly, all values can be scaled up by 6 to preserve integrality). It is not hard to check that the treewidth of this net is at most 3.

Now, suppose that $\mathcal{I}$ is a "yes"-instance of PARTITION and let C be such that $\sum_{i \in C} a_i = B$. Then the coalition partition $\pi = \{\{n+3, n+4\}, C \cup \{n+1, n+2\}, \{i\}_{i \in \{1,\dots,n\} \setminus C}\}$ is stable. Indeed, $n+1$ does not want to deviate from π: as argued above, $n+1$ cannot obtain more than B in any coalition containing $n+2$, and obtains at most $B - 1/2$ in any coalition not containing $n+2$, while it gets B in π. The only profitable deviation for $n+4$ involves $n+1$, so $n+4$ does not want to deviate either; the same is true for $n+2$ and for all $i \in \{1, \dots, n\} \setminus C$. Finally, the players in C as well as $n+3$ are getting their maximal utility.

Conversely, suppose that $\mathcal{I}$ is a "no"-instance of PARTITION and consider an arbitrary partition π. Suppose for contradiction that π is in the core. If π_{n+1} does not contain $n+2$, $n+3$ or $n+4$, player $n+1$ gets 0, so he prefers to form $\{n+1, n+4\}$, which $n+4$ also prefers to its current situation.

If $\pi_{n+1} = \pi_{n+2}$, then, since for any $C \subseteq \{1, \dots, n\}$ we have $\sum a_i \neq B$, π_{n+1} either has value at most $B-1$ for $n+1$, or has a negative value for $n+2$. As the latter is impossible, it follows that $n+1$ would prefer $\{n+1, n+4\}$ to π_{n+1}, and $n+4$ also prefers $\{n+1, n+4\}$ to π_{n+4} (as either $n+1$ is not in π_{n+4}, or both $n+1$ and $n+2$ are in π_{n+4}). Hence, any such coalition is unstable. So, from now on we can assume that $n+2 \notin \pi_{n+1}$.

If π_{n+1} contains $n+3$ but not $n+4$, $n+4$ has value 0, so he prefers to form a coalition with $n+3$, which is also the most preferred outcome for $n+3$. Similarly, if π_{n+1} contains $n+4$ but not $n+3$, $n+3$ has value 0, so he prefers to form $\{n+1, n+3\}$, which $n+1$ also prefers to his current situation (in which he gets $B - 2/3$). Finally, if $n+1$, $n+3$ and $n+4$ appear in the same coalition then $n+3$ and $n+4$ have value 0, and would prefer to deviate by forming $\{n+3, n+4\}$. □

Observe that Theorems 4 and 5 are based on a reduction from PARTITION and therefore rely on the RHSs of the rules being given in binary. We will now show that if we stipulate that all RHSs are given in unary (or, alternatively, are known to be at most polynomial in n) *and* the agent graph has bounded treewidth, CORE MEMBERSHIP can be solved in polynomial time.

THEOREM 6. *There is an algorithm that, given a hedonic net $H = \langle N, R_1, \dots, R_n \rangle$, and a partition π, correctly decides whether π is in the core of G_H and runs in time $\mathrm{poly}(n)2^s(2mM+1)^{2s}$, where $M = \max\{|\beta| \mid (\varphi, \beta) \in R_i, i = 1, \dots, n\}$, $m = |R_1| + \dots + |R_n|$, and $s = \mathrm{Tr}(\mathcal{G}_H)$.*

PROOF. We will work with a nice tree decomposition $(\mathcal{S}, T)$ of $\mathcal{G}_H$ such that $|\mathcal{V}| = \mathrm{poly}(n)$. For every node $v \in \mathcal{V}$, let T_v be the subtree of T that is rooted at v, and let I_v be the set of *inactive* agents at v, i.e., set $I_v = (\cup_{v' \in T_v} S_{v'}) \setminus S_v$. Observe that by definition the agents in I_v do not appear in any nodes outside of T_v. Recall that $u_i(C)$ denotes the utility that agent i assigns to a coalition C. We will say that a vector $\mathbf{z} = (z_1, \dots, z_n)$ is *acceptable* for C and write $\mathbf{z} \in a(C)$ if $z_j \in \{-mM, \dots, mM\}$ for $j \in C$ and $z_j = 0$ for $j \notin C$.

Now, for any $v \in \mathcal{V}$, any $C' \subseteq S_v$, and any $\mathbf{z} \in a(C')$, let $X(v, C', \mathbf{z}) = 1$ if there is a $C'' \subseteq I_v$ such that for $C = C' \cup C''$ we have (a) for all $j \in C'$ it holds that $u_j(C) = z_j$ and (b) for all $j \in C''$ it holds that $u_j(C) > u_j(\pi_j)$, and let $X(v, C', \mathbf{z}) = 0$ otherwise. In what follows, we will show how to compute all values of $X(v, C', \mathbf{z})$ inductively. Note that for any $v \in \mathcal{V}$ and any $C' \subseteq S_v$, the vectors in $a(C')$ have at most s non-zero coordinates. Hence, the values $X(v, C', \mathbf{z})$ are only defined for at most $\mathrm{poly}(n)2^s(2mM+1)^s$ triples of the form $(v, C', \mathbf{z})$.

Observe that if $X(v, C', \mathbf{z}) = 1$ for some $\mathbf{z}$ such that $z_j > u_j(\pi_j)$ for all $j \in C'$, then the corresponding coalition C can successfully deviate from π. Conversely, suppose that there is a coalition C that can successfully deviate from π, i.e., $C \succ_j \pi_j$ for all $j \in C$. We can assume that the set $\mathcal{V}' = \{v \mid S_v \cap C \neq \emptyset\}$ is connected in T. Indeed, if it is not, we can take a connected component $\mathcal{V}''$ of $\mathcal{V}'$ and replace C with $\bar{C} = C \cap (\cup_{v \in \mathcal{V}''} S_v)$ in our reasoning: as the agents in $\bar{C}$ are indifferent to being in the same coalition with agents in $C \setminus \bar{C}$ (there is no rule that involves both an agent in $\bar{C}$ and an agent in $C \setminus \bar{C}$), we have $\bar{C} \succ_j \pi_j$

for all $j \in \bar{C}$. Now, let v be the least common ancestor of all nodes in $\mathcal{V}'$; as $\mathcal{V}'$ is connected, $v \in \mathcal{V}'$. Let $C' = C \cap S_v$, and let $C'' = C \setminus C'$. By construction, we have $C'' \subseteq I_v$. Let $\mathbf{z}$ be given by $z_j = u_j(C)$ for $j \in C'$, $z_j = 0$ for $j \in N \setminus C'$. We have $X(v, C', \mathbf{z}) = 1$ and $z_j > u_j(\pi_j)$ for all $j \in C'$. We conclude that to check whether π is stable it suffices to compute $X(v, C', \mathbf{z})$ for all $v \in \mathcal{V}$, $C' \subseteq S_v$, and all $\mathbf{z} \in a(C')$, and check if there exists a triple $(v, C', \mathbf{z})$ such that $X(v, C', \mathbf{z}) = 1$ and $z_j > u_j(\pi_j)$ for all $j \in C'$. Moreover, the successful deviation itself (i.e., the coalition C) can then be computed in polynomial time using standard dynamic programming techniques. We will now show how to compute $X(v, C', \mathbf{z})$ for a given vertex $v \in \mathcal{V}$, all $C' \subseteq S_v$ and all $\mathbf{z} \in a(C')$, given the values of X for the children of v. We have to consider 4 cases:

- **v is a leaf node:** Suppose $S_v = \{i\}$. Then $X(v, \{i\}, \mathbf{z}) = 1$ if $\mathbf{z} \in a(\{i\})$ and $z_i = u_i(\{i\})$, and $X(v, \{i\}, \mathbf{z}) = 0$ for all $\mathbf{z} \in a(\{i\})$ such that $z_i \neq u_i(\{i\})$.

- **v is an introduce node:** Suppose that v has one child x and $S_v = S_x \cup \{i\}$. Observe that $I_v = I_x$. Consider some $C' \subseteq S_v$. If $i \notin C'$, we obviously have $X(v, C', \mathbf{z}) = X(x, C', \mathbf{z})$. Now, suppose that $i \in C'$. Set $\hat{C}' = C' \setminus \{i\}$. Consider a vector $\hat{\mathbf{z}} \in a(\hat{C}')$ such that $X(x, \hat{C}', \hat{\mathbf{z}}) = 1$. This means that there exists a $C'' \subseteq I_x$ such that $u_j(\hat{C}' \cup C'') = z_j$ for all $j \in \hat{C}'$ and $u_j(\hat{C}' \cup C'') > u_j(\pi_j)$ for any $j \in C''$. Since i is indifferent about being in the same coalition with players from C'' and vice versa, we have $u_i(C' \cup C'') = u_i(C')$ and $u_j(C' \cup C'') > u_j(\pi_j)$ for any $j \in C''$. Finally, consider a $j \in \hat{C}'$. For any rule $(\varphi, \beta) \in R_j$ affected by i, i.e., such that $\hat{C}' \cup C'' \models \varphi$ and $C' \cup C'' \not\models \varphi$ or, alternatively, $\hat{C}' \cup C'' \not\models \varphi$ and $C' \cup C'' \models \varphi$, it has to be the case that i appears in φ and hence the agents from C'' do not. Hence, the change in j's utility from adding i, i.e., $u_j(C' \cup C'') - u_j(\hat{C}' \cup C'')$ is equal to $u_j(C') - u_j(\hat{C}')$, i.e., is independent of C''. We conclude that we have $X(v, C', \mathbf{z}) = 1$ for $\mathbf{z}$ given by $z_i = u_i(C')$, $z_j = \hat{z}_j + (u_j(C') - u_j(\hat{C}'))$ for $j \in \hat{C}'$.

  By a similar argument, if $X(v, C', \mathbf{z}) = 1$ for some $\mathbf{z} \in a(C')$ and $i \in C'$, it has to be the case that $z_i = u_i(C')$ and $X(x, \hat{C}', \hat{\mathbf{z}}) = 1$, where $\hat{C}' = C' \setminus \{i\}$ and $\hat{\mathbf{z}} \in a(\hat{C}')$ is given by $\hat{z}_i = u_i(C')$, $\hat{z}_j = z_j - (u_j(C') - u_j(\hat{C}'))$ for $j \in \hat{C}'$. Hence, given an introduce node v, we can compute all $X(v, C', \mathbf{z})$, $C' \subseteq S_v$, $\mathbf{z} \in a(C')$ given the values $X(x, C', \mathbf{z})$ for its child x in time $2^s(2mM + 1)^s$.

- **v is a forget node:** Suppose that v has one child x and $S_v = S_x \setminus \{i\}$. Observe that $I_v = I_x \cup \{i\}$. Now, it is easy to see that $X(v, C', \mathbf{z}) = 1$ iff $X(x, C', \mathbf{z}) = 1$ or $X(x, C' \cup \{i\}, \hat{\mathbf{z}}) = 1$ for some $\hat{\mathbf{z}} \in a(C' \cup \{i\})$ such that $\hat{z}_j = z_j$ for $j \in C'$ and $\hat{z}_i > u_i(\pi_i)$. Hence, we can evaluate $X(v, C', \mathbf{z})$ for all $C' \subseteq S_v$ and all $\mathbf{z} \in a(C')$ in time $O(2^s(2mM + 1)^s)$.

- **v is a merge node:** Suppose that v has two children x and y, and $S_v = S_x = S_y$. Observe that $I_v = I_x \cup I_y$.

  Suppose that we have $X(x, C', \hat{\mathbf{z}}) = 1$ and $X(y, C', \check{\mathbf{z}}) = 1$ for some $C' \subseteq S_x = S_y = S_v$ and $\hat{\mathbf{z}}, \check{\mathbf{z}} \in a(C')$. Let $\hat{C}'' \subseteq I_x$ and $\check{C}'' \subseteq I_y$ be the corresponding sets of inactive agents, i.e., for $\hat{C} = C' \cup \hat{C}''$ and $\check{C} = C' \cup \check{C}''$ we have $u_j(\hat{C}) = \hat{z}_j$ and $u_j(\check{C}) = \check{z}_j$ for $j \in C'$, $u_j(\hat{C}) > u_j(\pi_j)$ for $j \in \hat{C}''$, $u_j(\check{C}) > u_j(\pi_j)$ for $j \in \check{C}''$. Consider the coalition $C = C' \cup \hat{C}'' \cup \check{C}''$. For any player $j \in \hat{C}'' \cup \check{C}''$ we have $u_j(C) > u_j(\pi_j)$, because players in $\hat{C}''$ are indifferent about the presence of agents in $\check{C}''$ and vice versa. Now consider a player $j \in C'$. Our goal is to evaluate $u_j(C)$. Fix a rule $(\varphi, \beta) \in R_j$; we will compute its contribution to $u_j(C)$. There are four cases to be considered.

  First, suppose that $\hat{C} \models \varphi$, $\check{C} \models \varphi$. As elements of $\hat{C}''$ and $\check{C}''$ cannot both appear in φ, it has to be the case that $C' \models \varphi$ and also $C \models \varphi$: indeed, if, for example, φ does not contain elements of $\hat{C}''$, then $\hat{C} \models \varphi$ implies $C' \models \varphi$, and $\check{C} \models \varphi$ implies $C \models \varphi$. Second, suppose that $\hat{C} \models \varphi$, $\check{C} \not\models \varphi$. If φ contains elements of $\hat{C}''$, but not $\check{C}''$, it holds that $C \models \varphi$, $C' \not\models \varphi$. Otherwise, φ contains elements of $\check{C}''$, but not $\hat{C}''$. Then $C' \models \varphi$, $C \not\models \varphi$. The third case, namely, $\hat{C} \not\models \varphi$, $\check{C} \models \varphi$ is similar to the previous one. Finally, suppose $\hat{C} \not\models \varphi$, $\check{C} \not\models \varphi$. Similarly to the first case, we can conclude that $C' \not\models \varphi$, $C \not\models \varphi$.

  Summing over all rules in R_j, we obtain

  $$u_j(\hat{C}) + u_j(\check{C}) = u_j(C') + u_j(C).$$

  We conclude that we have $X(v, C', \mathbf{z}) = 1$ for $\mathbf{z} \in a(C')$ given by $z_j = \hat{z}_j + \check{z}_j - u_j(C')$ for all $j \in C'$

  Conversely, suppose that $X(v, C', \mathbf{z}) = 1$ for some $\mathbf{z} \in a(C')$. Let $C'' \subseteq I_v$ be a corresponding set of inactive players, i.e., for $C = C' \cup C''$ it holds that $u_j(C) = z_j$ for $j \in C'$, $u_j(C) > u_j(\pi_j)$ for $j \in C''$. Set $\hat{C}'' = C'' \cap I_x$, $\check{C}'' = C'' \cap I_y$. For $\hat{C} = C' \cup \hat{C}''$ and $\check{C} = C' \cup \check{C}''$ we have $u_j(\hat{C}) > u_j(\pi_j)$ for $j \in \hat{C}''$ and $u_j(\check{C}) > u_j(\pi_j)$ for $j \in \check{C}''$. Also, by the argument above we have $u_j(\hat{C}) + u_j(\check{C}) = u_j(C') + u_j(C)$. Hence, it must be the case that $X(x, C', \hat{\mathbf{z}}) = 1$, $X(y, C', \check{\mathbf{z}})$ for some $\hat{\mathbf{z}}, \check{\mathbf{z}} \in a(C')$ that satisfy $\hat{z}_j + \check{z}_j = z_j + u_j(C')$ for all $j \in C'$.

  We conclude that we can compute $X(v, C', \mathbf{z})$ for a merge node $v \in \mathcal{V}$, $C' \in S_v$ and all $\mathbf{z} \in a(C')$ by considering all pairs of vectors $\hat{\mathbf{z}}$, $\check{\mathbf{z}}$ such that $X(x, C', \hat{\mathbf{z}}) = 1$, $X(y, C', \check{\mathbf{z}}) = 1$ and setting $X(v, C', \mathbf{z}) = 1$ if $z_j = \hat{z}_j + \check{z}_j - u_j(C')$ for some such $\hat{\mathbf{z}}, \check{\mathbf{z}}$. This can be done in time $O((2mM + 1)^{2s})$; therefore, computing $X(v, C', \mathbf{z})$ for a fixed merge node v and all $C' \subseteq S_v$, $\mathbf{z} \in a(C')$ takes $O(2^s(2mM + 1)^{2s})$ steps.

We have argued that for each $v \in \mathcal{V}$ the quantities $X(v, C, \mathbf{z})$ can be computed in time $O(2^s(2mM + 1)^{2s})$. Hence, the running time of our algorithm meets the stated bound. □

The algorithm presented above can be adapted to check if a given outcome is in the core of a TU game with bounded treewidth in polynomial time (the resulting algorithm will be similar, though not identical to that of [12]). Intuitively, the reason why in the bounded treewidth setting there exists a polynomial-time algorithm for TU games, but for hedonic games the best we can do is pseudopolynomial time is as follows: In TU games, the members of each coalition can distribute the payoff between themselves in an arbitrary way, so we only have to keep track of one number: the largest amount the players C' can obtain by forming a coalition with inactive players in their subtree and keeping those players happy. In hedonic games, there are no transfers between the players in C', so we have to keep track of the entire Pareto frontier, which is potentially linear in the maximal individual payoff.

Core Characterisation: We know that the core can be empty in hedonic games, and moreover that it can contain multiple coalition structures; in fact, it is easy to see that it in the worst case, the core contains every possible coalition structure. It is thus interesting to ask whether there is a compact way of classifying the core, or at least the coalitions in the core of which a particular player is a member [10]. One approach is to classify these coalitions as a formula $\varphi \in \mathcal{L}_N$ of propositional logic over the set of players N. Formally, given a hedonic net H, a player $i \in N$, and a formula $\varphi \in \mathcal{L}_N$, the decision problem CORE CONTAINMENT is the problem of deciding whether $\forall \pi \in core(G_H) : \pi_i \models \varphi$.

THEOREM 7. CORE CONTAINMENT *is* Π_2^p*-complete.*

PROOF. Expanding out, our aim is to check whether

$$\forall \pi \in \Pi : (\forall C \subseteq N : [\forall i \in C : \pi_i \succeq_i C]) \rightarrow \pi_i \models \varphi.$$

We work with the complement problem, i.e., the problem of checking whether $\exists \pi \in \Pi : (\forall C \subseteq N : [\forall i \in C : \pi_i \succeq_i C]) \wedge \pi_i \not\models \varphi$. Clearly the problem is in Σ_2^p. For hardness, we reduce the problem of checking core non-emptiness for hedonic nets, which from [13, Theorem 6.3] is Σ_2^p-complete. Given a hedonic net H over players N, let i be any arbitrary player and set $\varphi = \bot$. Then we will be checking whether $\exists \pi \in \Pi :(\forall C \subseteq N : [\forall i \in C : \pi_i \succeq_i C]) \wedge \pi_i \models \top$, which reduces to $\exists \pi \in \Pi(\forall C \subseteq N : [\forall i \in C : \pi_i \succeq_i C])$, which is exactly the question of whether the core is non-empty. □

The converse of this problem is rather more delicate: we are given hedonic net H, player $j \in N$, and formula $\varphi \in \mathcal{L}_N$, and asked whether $\forall C \subseteq N : C \models \varphi \rightarrow \exists \pi \in core(G_H) : C = \pi_i$. However, the final equality makes this version of the problem rather strong, and so we consider the following, weaker version of the problem: we are given H, i, φ and asked whether $\forall C \subseteq N : C \models \varphi \rightarrow \exists \pi \in core(G_H) : C \subseteq \pi_i$. Referring to this problem as CORE CHARACTERISATION, we have:

THEOREM 8. CORE CHARACTERISATION *is* Σ_2^p*-hard.*

PROOF. We again reduce CORE NON-EMPTINESS for hedonic nets. Given a hedonic net H, pick an arbitrary i, and define $\varphi = (i \wedge_{j \in N, j \neq i} \neg j)$. Notice that φ has exactly one satisfying assignment, *viz* $\{i\}$. Now, CORE CHARACTERISATION is $\forall C \subseteq N : C \models \varphi \rightarrow \exists \pi \in core(G_H) : C \subseteq \pi_i$, or, equivalently, if $\exists \pi \in core(G_H) : \{i\} \subseteq \pi_i$; this is true iff the core is non-empty. □

6. CONCLUSIONS AND FUTURE WORK

We have introduced hedonic coalition nets, a complete, rule-based representation scheme for hedonic games. We have compared hedonic nets to other representation formalisms for hedonic games, and showed that they provide a good tradeoff between expressivity and succinctness. We then characterised the computational complexity of classic solution concepts, such as the core or the Nash stable set, under this representation, and identified a natural special class of games that admits a polynomial-time algorithm for CORE MEMBERSHIP. Our results show that TU games and hedonic games can be very different from computational perspective: while in the former, bounded treewidth implies existence of a polynomial-time algorithm for CORE MEMBERSHIP, in hedonic games one needs both bounded treewidth and a polynomial bound on coalitional values to solve CORE MEMBERSHIP in polynomial time.

Given our results on the complexity of CORE MEMBERSHIP for hedonic nets with bounded treewidth, it is natural to ask if our approach can be extended to decide CORE NON-EMPTINESS in this setting in polynomial time. Indeed, Ieong and Shoham [12] develop a polynomial-time algorithm for CORE NON-EMPTINESS in TU games of bounded treewidth, using their algorithm for CORE MEMBERSHIP as a separation oracle for the corresponding linear program. In our setting this approach does not seem to work, as now the outcomes are partitions rather than vectors, and it is not clear if one can reduce CORE NON-EMPTINESS to solving a linear program. The complexity of CORE NON-EMPTINESS for hedonic games with bounded treewidth remains an interesting open problem.

Complexity results for various solution concepts in hedonic games have been obtained for a number of representations, such as the IRCL representation [1], anonymous preferences [1], $\mathcal{B}$- and $\mathcal{W}$-preferences [5, 6], additively separable preferences [15, 14], as well as some special classes of additively separable preferences [8]. These results indicate that many of the problems considered in the paper become easier if the players' preferences have a special structure: for instance, checking non-emptiness of the core is NP-complete for anonymous preferences, whereas we have seen that for hedonic nets the problem is Σ_2^p-complete. Therefore, it would be interesting to investigate the complexity of checking that a given player's preferences, represented as a hedonic net, fit into one of the preference classes described in Section 3.1.

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