k-Coalitional Cooperative Games

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ABSTRACT

In most previous models of coalition structure generation, it is assumed that agents may partition themselves into any coalition structure. In practice, however, there may be physical and organizational constraints that limit the number of co-existing coalitions. In this paper, we introduce k-coalitional games: a type of partition function game especially designed to model such situations. We propose an extension of the Shapley value for these games, and study its axiomatic and computational properties. In particular, we show that, under some conditions, it can be computed in polynomial time given two existing representations of coalitional games with externalities. Finally, we use k-coalitional games to analyse the relative importance of geographical locations in the game of Diplomacy.

1. INTRODUCTION

Coalitional games have been widely studied in both the game theory and computer science literature [2, 15]. One of the most important problems in coalitional games is that of *coalition structure formation*: partitioning a group of agents into disjoint teams, typically with the goal of maximising some social welfare measure. In most studies of coalition structure generation, it is assumed that *all cooperative arrangements are feasible*, and in particular, that agents may be partitioned into *any* coalition structure. In characteristic function games – the most-studied model of cooperative games – it is further assumed that no coalition influences the value of any other co-existing coalition. This latter assumption is dropped in the more general model of partition function games [32], where the value of any coalition may be subject to *externalities* and, therefore, depend on co-existing coalitions.

Despite the popularity of the above models, it has long been recognized that many potential applications of coalitional games impose restrictions of various kinds on the coalitions and coalition structures that may be formed. This consideration gave rise to formalisms such as graph-restricted games [20], which have been studied by various authors in artificial intelligence (Meir et al. [17], Voice et al. [34], Skibski et al. [28]). Another model was proposed by Rahwan et al. [23], where restrictions on feasible coalitions are defined by logical constraints, while Meir et al. [17] consider games with coalitions of restricted size.

A common feature of all the above models is that they focus on restricting only feasible *coalitions*, rather than *coalition structures*, which are restricted only indirectly, (*i.e.*, by the fact that they cannot contain infeasible coalitions). However, one can see many physical and organizational restrictions that place constraints on feasible coalition structures rather than coalitions. For instance, during the Cold War, any country in Europe belonged either to NATO or the Warsaw Pact, or else remained neutral. Thus, if we consider all neutral countries as a coalition, then any feasible coalition structure would have at most three coalitions. Similar restrictions are likely to occur in a multi-agent system, where, due to cost considerations, only a few agents may be sophisticated enough to play the role of coalition coordinators/leaders (see the work by Coviello and Franceschetti [3] for an analysis of the problem of assigning followers to leaders). Thus, any coalition is feasible, but no coalition structure may contain more coalitions than there are leaders in the system. The only model that we are aware of in which restrictions are placed on the *number* of coalitions in a coalition structure is due to Sless et al. [31], who consider symmetric, additively separable hedonic games over a network with the assumption that any coalition structure should contain *exactly* k coalitions. Furthermore, an implicit assumption that limits the number of coalitions in a coalition structure can be found in the literature on well-known assignment games [25, 21, 13], where, for instance, m workers are hired by n companies [9]. All these models, however, focus on some very specific forms of the characteristic function and no more general transferable-utility model has been developed so far.

Against this background, we introduce *k-coalitional cooperative games* – a subclass of partition function games that are intended to model environments with the characteristics described above. In particular, we assume that any coalition may form, but the number of coalitions in any coalition structure is *limited* by a constant, k. For these new games, we develop a dedicated extension of the Shapley value [24] – a fundamental solution concept for coalitional games. We present two axiom systems that uniquely define the new value. Furthermore, we analyse the computational properties of the new value and show that it can be computed in polynomial time if a game is represented by Partition Decision Trees [29]. Finally, we use our approach to analyse the importance of geographical locations in the well-known board game of Diplomacy.

2. PRELIMINARY DEFINITIONS

Let $N = \{1, 2, \dots, n\}$ be a finite set of agents with $|N| = n$. A *coalition*, S, is any non-empty subset of N. A game without externalities (i.e., in characteristic function form) is given by a function $\hat{v}$ that associates a real number with every coalition of agents: $\hat{v} : 2^N \to \mathbb{R}$, with the assumption $\hat{v}(\emptyset) = 0$. Formally, a game is a pair $(N, \hat{v})$, but we frequently just write $\hat{v}$. We denote the set of all games without externalities by $\mathcal{CG}$.

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In a game with externalities, the value of a coalition depends not just on the members of the coalition, but also on co-existing coalitions. A *partition* of N (also known as a *coalition structure*) is a set of disjoint coalitions that collectively cover N. A pair (S, P), where P is a partition and $S \in P$, is called an *embedded coalition*. The set of all partitions is denoted by $\mathcal{P}$, and all embedded coalitions by $\mathcal{EC}$. Now, in a *game with externalities* (in partition function form) (N, v), the function v associates a real number with every embedded coalition in every partition, *i.e.*, $v : \mathcal{EC} \to \mathbb{R}$ (with $v(\emptyset, P) = 0$ for every partition $P \in \mathcal{P}$). We denote the set of all games with externalities by $\mathcal{PG}$.

We use a shorthand notation for set subtraction and set union operations: $N_{-S} = N \setminus S$ and $S_{+\{i\}} = S \cup \{i\}$. Often, we omit brackets and write S_{+i}. We denote the partition obtained by the transfer of agent i to coalition T in partition P as:

$$\tau_i^T(P) = P \setminus \{P(i), T\} \cup \{P(i)_{-i}, T_{+i}\},$$

where $P(i)$ denotes i's coalition in P. For function $f : N \to X$ and subset $S \subseteq N$ we define $f(S) = \{f(i) \mid i \in S\}$, and, in the same manner, for a set of sets $P = \{S_1, \ldots, S_m\}$, we have $f(\{S_1, \ldots, S_m\}) = \{f(S_1), \ldots, f(S_m)\}$. Also, $f^{-1} : X \to 2^N$ is an inverse function, *i.e.*, $f^{-1}(x) = \{i \in N \mid f(i) = x\}$ for $x \in X$. In particular, combining the above definitions, $f^{-1}(X)$ forms a partition of N.

The Shapley value: A *value* of a game is a vector that divides among agents the payoff of the grand coalition, N. The value of agent i in game v will be denoted $\varphi_i(v)$. In his seminal work, Shapley [24] proved that there exists a unique division scheme in games without externalities that satisfies the following four axioms:

- *Efficiency* (the entire payoff of the grand coalition is distributed): $\sum_{i \in N} \varphi_i(\hat{v}) = \hat{v}(N)$ for every $\hat{v} \in \mathcal{CG}$;
- *Symmetry* (payoffs do not depend on the agents' names): $\varphi_i(f(\hat{v})) = \varphi_{f(i)}(\hat{v})$ for every $\hat{v} \in \mathcal{CG}$ and $f : N \to N$;
- *Additivity* (the sum of payoffs in two separate games equals the payoff in the combined game): $\varphi(\hat{v}_1 + \hat{v}_2) = \varphi(\hat{v}_1) + \varphi(\hat{v}_2)$ for all $\hat{v}_1, \hat{v}_2 \in \mathcal{CG}$;
- *Null-player Axiom* (agents that make no contribution to any coalition receive nothing): if $\hat{v}(S) - \hat{v}(S \setminus \{i\}) = 0$ for every $S \subseteq N, i \in S$, then $\varphi_i(\hat{v}) = 0$, for every $\hat{v} \in \mathcal{CG}$;

Here, games $f(\hat{v})$ (for bijection f) and $\hat{v}_1 + \hat{v}_2$ are defined as follows: $f(\hat{v})(S) = \hat{v}(f(S))$, and $(\hat{v}_1 + \hat{v}_2)(S) = \hat{v}_1(S) + \hat{v}_2(S)$. This unique solution is known as *the Shapley value*:

$$SV_i(\hat{v}) = \sum_{S \subseteq N, i \notin S} \beta(S, N) \cdot (\hat{v}(S \cup \{i\}) - \hat{v}(S)). \qquad (1)$$

where $\beta(S, N) = (|S|!(|N| - |S| - 1)!)/(|N|!)$. As an intuition, Shapley provided the following process that leads to his value. Assume that the agents enter the game in a random order with an aim to form the grand coalition. As agent i enters, he receives a payoff that equals his marginal contribution to the group of agents that she joins: $[mc_i(v)](S) = \hat{v}(S) - \hat{v}(S \setminus \{i\})$. The Shapley value is the expected value of agents' contributions over all possible orders.

To formalize this description, we need additional notation. A set of all permutations (orders) of N is denoted by Π. For a given permutation π, the set of agents that appear in π before i is denoted A_i^π, and after Z_i^π. If we include i in these sets, we write A_{i+}^π and Z_{i+}^π. Now, the Shapley value is given by the following formula:

$$SV_i(\hat{v}) = \frac{1}{|N|!} \sum_{\pi \in \Pi} \hat{v}(A_{i+}^\pi) - \hat{v}(A_i^\pi).$$

Extended Shapley values: Translating Shapley's axioms to games with externalities is problematic. While Efficiency, Symmetry, and Additivity can be easily adapted, the Null-player Axiom poses a problem – how should the contribution of an agent to an embedded coalition be defined? In games without externalities, this is the difference between the value of a coalition with and without an agent. But in games with externalities, when agent leaves coalition $S \cup \{i\}$ in partition $\{S \cup \{i\}, T_1, \ldots, T_k\}$, the effect of his move depends on what other coalition he joins, as the value of S when i joins T_1 may differ than the one when i joins T_2. A change associated with the transfer of agent i from coalition $S \cup \{i\}$ that results in partition P is denoted by

$$[emc_i(v)](S, P) = v(S_{+i}, \tau_i^S(P)) - v(S, P),$$

and called the *elementary marginal contribution*.

In the standard, most strict definition of a null-player, we assume that *every* transfer does not change the value of a coalition, *i.e.*, every elementary marginal contributions has value zero:

- *Null-player Axiom (for $\mathcal{PG}$)*: if $[emc_i(v)](S, P) = 0$ for every $(S, P) \in \mathcal{EC}, i \notin S$, then $\varphi_i(v) = 0$, for every $v \in \mathcal{PG}$;

Unfortunately, the (Strict) Null-player Axiom combined with the three other axioms (the so called *standard translation*) is too weak to imply uniqueness. To overcome this problem, a number of singular extensions and two more general approaches were proposed. In the first, (proposed by Macho-Stadler et al. [14] and called the *average approach*), the unique value of each coalition is calculated as a weighted average of its values in various partitions. Then, the Shapley value is applied to the resulting game without externalities:

$$\varphi_i(v) = SV_i(\hat{v}), \text{ where } \hat{v}(S) = \textstyle\sum_{(S,P)} a(S, P)v(S, P), \qquad (2)$$

for some weights a.

In the *marginality approach*, used by a number of researchers the marginal contribution of an agent i is defined as a weighted average over elementary marginal contributions associated with leaving a given embedded coalition (S, P). Specifically, $[mc_i(v)](S, P)$ is defined as

$$\sum_{T \in P \setminus S \cup \{\emptyset\}} \alpha_i(S_{-i}, \tau_i^T(P)) \cdot [emc_i(v)](S_{-i}, \tau_i^T(P)), \qquad (3)$$

for some weights α. Here, the empty set in the sum corresponds to the creation of a new coalition. Now, agent i is an α-null-player if all his marginal contributions equal zero:

- *α-Null-player Axiom (for $\mathcal{PG}$)*: if $[mc_i(v)](S, P) = 0$ for every $(S, P) \in \mathcal{EC}, i \in S$, then $\varphi_i(v) = 0$, for every $v \in \mathcal{PG}$;

Skibski et al. [30] proved that for every α, Efficiency, Symmetry, Additivity, and the α-Null-player Axiom implies a unique value.

3. K-COALITIONAL GAMES

In this section, we formally define k-coalitional games. Let $\mathbf{K} = \{1, 2, \ldots, k\}$, where $k \leq |N|$. The *size*, $|P|$, of a partition P is simply the number of coalitions it contains. We denote the set of all partitions of size at most k by $\mathcal{P}_k$, and the set of all coalitions embedded in them by $\mathcal{EC}_k$. Now, a *k-coalitional game* is a pair (N, v), where v is a value function $v : \mathcal{EC}_k \to \mathbb{R}$ that associates a real number with every element of $\mathcal{EC}_k$. We denote the set of all k-coalitional games by $\mathcal{PG}_k$.

Let us now consider whether the existing extensions of the Shapley value to games with externalities could be applied to k-coalitional games.[1] Of course, as long as the domains are differ-

[1]Note also that there exist extensions of the Shapley value to games with restricted space of coalitions (see, e.g., [4]) but not restricted space of coalition structures.

ent (i.e., $\mathcal{EC}_k$ vs. $\mathcal{EC}$) the answer is no. Next, consider the following procedure. Let $v^\star$ be a partition-function form game created from $v \in \mathcal{PG}_k$ by assigning zero values to all coalitions embedded in partitions with more than k coalitions: $v^\star(S,P) = v(S,P)$ if $|P| \leq k$ and $v^\star(S,P) = 0$, otherwise. Now, mathematically, every value for partition-function form games can be computed. While at first such an indirect procedure seems appealing, it does not satisfy the Null-player Axiom translated (in a standard way) to k-coalitional games:

- $\mathcal{EC}_k$*-restricted Null-player Axiom* (agents with no effect on the value of any coalition should get nothing): if $[emc_i(v)](S,P) = 0$ for every embedded coalition $(S,P) \in \mathcal{EC}_k$ such that $i \notin S$, then $\varphi_i(v) = 0$, for every $v \in \mathcal{PG}_k$.

In particular, an agent who in game $v \in \mathcal{PG}_k$ does not affect any coalition value (with all elementary marginal contributions zero) may affect it in $v^\star$ by forming a $k+1$'th coalition. Hence, existing solutions for games with externalities – even those satisfying the standard Null-player Axiom – would assign non-zero payoff to such an agent and violate the $\mathcal{EC}_k$-restricted Null-player Axiom:[2]

PROPOSITION 1. *Values proposed by Bolger [1], Pham Do and Norde [22], Hu and Yang [6], Macho-Stadler et al. [14] and Myerson [19] violate the* $\mathcal{EC}_k$*-restricted Null-player Axiom.*

PROOF. Let $(S,P) \in \mathcal{EC}_k$ be an embedded coalition such that $i \in S$ and $|P| = k$. We define $v \in \mathcal{PG}_k$ as follows: $v(S,P) = 1 = v(S_{-i}, \tau_i^T(P))$ for every $T \in P$, and $v(S',P') = 0$, otherwise. Thus, agent i is a null-player in v. Consider an extension $v^\star$ of game v to general games with externalities. Now, $[emc_i(v)](S_{-i}, \tau_i^\emptyset(P)) = 1$. As the Hu and Yang, Macho-Stadler et al. and the Myerson value take $[emc_i(v)](S_{-i}, \tau_i^\emptyset(P))$ into account, player i will have non-zero payoff. Regarding Pham Do and Norde, the same argument applies if and only if $P \setminus S$ is the set of singletons. □

Given this result, in the next section, we propose a dedicated extension of the Shapley value to k-coalitional games.

4. K-COALITIONAL SHAPLEY VALUE

We begin by formulating a process similar to the one proposed by Shapley [24], which yields the value in expectation. To this end, assume that agents leave (rather than enter) the grand coalition in a random order through one of the $k-1$ exits/doors. We assume that agents that left through the same exit form a coalition; thus, all agents are partitioned according to their selected exits. As agent i leaves, he chooses each exit with the same probability (*i.e.,* $\frac{1}{k-1}$) and receives a payoff that equals his (elementary) marginal contribution to the group of agents that he left. More formally, assuming the agents in S have not yet left, other agents are partitioned into $P \setminus S$, and that i chose the same exit as coalition T, i's elementary marginal contribution equals $v(S,P) - v(S_{-i}, \tau_i^T(P))$. Now, the *$k$-coalitional Shapley value* is the expected outcome of the agent's elementary marginal contributions over all orders. Formally, this value is a function $\varphi^k : \mathcal{PG}_k \to \mathbb{R}^N$:

$$\varphi_i^k(v) = \sum_{\pi \in \Pi} \sum_{f: A_{i+}^\pi \to \mathbf{K}_{-1}} \frac{[emc_i(v)](Z_i^\pi, Z_i^\pi \cup f^{-1}(\mathbf{K}_{-1}))}{|N|!(k-1)^{|A_i^\pi|}} \qquad (4)$$

EXAMPLE 1. *Let $N = \{1,2,3\}$ be the set of agents and consider the following 2-coalitional game:*

$$v(N,\{N\}) = 4, v(S,\{S, N \setminus S\}) = |S| \text{ for every } S \subsetneq N.$$

Note that the value $v(\{1\},\{\{1\},\{2\},\{3\}\})$ is not specified, as in a 2-coalitional game 3 coalitions cannot coexist. Consider a permutation $(1,2,3)$. Since there is only one exit, as agents leave the grand coalition, they obtain marginal contributions $2,1,1$. Now, 2-coalitional Shapley value is an average over all possible permutations.

We now propose two axiomatizations of the k-coalitional Shapley value: one that follows the marginality approach (i.e., with an α-Null-player Axiom, see Preliminaries) and one with the standard translation.

Axiomatization with the α-Null-player Axiom: We begin our analysis by deriving a more concise version of formula (4).

LEMMA 1. *The k-coalitional Shapley value satisfies:*

$$\varphi_i^k(v) = \sum_{(S,P) \in \mathcal{EC}_k, i \notin S} \beta(S,N) p(P \setminus S) [emc_i(v)](S,P), \qquad (5)$$

where $p(P \setminus S) = \frac{(k-1)!}{(k-|P|)! \cdot (k-1)^{|N|-|S|}}$.

PROOF. Let us calculate the probability of a given transfer that corresponds to elementary marginal contribution $[emc_i(v)](S,P)$. Firstly, exactly agents from S must have not left. Thus, in a permutation, i has to be exactly before the agents from S and after the agents from $N \setminus (S \cup \{i\})$. This happens with the probability $\frac{(|S|)!(|N|-|S|-1)!}{|N|!} = \beta(S,N)$. Secondly, the exit of agent i should lead to partition P. Thus, the first agents (in the permutation) from each coalition in $P \setminus S$ can choose arbitrary – but different – exits (they have exactly $(k-1)\cdots(k-|P|+1)$ such choices). Then, each of the remaining agents must choose exactly the same exit as the first one from his coalition. □

COROLLARY 1. *The k-coalitional Shapley value can be obtained using formula (2) with weights $a(S,P) = p(P \setminus S)$ for $|P| \leq k$, and $a(S,P) = 0$, otherwise.*

Note that for a given S these weights are equal for all partitions P with the same number of coalitions. This property is violated by the well-known value proposed by Macho-Stadler et al. [14].

Let us formalize the notion of the marginal contribution in our value. Assume that $i \in S \in P$. Based on the process described above, the probability that agent i will join any coalition $T \in P$ equals $1/(k-1)$. Thus, the chance of creating a new coalition equals $(k-|P|)/(k-1)$. This leads to the following theorem and first axiomatization of our value.

THEOREM 1. *The k-coalitional Shapley value can be obtained using the marginality approach (Formula (3)) with weights*

$$a_i^k(S,P) = \begin{cases} 1/(k-1) & \text{if } P(i) \neq \{i\}, \\ (k-|P|)/(k-1) & \text{otherwise.} \end{cases}$$

Thus, it is the only value that satisfies Efficiency, Symmetry, Additivity, and the α^k-Null-player Axiom.

PROOF. First, we show that the k-coalitional Shapley value satisfies all four axioms. To argue that Efficiency is satisfied, consider Formula (4). Let us fix permutation $\pi \in \Pi$ and function f. Considering the sum of elementary marginal contributions: $\sum_{i \in N} [emc_i(v)](Z_i^\pi, Z_i^\pi \cup f^{-1}(\mathbf{K}_{-1}))$ we see that it sums up to $v(N,\{N,\emptyset\})$ (as the grand coalition, N, dissolves sequentially to the empty coalition, $\emptyset$, with zero value). Satisfying Additivity and Symmetry can be easily seen, as based on formula (4) $\phi(v_1+v_2) = \phi(v_1) + \phi(v_2)$, and permuting the players names in game will result in the corresponding permutation of payoffs. The fact, that the

[2]The only exception is the McQuillin value [16] which is the special case ($k=2$) of our dedicated extension.

value satisfies the α^k-Null-player Axiom follows from Lemma 1. To see this, note that if $i \notin S$, then $v(S, P)$ appears in the formula only once. On the other hand, if $i \in S$, then $v(S, P)$ appears once for each possible transfer of i from S outside with weight $(|S| - 1)!(|N| - |S|)!/|N|!$ times $(k-1)!/((k-|P|)!(k-1)^{|N|-|S|+1})$ for transfers to existing coalitions, and with weight $k-|P|$ replaced by $k - |P| + 1$ for a transfer to a new coalition. The proper reformulation leads to the following formula:

$$\varphi_i^k(v) = \sum_{\substack{(S,P)\in\mathcal{EC} \\ i\in S}} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} p(P\setminus S)[mc_i^k(v)](S,P),$$

which proves that if all marginal contributions of player i equal zero, then he gets zero payoff.

The proof of uniqueness follows from [30, Theorem 1] that states Efficiency, Symmetry, Additivity and α-Null-player Axiom imply a unique value for every α. $\square$

In this axiomatization we used a notion of α^k-null-player, where each player that had the sums in Formula (3) equal zero is called a null-player. Our second axiomatization will use only the direct translation of the standard Null-player Axiom to k-coalitional games (see section 'k-Coalitional Games').

Axiomatization with the standard Null-player Axiom: Interestingly, the probability of a transfer to an existing coalition is always the same, regardless of coalition joined/left, and even the size of the partition. We will use this observation to provide our alternative axiomatization. Apart of two special cases, where joining an existing coalition has weight zero or one (proposed by Pham Do and Norde [22] and Skibski [27], respectively), the k-coalitional Shapley value is the first value with this property proposed in the literature.

As already mentioned, the standard translation of the Null-player Axiom is too weak to imply uniqueness. Hence, we obtain uniqueness by introducing two additional axioms – *Sensitivity* and *Fixed Impact*. The former one analyses how increasing the value of one specific coalition influences the agents. The latter one is a very basic condition – we require that increasing the value of a coalition has at least a slight effect on the payoffs of its members.

- $\mathcal{EC}_k$*-restricted Sensitivity* (the value of all coalitions of an agent affects his payoffs): if $v(S, P) \neq v'(S, P)$ for some $(S, P) \in \mathcal{EC}_k$ with $i \in S$, and $v(S', P') = v(S', P')$ for $(S', P') \neq (S, P)$, then $\varphi_i(v) \neq \varphi_i(v')$, for every $v, v' \in \mathcal{PG}_k$.

Note that assuming that the increase of a value of a coalition increases the payoffs of members would lead to the strict version of coalitional monotonicity proposed by Young [35]. The above condition is, however, weaker, and hence more general.

Let us consider how the values of coalitions without player i affect his payoff. To this end, first consider a game without externalities. Now, an increase of the value of both coalitions $S \cup \{i\}$ and S does not change the vector of marginal contributions of agent i, and does not affect his payoff. In games with externalities, we argue that there should exist a constant c such that increase of (S, P) with $i \notin S$ by the value ω is balanced by an increase of $(S_{+i}, \tau_i^S(P))$ by $\omega \cdot c$.

- $\mathcal{EC}_k$*-restricted Fixed Impact* (values of coalitions without an agent affect him similarly to values of coalitions with him): there exists a constant c such that if $v'(S, P) = \omega + v(S, P)$, $v'(S_{+i}, \tau_i^S(P)) = \omega \cdot c + v(S_{+i}, \tau_i^S(P))$ for some $(S, P) \in \mathcal{EC}_k$ such that $\{i\} \notin P$ and $v'(S', P') = v(S', P')$ otherwise, then $\varphi_i(v) = \varphi_i(v')$.

Finally, we show that both new axioms coupled with the standard translation of the original Shapley axioms imply uniqueness.

THEOREM 2. *The k-coalitional Shapley value is the only value that satisfies Efficiency, Symmetry, Additivity, the $\mathcal{EC}_k$-restricted Null-player Axiom, $\mathcal{EC}_k$-restricted Sensitivity, and $\mathcal{EC}_k$-restricted Fixed Impact.*

PROOF. Theorem 1 implies that our value satisfies standard axioms – Efficiency, Symmetry, Additivity and the $\mathcal{EC}_k$-restricted Null-player Axiom. Furthermore, Lemma 1 implies that it satisfies $\mathcal{EC}_k$-restricted Sensitivity, i.e., the change in the value of a embedded coalition, (S, P), affects the corresponding elementary marginal contribution $[emc_i(v)](S, P)$ or $[emc_i(v)](S_{+i}, \tau_i^S(P))$ if $i \notin S$ which a has non-zero weight in Formula 5. $\mathcal{EC}_k$-restricted Fixed Impact is also satisfied based on the same formula for $c = \frac{1}{k-1}$.

Let us now prove uniqueness. Let φ be the value which satisfies all six axioms. An *elementary game* $e^{(S,P)}$ is a game in which only embedded coalition (S, P) has value 1, and is the only coalition with a non-zero value:

$$e^{(S,P)}(S', P') = \begin{cases} 1 & \text{if } (S', P') = (S, P), \\ 0 & \text{otherwise.} \end{cases}$$

From Additivity, we know that

$$\varphi_i(v) = \sum_{(S,P)\in\mathcal{EC}_k} \varphi_i(e^{(S,P)}).$$

Consider $\mathcal{EC}_k$-restricted Sensitivity and $\mathcal{EC}_k$-restricted Fixed Impact. $\mathcal{EC}_k$-restricted Sensitivity implies that in the formula for φ_i every embedded coalition counts:

$$\varphi_i(e^{(S,P)}) \neq 0. \quad \text{for every } (S, P) \in \mathcal{EC}_k. \tag{6}$$

Furthermore, $\mathcal{EC}_k$-restricted Fixed Impact implies that there exist a constant c such that

$$c \cdot \varphi_i(e^{(S,P)}) + \varphi_i(e^{(S_{-i},\tau_i^T(P))}) = 0 \tag{7}$$

for every $(S, P) \in \mathcal{EC}_k$ such that $i \in S$, and for every $T \in P \setminus \{S\}$. Consider an arbitrary embedded coalition (S, P) such that $i \in S$ and $|P| = k$. Note that player i is a null-player in the following game:

$$e^{(S,P)} + \sum_{T\in P\setminus\{S\}} e^{(S_{-i},\tau_i^T(P))}.$$

$T = \emptyset$ is not included in the sum, as $(S_{-i}, \tau_i^{\emptyset}(P)) \notin \mathcal{EC}_k$. Thus, $\mathcal{EC}_k$-restricted Null-player Axiom implies that

$$\varphi_i(e^{(S,P)}) = -\sum_{T\in P\setminus\{S\}} \varphi_i(e^{(S_{-i},\tau_i^T(P))}).$$

This fact combined with (7) implies that

$$\varphi_i(e^{(S,P)}) = c \cdot |P \setminus \{S\}| \cdot \varphi_i(e^{(S,P)}).$$

Finally, Formula (6) implies $c = \frac{1}{k-1}$. $\square$

From the set of all extensions of the Shapley value to games with externalities only one – the McQuillin value [16] – can be applied to our restricted setting. We characterize it using the average approach: let us define the game without externalities as follows: $\hat{v}^{McQ}(S) = v(S, \{S, N \setminus S\})$. The McQuillin value is the Shapley value for this game: $\varphi_i^{McQ}(v) = SV_i(\hat{v}^{McQ})$. We show below that McQuillin's value is a special case of the k-coalitional Shapley value.

PROPOSITION 2. *The McQuillin value is the* 2*-coalitional Shapley value.*

PROOF. Consider the k-coalitional Shapley value with $k = 2$. As we recall the interpretation of our process, we see there is only one exit and all agents leaving coalition S form one group – $N \setminus S$. Thus, the marginal contribution for every permutation coincides with the marginal contributions in the game $\hat{v}^{McQ}$; hence, both values are equal. □

k-coalitional Shapley value, although proposed for a restricted environments, can be directly applied also for an arbitrary game with externalities. It is easy to verify that the value is uniquely characterize by both axiomatization also in this environment. In the same manner, k-coalitional Shapley value can be applied to m-coalitional games with $m > k$. However, as it ignores values of coalitions embedded in partition bigger than k, it will not satisfy $\mathcal{EC}_m$-restricted Sensitivity. Finally, for n players, the n-coalitional Shapley value is uniquely characterized by non-restricted (as they are restricted to all embedded coalitions: $\mathcal{EC}_n = \mathcal{EC}$) versions of the axioms.

COROLLARY 2. *n-coalitional Shapley value is the only value that satisfies Efficiency, Symmetry, Additivity, Null-player Axiom, Sensitivity, and Fixed Impact.*

We note that only for $n = 2$ this value coincides with an existing one, i.e., with the already mentioned McQuillin's value.

5. COMPUTATIONS UNDER SPECIFIC REPRESENTATIONS

In this section, we discuss the complexity of calculating the k-coalititional Shapley value based on two representations proposed for games with externalities – *Embedded MC-Nets* [18] and *Partition Decision Trees* [29]. As we will prove, there exists an interesting connection between computing the k-coalitional Shapley value and the problem of *k-colorings in a graph*.

We start with a property of the k-coalitional Shapley value that is crucial from a computational perspective.

LEMMA 2. *The k-coalitional Shapley value satisfies the* Strong Null-player Axiom*: if an agent is a null-player (in a strict sense), then he has no impact on the payoffs of others,* i.e.*, he can be removed from the game.*

PROOF. Let player j be a null-player. For an embedded coalition (S, P) such that $j \in S$, consider the game $\tilde{v}^{(S,P)}$ defined as:

$$\tilde{v}^{(S,P)} = e^{(S,P)} + \sum_{T \in P \setminus \{S\} \cup \{\emptyset\}} e^{(S_{-j}, \tau_j^T(P))}.$$

See proof of Theorem 2 for a definition of elementary game $e^{(S,P)}$. We can easily check that agent j is a null-player in $\tilde{v}^{(S,P)}$, and the collection of games $\langle \tilde{v}^{(S,P)} \rangle_{(S,P) \in \mathcal{EC}, j \in S}$ forms a basis for games where agent j is a null-player. Now, the game $\tilde{v}^{(S,P)}$ with agent j removed simplifies to the elementary game $e^{(S_{-j}, P_{-j})}$ where only coalition (S_{-j}, P_{-j}) has a non-zero value which is 1 (P_{-j} is a partition of players over $N \setminus \{j\}$). Thus, it is enough to show that $\varphi_i(N, \tilde{v}^{(S,P)}) = \varphi_i(N_{-j}, e^{(S_{-j}, P_{-j})})$ holds for every $i \in N \setminus \{j\}$.

Assume that $i \notin S$. From Lemma 1 we have:

$$\varphi_i^k(\tilde{v}^{(S,P)}) = \frac{|S|!(|N| - |S| - 1)!}{|N|!} p(P \setminus S) + \\ + \sum_{T \in P \setminus \{S\} \cup \{\emptyset\}} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} p(\tau_j^T(P) \setminus S_{-j}),$$

for $p(P \setminus S) = \frac{(k-1)!}{(k-|P|)! \cdot (k-1)^{|N|-|S|}}$. Simple calculations give:

$$\sum_{T \in P \setminus \{S\} \cup \{\emptyset\}} p(\tau_j^T(P) \setminus S_{-j}) = \\ = \frac{(k-1)!}{(k-|P|)! \cdot (k-1)^{|N|-|S|+1}} ((|P| - 1) + (k - |P|)),$$

which equals $p(P \setminus S)$. Thus,

$$\varphi_i^k(\tilde{v}^{(S,P)}) = \\ = p(P \setminus S) \left(\frac{(|S|-1)!(|N|-|S|)!}{|N|!} + \frac{|S|!(|N|-|S|-1)!}{|N|!} \right) \\ = p(P \setminus S) \frac{(|S|-1)!(|N|-|S|-1)!}{(|N|-1)!} = \varphi_i^k(e^{(S_{-j}, P_{-j})}).$$

As payoffs of all agents not in S remain the same after removing player j, then based on efficiency and symmetry, the payoffs of agents from S also do not change. □

The importance of this lemma comes from the fact that most representations (the aim of which is to provide a concise description of the game) usually focus on modelling relationships between subsets of agents. Following Lemma 2, when calculating the value, we can limit ourselves to agents that matter for a given relationship.

The first representation that we discuss, *Embedded MC-Nets*, is an extension of the MC-Nets representation [7]. The basic building block of the original MC-Nets is a boolean expression over N of the form: $p_1 \wedge p_2 \wedge \ldots \wedge p_k \wedge \neg n_1 \wedge \neg n_2 \wedge \ldots \wedge \neg n_l$, with $p_1, p_2, \ldots p_k \in N$ being positive literals, and $n_1, n_2, \ldots, n_l \in N$ being negative literals. Coalition S satisfies a given boolean expression if it contains all agents corresponding to positive literals, and does not contain any agent corresponding to a negative literal.

Now, a single rule of Embedded MC-Nets is of the form:

$$\alpha \mid \beta_1, \beta_2, \ldots, \beta_m \to w,$$

where $w \in \mathbb{R}$, and $\alpha, \beta_1, \beta_2, \ldots, \beta_m$ are the standard MC-Nets boolean expressions. An embedded coalition (S, P) satisfies the entire rule if S satisfies α and, for every β_i, there exists a coalition $T \in P$ such that T satisfies β_i. We assume also that agents that appear in a negative part of the rules appear also somewhere in the positive ones.

THEOREM 3. *Calculating the k-coalitional Shapley value from a single Embedded MC-Nets rule is equivalent to the problem of counting $(k-1)$-colorings of a graph. Thus, it is #P-complete.*

PROOF. First, we assume that: α does not contain negative literals (each can be added as a separate boolean expression on the right-hand side), sets of positive literals in all boolean expressions $\alpha, \beta_1, \ldots, \beta_m$ do not overlap (if they do, they can be combined, or if not – for example if one of these expressions is α – the rule is self-contradictory), and that each expression is not contrary. Now, based on the Strong Null-player Axiom (Lemma 2), we assume that the game only consists of agents that appear in the rule – N. Thus, the only coalition with a non-zero value in the game described using this rule is a coalition formed by agents from positive literals from α. We will denote it by S. The value, however, is non-zero only in partitions that satisfy boolean expressions from the right-hand side.

Thus, our goal is to compute the sum of weights of all such partitions. Recall that the weight of a partition is the number of functions $f : P \setminus S \to \mathbf{K}_{-1}$ such that $f^{-1}(\mathbf{K}_{-1}) = P \setminus S$ divided by $(k-1)^{|N|-|S|}$ (and using the process interpretation, the number of ways agents can leave the grand coalition and form partition

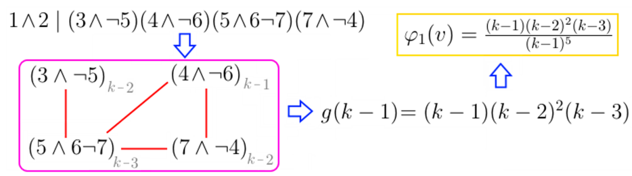

Figure 1: Computing the k-coalitional Shapley value based on an Embedded MC-Nets rule using k-coloring. Here, $g(k)$ is a chromatic polynomial.

$P \setminus S$ outside using k exits). To achieve our goal, consider a graph with m nodes that corresponds to boolean expressions $\beta_1, \ldots, \beta_m$. Now, two expressions are connected if there exists a negative literal in one of them, with agent from the other one. Thus, an edge means that these expressions cannot be merged. Now, each $(k-1)$-coloring of this graph (assigning colors from set $\{1, 2, \ldots, k-1\}$ in such a way that connected nodes have different colors) corresponds to one partition of agents that satisfies the boolean expressions, *i.e.*, one function f that meets the above criteria and the number of colorings divided by $(k-1)^{|N|-|S|}$ is equal to the sum of weights of partitions. □

The intuition behind the above proof is depicted in Figure 1. Rule $(3\wedge\neg5)(4\wedge\neg6)(5\wedge6\wedge\neg7)(7\wedge\neg4)$ can be represented by the graph (see the frame) in which every node corresponds to one boolean expression. For this graph, we calculate the chromatic polynomial for $k-1$, i.e., the number of $k-1$-colorings. This value, as argued in the proof, is equal to the k-coalitional Shapley value of player 1.

Although the above result shows that computing our value is hard in general, the relation to k-coloring shows that whenever rules can be modelled by a graph for which counting the number of k-colorings is simple, we have a polynomial algorithm for our problem. We state the two following corollaries.

COROLLARY 3. *The 2-coalitional and 3-coalitional Shapley values can be calculated in polynomial time under the Embedded MC-Nets representation.*

This result comes from trivial algorithms that count k-colorings for $k = 1, 2$.[3] Another result comes from the restriction of the Embedded MC-Nets.

COROLLARY 4. *The k-coalitional Shapley value can be calculated in polynomial time under the Embedded MC-Nets representation restricted to rules without negative literals.*

Note that, without negative literals, the graph that represents restrictions in connecting formulas does not have edges. Another example of the application of Theorem 3 is Corollary 5.

With Partition Decision Trees [29], a game is represented as a set of rooted directed trees, called a *PDT rule*. Each PDT rule is a tree (V, E), with root x and two label functions f_V and f_E – non-leaf nodes are labelled with agents' names, leaf nodes are labelled with payoff vectors, and edges are labelled with numbers that correspond to coalitions. Thus, one path defines a partition of agents and their value. Conciseness of this representation comes from two features. Firstly, not all agents have to appear on a path. Secondly, trees are additive (the value of embedded coalition is the sum of values of this coalition in every tree), thus separate concise rules can be independently described.

[3] For $k = 1$ there exists (only one) k-coloring if a graph have no edges. For $k = 2$, each connected component has two k-colorings if it is bipartite (otherwise, zero k-colorings exists for the whole graph).

COROLLARY 5. *The k-coalitional Shapley value can be calculated from Partition Decision Trees in polynomial time.*

PROOF. Each path of the Partition Decision Trees can be translated into a set of Embedded MC-Nets rules. In such rules, boolean expressions cannot be merged and the graph of restrictions is a clique. As for cliques, the chromatic polynomial (and the number of colorings) is known; hence, based on Theorem 3, we get a polynomial algorithm. □

6. DIPLOMACY AND THE GEOSTRATEGIC IMPORTANCE OF EUROPEAN REGIONS

Diplomacy is a popular board game, created in the 1950s, that was inspired by national rivalries at the outset of World War I [26]. Diplomacy not only achieved enormous success among the general public, but was also played by politicians and diplomats, such as John F. Kennedy and Henry Kissinger. Aspects of the game have proved to be of interest both for game theorists [33] and computer scientists [12, 8, 10]. In this section, we will show how the concepts and techniques developed in this paper may be used to analyse the relative importance of provinces in Diplomacy.

The Diplomacy board is an approximate map of Europe, divided into 75 provinces, with seven players corresponding to empires that existed before WWI. Each player initially controls a few provinces and with his/her armies tries to expand this territory, with the aim of dominating the continent. Conflicting interests lead to battles for contested provinces.

The *attack/defence rules* are as follows: a player that attacks a province wins a battle if his/her attack is stronger than the defence forces and stronger than any other simultaneous attack by other parties. The strength of an attack is measured by the number of armies that attack the province from adjacent provinces. Similarly, the strength of the defence is the number of armies that support the province from the adjacent provinces, plus one if there is an army already in the province.

One particularly interesting feature of the game is that the 75 provinces preserve the geographic advantages of particular locations. Hence, a natural question arises as to the relative importance of each of the provinces, given its strategic position on the map of Europe: Should a player try to position her army in more central provinces, from which it is possible to attack many neighbours but, as such, which are more exposed to enemy attacks? Or is it better to focus on more peripheral provinces, with fewer neighbours?

In what follows, we show that, *under certain simplifying assumptions, it is possible to construct value function that corresponds to the aforementioned attack/defence rules in the game of Diplomacy. Moreover, we show that this value function admits a polynomial algorithm for the k-coalitional Shapley value.* In other words, we develop a scalable method of measuring relative importance of provinces in the context of attack/defence rules. To this end, we model the Diplomacy game board as a graph $G = (V, E)$ in which nodes V represent provinces and edges E connect neighbouring provinces. Next, we construct a k-coalitional game on this graph. In particular, we assume that each province/node is a player in a cooperative game in which at most 7 coalitions can be created. Each such coalition should be interpreted as the territory controlled by a single empire.

Before proceeding with the definition of the value function, let us introduce the following notation. We will denote by $M(i)$ the set of neighbors of node $i \in V$, and by $M_+(i) = M(i) \cup \{i\}$ – the set of neighbors including node i. We define the *strength of the attack* of nodes $S \subseteq V$ on node i as the number of i's neighbors

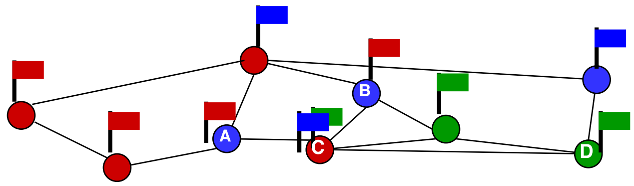

Figure 2: A sample network. Each color denotes a different coalition and flags denote the strongest attackers' colors. Here, node A has 3 red neighbors, so it is counted to the red coalition valuation. Node B has 2 red neighbors, only 1 green, and it is blue itself; hence, B is also captured by the red. Node C has 2 neighbors blue and 2 green. Thus, it has both green and blue flags. Node D is green and has 1 green neighbor, while all the opponents have attack strength of 1; hence, D has a green flag. We have $\nu(red)=4$, $\nu(blue)=3$, $\nu(green)=3$.

(including i) in S: $\text{infl}_i(S) = |M_+(i) \cap S|$. Furthermore, denoting by P a partition of nodes V, let the *strongest attackers* on node i, denoted $\max \text{infl}_i(P)$, be the coalitions in P that have highest attack strength on i. Formally:

$$\max \text{infl}_i(P) = \{S \in P \mid \text{infl}_i(S) = max_{T \in P}\, \text{infl}_i(T)\}.$$

We are ready now to formally define a k-coalitional cooperative game over graph $G = (V, E)$, where $k = 7$, G corresponds to the board of the Diplomacy game, and where the value function corresponds to the attack/defence rules in this game. In particular, let $(G, \nu) \in \mathcal{PG}_k$ with $\nu : \mathcal{EC}_k \to \mathbb{R}$ be defined as:

$$\nu(S, P) = |\{i \in V \mid S \in \max \text{infl}_i(P)\}|.$$

In words, the value of a coalition in our game is the sum of *elements* gained from attacking adjacent provinces: coalition S captures node j only if S is one of its strongest attackers (see Figure 2 for an example). We call (G, ν) the *Network Control Game*. We can also rewrite it as:

$$\nu(S, P) = \textstyle\sum_{j \in V} \mathbb{1}(S \in \max \text{infl}_j(P)),$$

where $\mathbb{1}(\sigma)$ denotes a function that takes values one 1 if the condition σ is satisfied, and 0 otherwise. We will call expression $\mathbb{1}(S \in \max \text{infl}_j(P))$ *j's component of value of S in P*.

Note that our value function assumes the following simplification of Diplomacy attack/defence rules: we do not differentiate between sea and land provinces, we ignore the fact that some provinces are supply centers, and we assume that any configuration of provinces that belong to a single player is feasible.[4]

The key computational result of this section is that the k-coalitional Shapley value for the above game can be computed in polynomial time. To this end, let us focus on the marginal contribution of an arbitrary node. In particular, node i, by leaving coalition S, can decrease its value by changing the value of any of the components from 1 to 0. Because the attack is only local, i can only affect either his own component or any of his neighbors. Moreover, node i can decrease j's component only if coalition $S \not\ni j$ was the strongest attacker on j. If this holds, there are two ways i can decrease the component of node $j \in M_+(i)$:

(a) if there is another coalition that has the same attack strength as S, j's value will be decreased independently of the coalition that i moves to,

(b) if there is no coalition that has the same attack strength on j as S, but there is another coalition T that has attack strength smaller by 1, then i will decrease value of S if and only if he moves to T.

We formalize this observation in the following proposition.

PROPOSITION 3. *Let (G, ν) be the Network Control Game. An elementary marginal contribution $[emc_i(v)](S, P)$ (i.e., the effect that node i has on the value of coalition S if he moves to S in partition P) can be decomposed as follows:*

$$[emc_i(\nu)](S,P) = \nu(S_{+i}, \tau_i^S(P)) - \nu(S,P) =$$
$$= \sum_{j \in M_+(i)} \mathbb{1}(S_{+i} \in \max \text{infl}_j(\tau_i^S(P))) - \mathbb{1}(S \in \max \text{infl}_j(P)).$$

Now, we have the following theorem:

THEOREM 4. *The k-coalitional Shapley value of the Network Control Game can be computed in polynomial time assuming that k is constant.*

PROOF. The proof is based on the following lemma.

LEMMA 3. *Assume we have n numbered items and k colors, and we color each item with a single color. Let's denote $\kappa_k^d(n)$ the number of colorings such that each of the colors is used at most d times. Then, assuming that k is constant, $\kappa_k^d(n)$ can be computed in polynomial time.*

PROOF. First, observe that $\kappa_k^d(n) = 0$ if $n > k \cdot d$. Otherwise, we use exponential generating functions to count the colorings. In this approach, we consider a power series $K(x) = \sum_{i>0} \kappa_k^d(i) \frac{x^i}{i!}$ and using analytical transformation resolve its n-th coefficient.

For $k = 1$, there is only one coloring for each n if $n \le d$; hence, the corresponding exponential generating function is $\sum_{i=0}^{d} \frac{x^i}{i!}$. Multiplying this function k times (*i.e.*, taking the convolution of the sequence) and expanding using multinomial theorem, we get:

$$K(x) = \left(\sum_{i=0}^{d} \frac{x^i}{i!}\right)^k = \sum_{a_0+\ldots+a_d=k} \binom{k}{a_0, \ldots, a_d} \prod_{i=0}^{d} \left(\frac{x^i}{i!}\right)^{a_i}.$$

We can now extract $(\frac{x^n}{n!}) \Leftrightarrow \sum(i \cdot a_i) = n$ coefficient, that is:

$$n! \sum \binom{k}{a_0, \ldots, a_d} \prod_{i=0}^{d} \left(\frac{1}{i!}\right)^{a_i},$$

where we sum over all non-negative integers $a_0, \ldots, a_d$ such that $\sum_{i=0}^{d} a_i = k$ and $\sum_{i=0}^{d} i \cdot a_i = n$. To the best of our knowledge this is not computable in polynomial time in general case, as there is an exponential number of all partitions $(a_0, \ldots, a_d)$ of k. However, for constant (or small) k, this can be done efficiently. □

We use the interpretation of the process formalized in Formula (4) to compute the k-coalitional Shapley value. Recall, that according to this interpretation, this value equals to the change in the value of the grand coalition caused by player i in the process of players leaving the grand coalition in a random order and randomly dividing themselves into partition of $k-1$ coalitions. Let π be a permutation and P a partition. We will denote by $P(i, \pi)$ a partition formed before player i leaves, assuming preceding players are partitioned according to P, *i.e.*, $P(i, \pi) = \{S \setminus Z_{i+}^{\pi} \mid S \in P\} \cup \{Z_{i+}^{\pi}\}$. Now, to facilitate calculation of the expected value, we define the

[4]We believe that to some extent all such assumptions could be waived, without compromising polynomial complexity of the algorithm. This extension, however, is left for future work.

following events in the space $\Omega = \Pi \times \bigcup_{m \in \mathbf{K}} \mathcal{P}_m$ (*i.e.*, space of all permutations of nodes and all possible partitions) that corresponds to attack strength of player i on player j, and, in particular, cases (a) and (b) from the analysis of marginal contribution.

(a) $\mathrm{CA}^i_j(\pi, P) = ((Z^{\pi}_{i+}) \in \max \mathrm{infl}_j(P(i,\pi))) \wedge (\exists_{S \in P(i,\pi)} : S \neq Z^{\pi}_{i+} \wedge \mathrm{infl}_j(S) = \mathrm{infl}_j(Z^{\pi}_{i+}))$ means that the nodes that follow i in π and i itself are a strongest attacker on node j and there is another coalition with the same attack strength on node j as Z^{π}_{i+};

(b) $\mathrm{CB}^i_j(\pi, P) = ((Z^{\pi}_{i+}) \in \max \mathrm{infl}_j(P(i,\pi))) \wedge (\mathrm{infl}_j(P(i) \setminus Z^{\pi}_{i})) + 1 = \mathrm{infl}_j(Z^{\pi}_{i+})) \wedge \neg \mathrm{CA}^i_j(\pi, P)$ means that the nodes that follow i in π and i itself are a strongest attacker on node j and there is no other coalition with the same attack strength on node j, but the coalition that i leaves to (coalition of i in P without nodes that succeed i) has attack strength smaller only by one.

Combining Proposition 3 with formula for k-coalitional Shapley value (Formula (4)), we have:

$$\varphi^k_i(v) = \sum_{j \in M_+(i)} \mathrm{prob}(\mathrm{CA}^i_j(\pi, P)) + \mathrm{prob}(\mathrm{CB}^i_j(\pi, P)).$$

Here we used additivity of the expected value and the fact that events $\mathrm{CA}^i_j(\pi, P)$ and $\mathrm{CB}^i_j(\pi, P)$ are disjoint.

Let us compute $|\mathrm{CA}^i_j(\pi, P)|$. This event means that Z^{π}_{i+} is one of the strongest attackers of node j, but $Z^{\pi}_{i+} \setminus \{i\}$ is not (independently of the coalition i joins). First, we observe that only nodes that are j's neighbors matter (let $n = |M_+(j)|$ be the number of them). Among them: (i) the order and partition of nodes succeeding i in π does not matter; (ii) among nodes proceeding i in π there is a coalition S such that $\mathrm{infl}_j(S) = \mathrm{infl}_j(Z^{\pi}_{i+})$ and there is no stronger attacker on j. We sum over the number of j's neighbors before i in π, denoted by l. From (i) we get $(n-l-1)!(k-1)^{n-l}$ and from (ii) and Lemma 3 $\kappa^{n-l+1}_{k-1}(l) - \kappa^{n-l}_{k-1}(l)$, thus:

$$|\mathrm{CA}^i_j(\pi, P)| = \sum_{l=0}^{n-1} (n-l-1)!(k-1)^{n-l}(\kappa^{n-l+1}_{k-1}(l) - \kappa^{n-l}_{k-1}(l)).$$

Similarly to the case of $\mathrm{CA}^i_j(\pi, P)$ we compute $|\mathrm{CB}^i_j(\pi, P)|$ (combinatorial details are omitted):

$$|\mathrm{CB}^i_j(\pi, P)| = \sum_{l=0}^{n-1} (n-l-1)!(k-1)^{n-l} \frac{l!}{(l-(n-l-1))!} \kappa^{n-l-1}_{k-2}(l-(n-l-1)).$$

At the end, we divide cardinalities of both events by $|\Omega| = n! \cdot (k-1)^n$ (all permutations and all divisions of $M_+(j)$ into $k-1$ coalitions) to obtain the required probabilities. Thus,

$$\varphi^k_i(v) = \sum_{j \in M_+(i)} \frac{|\mathrm{CA}^i_j(\pi, P)| + |\mathrm{CB}^i_j(\pi, P)|}{|M_+(j)|! \cdot (k-1)^{|M_+(j)|}}, \qquad (8)$$

which concludes the proof. □

Let us now compute the k-coalitional Shapley Value for the board of Diplomacy using the above formula. The results are presented in Table 1 and illustrated in Figure 3. They have the following interpretation in the Diplomacy context: *if we add the top province to our empire then, on average, the probability of launching a successful attack by our armies will increase most (assuming the simplified attack/defence rules)*. According to the k-coalitional Shapley value, we observe that the worst provinces are either peripheral and poorly connected (Portugal, Skagerrak, Barents Sea) or central on the map and very well connected (Ruhr, Budapest, Berlin). The top provinces, on the other hand, are those that are not central on the map but that are adjacent to more central provinces (Smyrna, Norway, St. Petersburg) and large sea provinces. Interestingly, these results are consistent with the widespread belief that Turkey and Russia have best geostrategic positions [5]. However, according to the same source, England is also believed to have the best position) but it is no identified as such by our method.

Worst Provinces	k-SV	Best Provinces	k-SV
Portugal	0.002	Smyrna	0.311
Skagerrak	0.005	Norway	0.294
Barents Sea	0.005	St. Petersbourg	0.265
Helgoland Bight	0.013	Norwegian Sea	0.262
Ruhr	0.014	Armenia	0.256
Budapest	0.014	W. Mediterranean	0.246
Berlin	0.014	M.-Atlantic Ocean	0.233

Table 1: The best and the worst provinces.

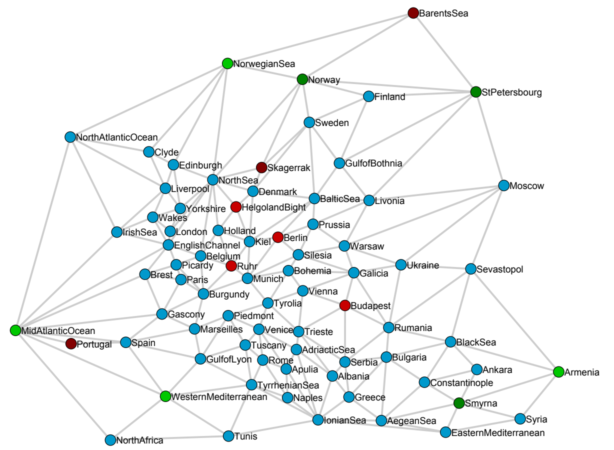

Figure 3: The Diplomacy board. Green provinces are the best according to k-coalitional Shapley value (dark green – top 3). Red provinces are the worst (dark red – top 3).

We emphasise that, our results do not represent a definite statement on which provinces are the most important in Diplomacy: Firstly, we made simplifying assumptions in the definition of the value function. Secondly, because the rules of Diplomacy are complex, the final results does not depend only on the map and attack/defence rules, but on various factors such as the players' level of experience or negotitation skills. Nevertheless, we believe our analysis provides useful strategic insights into the game.

7. CONCLUSIONS

In this paper, we introduced k-coalitional cooperative games – a class of cooperative games designed for settings where no more than k coalitions can co-exist. We proposed a natural extension of the Shapley value to these games and studied its computational properties. Finally, we applied our techniques to analyse the board game Diplomacy. To the best of our knowledge our work is the first study of network properties based on a cooperative games with externalities. In future work, we are keen to study other solution concepts, such as the core for games with externalities [11].

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