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Multilateral Bargaining for Resource Division

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Abstract. We address the problem of how a group of agents can decide to share a resource, represented as a unit-sized pie. We investigate a finite horizon non-cooperative bargaining game, in which the players take it in turns to make proposals on how the resource should be allocated, and the other players vote on whether or not to accept the allocation. Voting is modelled as a *Bayesian weighted voting game* with uncertainty about the players' weights. The agenda, (i.e., the order in which the players are called to make offers), is defined *exogenously*. We focus on impatient players with *heterogeneous* discount factors. In the case of a *conflict*, (i.e., no agreement by the deadline), all the players get nothing. We provide a Bayesian subgame perfect equilibrium for the bargaining game and conduct an *ex-ante* analysis of the resulting outcome. We show that, the equilibrium is *unique*, computable in polynomial time, results in an *instant Pareto optimal* agreement, and, under certain conditions provides a foundation for the *core* of the Bayesian voting game. Our analysis also leads to insights on how an individual's bargained share is influenced by his position on the agenda. Finally, we show that, if the *conflict point* of the bargaining game changes, then the problem of determining a non-cooperative equilibrium becomes NP-hard even under the perfect information assumption.

1 Introduction

We are concerned with one of the fundamental questions in multi-agent systems: There is a set of agents with complementary resources that can be combined to produce a joint gain, and all individually desire as large a portion of this gain as possible. How should they divide the gain between themselves? We assume the joint gain is a continuously divisible unit-sized "pie", so the problem is for the players to determine how to divide the pie amongst themselves. Conflict arises because each agent prefers to maximize its own share of the pie.

There are two key approaches for resolving such a conflict [12]. One approach is to model the situation as a cooperative game and distribute the pie as per its equilibrium, and the other is to model the scenario as a non-cooperative bargaining game, and distribute the pie as per its equilibrium solution. A study of the relation between the solutions generated by these two approaches forms the Nash program. In this paper, we follow the Nash program in the context of multilateral bargaining over a unit-sized divisible pie.

Over the years, a number of non-cooperative models of multilateral bargaining grew out of Rubinstein's bilateral bargaining game [13] in which the players decide how to split a unit-sized pie by exchanging a series of offers and counter-offers. These include [1, 14, 6, 5] and the focus of this research is on addressing issues such as determining their equilibria and studying the computational and economic properties of equilibrium outcomes. However, the scope of the results of this research is limited since they are based on assumptions such as perfect information, no deadline or a very constrained bargaining deadline, no time discounting or all the players having a common discount factor, or the agenda (the term agenda refers to the order in which the players are called to make proposals) being random (see Section 6 for details).

One of our aims therefore is to extend the scope of the existing results by dropping the above mentioned assumptions. Specifically, we consider a *new* and more *realistic* bargaining context with a *flexible* deadline, *heterogeneous* discount factors (i.e., discount factors may differ across agents), *imperfect information*, and an *exogenously* defined agenda. As we will show, these considerations lead to differences in terms of the type (i.e., stationary or subgame perfect) of equilibrium but also in terms of the resulting equilibrium shares.

Our second aim is to address new research issues (viz., analyze *how* a player's position on the agenda influences his equilibrium share, and how the *conflict point* of a bargaining game influences the complexity of computing its equilibrium) in the above context.

To this end, we propose a bargaining game that runs in a series of rounds. The agents take it in turns to propose a division of the pie: the order in which players make proposals is defined by an *exogenous* agenda. After a proposal is made, the remaining players vote on whether to accept or reject it. Voting takes place using a *weighted voting game*, in which each player has a weight, and a proposal is accepted if the sum of the weights of those in favour of the proposal meets or exceeds a certain quota. If a proposal is accepted, then it is implemented; otherwise we turn to the next player on the agenda to make a proposal in the next round. If no proposal is accepted by a fixed *deadline*, i.e., there is a *conflict*, all the players receive nothing. The agents have *imperfect information* about their weights. Imperfect information is modelled with a *Bayesian voting game* [9].

The **key results** of our research are as follows. For our bargaining game, we provide a Bayesian subgame perfect equilibrium (SPE) and conduct an *ex-ante* analysis of the resulting outcome. We show that the equilibrium is *unique*, is computable in polynomial time, results in an *instant Pareto optimal* agreement, and, under certain conditions (given in Section 4) provides a foundation for the *core* of the Bayesian voting game. In addition, our analysis generates the following key insights about the noncooperative equilibrium: i) a player's share is independent of his weight and depends only on whether he is a veto player or not, ii) every non-veto player who is not the first mover will get nothing regardless of the agenda, iii) every veto player will get a non-zero share that depends on his position on the agenda, iv) if there are no veto players, then the first mover will get the entire pie regardless of how the remaining players are arranged on the agenda. Finally, we show that changing the *conflict point* from one where all the players receive nothing to one where each player receives a constant share makes the problem of computing the non-cooperative equilibrium NP-hard even under the perfect information

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assumption.

The paper makes two **key contributions** i) we investigate some established research questions – i.e., how to obtain a noncooperative equilibrium, what are its properties, what is the link between it and the core – in a *new* and more *realistic* bargaining context, and ii) we address new research questions (i.e., *how* a player's position on the agenda impacts on his equilibrium share, and *how* the *conflict point* impacts on the complexity of computing an equilibrium) in the *new context*.

Section 2 introduces our bargaining game. In Section 3, we analyze its equilibrium. The relation between the non-cooperative equilibrium and the core is shown in Section 4. Section 5 shows how the conflict point impacts on the complexity of computing an equilibrium. Section 6 discusses related literature and Section 7 concludes.

2 The Model

There is a resource, modeled as a unit-sized divisible pie, that must be allocated between a set of p players. An *allocation* specifies how the pie is split between the players, and is represented as a vector $(x_1, \dots, x_p)$. The element x_i $(0 \leq x_i \leq 1)$ denotes player i's allocation, i.e., the amount of the resource that player i receives. A player's utility from an allocation depends both on his share of the pie and the time at which he receives his allocation. Time is divided into discrete time periods numbered $1, 2, \dots$. Player i's utility from an allocation x at time t is defined as:

$$u_i(x, t) \quad = \quad \delta_i^{t-1} \cdot x_i \tag{1}$$

where $0 < \delta_i \leq 1$ is i's *discount factor*. Thus, the discount factors are *heterogeneous*, and, at time t, i's utility gets discounted by δ_i^{t-1}.

The p players want to implement an allocation that has majority support. For this, we propose a *non-cooperative bargaining game*. This is a finite-horizon game of T discrete time periods, i.e., bargaining must end within T rounds. In each round, a chosen player makes an offer that specifies an allocation. The *outcome* of bargaining is an offer that has majority support. Majority support is modelled with a *weighted voting game*.

2.1 The Weighted Voting Game G

A weighted voting game (WVG) is a 3-tuple $G = (P, w, q)$ where $P = \{1, \dots, p\}$ is the set of players, w is the weight vector with $w_i > 0$ denoting the *weight* for player $1 \leq i \leq p$, and $q \in \mathbb{R}$ is the *quota*. The total weight of coalition $C \subseteq P$ is $w(C) = \sum_{i \in C} w_i$. The characteristic function $(v : 2^P \rightarrow \{0, 1\})$ of a game G is:

$$v(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

The set of all binary strings of length p will be denoted B. A coalition $C \in B$ is *winning* if $v(C) = 1$, otherwise it is *losing*. A player is called a *veto player* if a winning coalition cannot be formed without him, i.e., i is a veto player if for each $C \subseteq N$ such that $v(C) = 1$, we have $i \in C$. Let $Z \subseteq P$ denote the set of all veto players with $|Z| = z$, and $N \subseteq P$ the set of all non-veto players with $|N| = n$. Also, let S_W (S_L) be the set of all winning (losing) coalitions.

2.2 The Noncooperative Bargaining Game G_1

The proposed game (G_1) proceeds in a series of rounds. We let the bargaining deadline be $T = mp$ for a finite integer $m \geq 1$, i.e., an agreement must be reached within T rounds, otherwise all the players will get a *conflict share* of zero[3]. By letting T be a multiple of p, we give all the players an equal number of chances to make a proposal. Note that, m could be any positive integer and this is what we mean by *flexible deadline*. We suppose that $A = (1, \dots, p)$ is an exogenously defined bargaining agenda.

Bargaining begins at $t = 1$ with player 1 proposing an offer $x^t = (x_1^t, \dots, x_p^t)$ (with $\sum_{i=1}^{p} x_i^t \leq 1$) that specifies how to split the pie between the p players. All the remaining players then respond by either *accepting* or *rejecting* x^t. Let C_A^t denote the set of players that accept the proposal x^t and C_R^t those that reject it. If $\sum_{i \in C_A^t} w_i \geq q$, the game ends at t and the pie is split as per the offer x^t. But, if $\sum_{i \in C_A^t} w_i < q$, time is incremented and bargaining proceeds to the second round when player 2 will propose. The remaining players $1, 3, \dots, p$ respond. If 2 gets majority support, then the pie is split as per x^2 (i.e., player 2's offer) and the game ends. Otherwise, the process repeats. If no winning coalition is formed within T time periods, the game ends and all the players get a *conflict share* of zero. Let $\mathbb{O} = \mathbf{0}$ denote this conflict point where $\mathbf{0}$ denotes a vector of p zeros.

Note that, as $T = mp$, the order in which the players are called to make a proposal is $1, \dots, p, 1, \dots, p, \dots$. Thus, for a time $t = (m - X - 1)p + i$ (where $0 \leq X \leq m - 1$ and $1 \leq i \leq p$), player i will be the proposer.

3 Noncooperative Equilibrium Analysis

We first show how to obtain equilibrium for the perfect information setting and then for an imperfect information setting.

3.1 Perfect Information Setting

We will obtain an equilibrium for G_1 by using backward induction. For a time period $t \leq T$, $c_W^t \subseteq P$ will denote a winning coalition containing the proposing player and C_W^t will denote the set of all such winning coalitions. We will begin by showing how to obtain equilibrium in the context of Example 1 and then, in Theorem 1, characterize the equilibrium for any general bargaining situation.

Example 1 *We have $P = \{1, 2, 3\}$, $w = (2, 1, 2)$, and $q = 4$. The deadline is $T = 6$. The discount factors are $\delta_1 = 1/4$, $\delta_2 = 1/2$, and $\delta_3 = 3/4$. The agenda is $A = (1, 2, 3)$.*

The veto players are $Z = \{1, 3\}$. Since $A = (1, 2, 3)$, player 1 will propose in rounds 1 and 4, player 2 in rounds 2 and 5, and player 3 in rounds 3 and 6. Consider the last round $t = 6$ when player 3 will propose. Since this is the last round, player 3 will keep the entire pie by offering $x^6 = (0, 0, 1)$, and the others will accept. In the previous round $t = 5$, player 2 will give to player 3 his discounted share (i.e., $x_3^5 = \delta_3$) since player 3 is a veto player, nothing to player 1 since he is a non-veto player, and keep the remaining pie (i.e., $x_2^5 = 1 - \delta_3$) for himself. Then, at $t = 4$, player 1 will give $x_3^4 = \delta_3^2$ to player 3 (since he is a veto player), nothing to player 2 (since he is a non-veto player), and keep $x_1^4 = 1 - \delta_3^2$ for himself. Thus, player 1 will form a winning coalition with player 3. Then at $t = 3$, player 3 will give $x_1^3 = (1 - \delta_3^2) \cdot \delta_1$ to player 1, nothing to player 2, and keep $x_3^3 = (1 - \delta_1 + \delta_3^2 \cdot \delta_1)$ for himself. Continuing in the same way, we can see that an agreement will occur at $t = 1$ and result in the shares $(1 - \delta_3^2 + \delta_1 \cdot \delta_3^2 - \delta_1 \cdot \delta_3^4, 0, \delta_3^2 - \delta_1 \cdot \delta_3^2 + \delta_1 \cdot \delta_3^4)$.

[3] In Section 5, we will consider non-zero conflict shares.

Here, only the veto players 1 and 3 get a non-zero share while the non-veto player gets nothing. We can verify that if the non-veto player is the first mover (say we have the agenda $A = (2, 1, 3)$), then all the three players will get a non-zero share in the pie.

We are now ready to characterise equilibrium strategies. In the following text, x_i^t will denote player i's equilibrium share for time t.

Time (t)	Equilibrium strategy
$t = (m - X - 1)p + i$ $0 \le X \le m - 1$ $1 \le i \le p$ Proposer: player i	**Player i's offer will be an x^t that solves:** O_t : Minimize $\sum_{j \in P \backslash \{i\}} x_j^t \cdot c_j$ s.t. $\sum_{j \in P \backslash \{i\}} w_j \cdot c_j + w_i \ge q$ $c_j \in \{0, 1\}$ for $j \in P \backslash \{i\}$; $c_i = 1$ $x_j^t = \delta_j \cdot x_j^{t+1}$ for $j \ne i$ $x_i^t = 1 - \sum_{j \in P \backslash \{i\}} x_j^t$ **Each player $j \in P \backslash \{i\}$ responds to x^t:** If $u_j(x^t, t) \ge u_j(x^{t+1}, t+1)$ Accept Else Reject
$t = T$ Proposer: player p	Player p proposes to keep the whole pie and all the other players accept.

Table 1. Subgame perfect equilibrium strategies for G_1.

Time period (t)	Equilibrium shares (x)
$t = 1$ Proposer: player 1	$(\mathbf{x_1^1}, \ldots, \mathbf{x_p^1})$ where $x_j^1 = 0$ if ($j > 1$ and $j \notin Z$) $x_j^1 = x_j^2 \cdot \delta_j$ if $j \in Z \backslash \{1\}$ $x_1^1 = 1 - \sum_{j \in Z \backslash \{1\}} x_j^1$
$\cdots$	$\cdots$
$t = (m - X - 1)p + i$ $0 \le X \le m - 1$ $1 \le i \le p$ Proposer: player i	$(\mathbf{x_1^t}, \ldots, \mathbf{x_p^t})$ where $x_j^t = 0$ if ($j \ne i$ and $j \notin Z$) $x_j^t = x_j^{t+1} \cdot \delta_j$ if $j \in Z \backslash \{i\}$ $x_i^t = 1 - \sum_{j \in Z \backslash \{i\}} x_j^t$
$\cdots$	$\cdots$
$t = (m - 1)p + 1$ Proposer: player 1	$(\mathbf{x_1^t}, \ldots, \mathbf{x_p^t})$ where $x_j^t = 0$ if ($j > 1$ and $j \notin Z$) $x_j^t = x_j^{t+1} \cdot \delta_j$ if $j \in Z \backslash \{1\}$ $x_1^t = 1 - \sum_{j \in Z \backslash \{1\}} x_j^t$
$\cdots$	$\cdots$
$t = (m - 1)p + i$ $1 \le i \le p$ Proposer: player i	$(\mathbf{0}, .., \mathbf{0}, \mathbf{x_i^t}, .., \mathbf{x_p^t})$ where $x_j^t = 0$ if ($j < i$) or ($j > i$ and $j \notin Z$) $x_j^t = x_j^{t+1} \cdot \delta_j$ if $j \in Z \cap (\cup_{i+1}^p k)$ $x_i^t = 1 - \sum_{j \in Z \backslash \{i\}} x_j^t$
$\cdots$	$\cdots$
$t = mp - 1$ Proposer: player $p - 1$	$(\mathbf{0}, .., \mathbf{0}, \mathbf{x_{t-1}^t}, \mathbf{x_p^t})$ where $x_{p-1}^t = 1$ and $x_p^t = 0$ if $p \notin Z$ $x_{p-1}^t = 1 - \delta_p$ and $x_p^t = \delta_p$ if $p \in Z$
$t = mp$ Proposer: player p	$(\mathbf{0}, .., \mathbf{0}, \mathbf{1})$

Table 2. Equilibrium shares for the game G_1.

Theorem 1 *For a WVG G with $0 \le z < p$ veto players, the bargaining game G_1 with agenda $A = (1, \ldots, p)$ admits the subgame perfect equilibrium given in Tables 1 and 2, and results in an immediate agreement.*

Proof 1 *We use backward induction. For the last time period $t = mp$, the player p will propose to keep a hundred percent of the pie and all the remaining players will agree. Thus, the equilibrium shares will be $x = (0, \ldots, 0, 1)$ (see the last row in Table 2).*

In each of the previous time periods $t < T$, the proposer (say player i) will consider the set C_W^t of all possible winning coalitions containing i. For each $c \in C_W^t$, the optimal offer will maximize i's share $x_i^t = 1 - \sum_{j \in c \backslash \{i\}} \delta_j x_j^{t+1}$ while giving to each player in $c \backslash \{i\}$ his discounted share for $t + 1$, and nothing to those players that do not belong to $c \backslash \{i\}$. Between all these $|C_W^t|$ optimal offers, the one that maximizes x_i^t (or, equivalently, minimizes $\sum_{j \in c \backslash \{i\}} \delta_j x_j^{t+1}$) will be his equilibrium offer. Thus, we must find a coalition $c \in B$ that solves O_t (see Table 1).

Consider $t = mp - 1$ when player $p - 1$ will propose. His offer will depend on whether the last mover, i.e., player p, is a veto player or not. If $p \in Z$, then the proposer $p - 1$ needs player p to form a winning coalition and must give to p his discounted share for the next time period, i.e., $x_p^t = \delta_p$. But the proposer will give nothing to the players $1, \ldots, p - 2$, i.e., $x_j^t = 0$ for $\cup_1^{p-2} j$. Thus, $p - 1$ will keep $x_{p-1}^t = 1 - \delta_p$ for himself. Here, the winning coalition will be $c = P$. But if $p \notin Z$, $p - 1$ can form a winning coalition with some or all of the players in $\cup_1^{p-2} j$. Since all the players in $\cup_1^{p-2} j$ get nothing in the equilibrium for T, $p - 1$ will propose to keep the whole pie and the players $\cup_1^{p-2} j$ will accept. Thus, the winning coalition will be $c = P \backslash \{p\}$, and the equilibrium shares will be as given in row $mp - 1$ of Table 2. Note that, $p - 1$ will get a non-zero share regardless of whether he is a veto player or not, and the last mover will get a non-zero share only if he is a veto player. Thus, only the players in $\{p - 1\} \cup (Z \cap \{p\})$ will get a non-zero share while the rest get nothing.

***Backward induction for rows of Table 2 marked in red lines**: Consider $t = (m - 1)p + i$ (for $1 \le i \le p - 1$) when mover i will propose. We will assume that the equilibrium shares are $(0, \ldots, 0, x_i^t, \ldots, x_p^t)$ where $x_j^t = 0$ if ($j < i$) or ($j > i$ and $j \notin Z$), $x_j^t = x_j^{t+1} \cdot \delta_j$ if $j \in Z \cap (\cup_{i+1}^p k)$, and $x_i^t = 1 - \sum_{j \in Z \backslash \{i\}} x_j^t$. In words, we will assume that only the players in $Z \cap (\cup_{i+1}^p k) \cup \{i\}$ get a non-zero share while the rest get nothing at t. Given this assumption, we will prove that the equilibrium shares for $t - 1$ will be $(0, \ldots, 0, x_{i-1}^{t-1}, \ldots, x_p^{t-1})$ where $x_j^{t-1} = 0$ if ($j < i - 1$) or ($j > i - 1$ and $j \notin Z$), $x_j^{t-1} = x_j^t \cdot \delta_j$ if $j \in Z \cap (\cup_i^p k)$, and $x_i^{t-1} = 1 - \sum_{j \in Z \backslash \{i\}} x_j^{t-1}$. I.e., we will prove that only player $i - 1$ and the veto players in $\cup_i^p k$ will get a non-zero share while the rest get nothing at $t - 1$.*

Consider the time $t - 1$ when $i - 1$'s offer will be a solution to O_{t-1}. The proposer will be able to form a winning coalition with all the veto players and some non-veto players. We are given that, at t, only player i and the veto players in $\cup_{i+1}^p k$ get a non-zero share while the rest get nothing. This implies that, player $i - 1$ must give each veto player in $\cup_{i+1}^p k$ his discounted share for the next time period. Note that. he need not give anything to player i if i is a non-veto player, since $i - 1$ can form a winning coalition without i. Therefore, at $t - 1$, only the players $i - 1$ and the veto players in $\cup_i^p k$ will get a non-zero share and the rest get nothing. Here, the winning coalition will be $c = P$ if $i \in Z$, and $c = P \backslash \{i\}$ otherwise.

***Backward induction for rows of Table 2 marked in black lines**: Going further back, consider a time $t = (m - X - 1)p + i$ for some $0 \le X \le m - 1$ and some $1 \le i \le p$ when player i will be the proposer. We will assume that the equilibrium shares are $(x_1^t, \ldots, x_p^t)$ where $x_j^t = 0$ if ($j \ne i$ and $j \notin Z$), $x_j^t = x_j^{t+1} \cdot \delta_j$ if $j \in Z \backslash \{i\}$, and $x_i^t = 1 - \sum_{j \in Z \backslash \{i\}} x_j^t$. In words, we will assume that only the players in $Z \cup \{i\}$ get a non-zero share while the rest get nothing at t. Given this assumption, we will prove that the equilibrium shares for $t - 1$ will be $(x_1^{t-1}, \ldots, x_p^{t-1})$ where $x_j^{t-1} = 0$*

if ($j \neq i-1$ and $j \notin Z$), $x_j^{t-1} = x_j^t \cdot \delta_j$ if $j \in Z \backslash \{i-1\}$, and $x_i^{t-1} = 1 - \sum_{j \in Z\backslash\{i-1\}} x_j^{t-1}$. That is, we will prove that only the players in $\{i-1\} \cup Z$ will get a non-zero share while the rest get nothing at $t-1$.

Consider the time period $t-1$ when $i-1$ will propose an offer that solves O_{t-1} defined in Table 1. We know that, the proposer will be able to form a winning coalition with all the veto players and some non-veto players. In addition, we are given that, at t, only the players in $Z \cup \{i\}$ get a non-zero share while the rest get nothing. This implies that, the proposing player ($i-1$) must give each veto player his discounted share for the next time period. But he need not give anything to player i if i is a non-veto player, since $i-1$ can form a winning coalition without i. This means that, at $t-1$, only the players in $\{i-1\} \cup Z$ will get a non-zero share and the rest get nothing. The winning coalition will be $c = P$ if $i \in Z$, and $c = P\backslash\{i\}$ otherwise.

Thus, we get the equilibrium shares as given in Table 2. An agreement will result in the first time period. □

Theorem 1 leads to the following interesting observations.

Observation 1 *A player's equilibrium share depends not on his weight but on whether he is a veto player or not. For the agenda $A = (1, \ldots, p)$, the first mover (regardless of whether he is a veto player or not) will get a non-zero share in the pie. Every veto player will get a non-zero share. None of the non-veto players in $\cup_2^p i$ will get any share (indeed, this is what we saw in Example 1: player 2, the only non-veto player, got nothing and the pie was split between the two veto players). If there are no veto players (i.e., $z = 0$), then the first mover will get the entire pie and the rest will get nothing.*

Observation 2 *The equilibrium outcome for G_1 is 'unique'. It is 'individual rational' (IR) since each player gets a non-negative share in the pie. It is also 'Pareto optimal' (PO) since, in the equilibrium for each time period, the pie is fully allocated to some non-empty subset of players in P. In addition, there is instant agreement so the pie is not wasted through shrinkage (there would be wastage from shrinkage if an agreement were to result at $t > 1$).*

Theorem 2 *The equilibrium shares for the first time period can be computed in $\mathcal{O}(mp^2)$ time.*

Proof 2 *Which of the p players are veto can be determined in $\mathcal{O}(p^2)$ time [3]. The equilibrium shares for the last time period $t = T$ are $(0, \ldots, 0, 1)$. For each time period $t < T$, each player's discounted share can be computed in constant time. Since there are p players, the total time to compute all the shares for t will be $\mathcal{O}(p)$. There are $T = mp$ time periods in all, and we go backward from the last to the first time period. Thus, the equilibrium shares for the first time period can be computed in $\mathcal{O}(mp^2)$ time.* □

3.2 Imperfect Information Setting

We consider uncertainty over the players' weights in the WVG. In a multi-agent setting, a player's weight could represent the quality or quantity of resources it possesses. For example, in the domain of transportation logistics, the weight of a company could represent the number of deliveries it can make. But this number depends on uncertain traffic conditions. Thus, the players would be uncertain with regard to their 'weights' and we model this uncertainty with the following Bayesian WVG of *imperfect information*. This Bayesian WVG belongs to the class of Bayesian coalitional games defined in [9].

In the **Bayesian voting game**, there are b possible weight vectors and w_i^j denotes the weight of player i in the jth vector. The players have probabilistic beliefs over these vectors; B^j denotes the probability that the weight vector is w^j and $\sum_j^b B^j = 1$. For $b = 1$, this setting reduces to the perfect information setting. Thus, a Bayesian WVG is defined as the 4-tuple $G = (P, w, B, q)$. We assume that w, B, q, T, δ, and the agenda are known to all the players.

There are three distinct time frames at which one can perform the analysis of a game of imperfect information: the *ex-ante* stage, the *interim* stage, and the *ex-post* stage [7]. We conduct equilibrium analysis at the ex-ante stage, i.e., when no agent knows his weight. We define the Bayesian weight of a coalition $C \subseteq P$ as $\bar{w}(C) = \sum_{j=1}^b B^j \cdot (\sum_{i \in C} w_i^j)$. In words, $\bar{w}(C)$ is the expected weight of coalition C. Player i is a **Bayesian veto player** if $\sum_{j=1}^b B^j \cdot (\sum_{k \in P\backslash\{i\}} w_k^j) < q$, otherwise, he is not. In words, player i is a Bayesian veto player if the expected weight of the coalition $P\backslash\{i\}$ is lower than the quota (i.e., $P\backslash\{i\}$ is a losing coalition), otherwise i is Bayesian non-veto player. For this setting, let the $z = |Z|$ element set Z denote the set of Bayesian veto players. The characteristic function for $G = (P, w, B, q)$ is defined as:

$$\bar{v}(C) = \begin{cases} 1 & \text{if } \bar{w}(C) \geq q \\ 0 & \text{otherwise} \end{cases}$$

Thus, a coalition is ex-ante winning if its expected weight equals or exceeds the quota. Otherwise, it is losing. Here, the set S_W (S_L) will denote the set of all ex-ante winning (losing) coalitions. Before characterizing the equilibrium for this imperfect information setting, we will work out the equilibrium for Example 2.

Example 2 *There are three players $P = \{1, 2, 3\}$. There are $b = 2$ weight vectors: $w^1 = (1, 2, 3)$ with probability 1/3 and $w^2 = (2, 1, 2)$ with probability 2/3. The quota is $q = 4$. The deadline is $T = 6$. The discount factors are $\delta_1 = 1/4$, $\delta_2 = 1/2$, and $\delta_3 = 3/4$. The agenda is $A = (1, 2, 3)$.*

Here, the set of Bayesian veto players is $Z = \{1, 3\}$ since $\bar{w}(1, 2) < q$, $\bar{w}(1, 3) = q$ and $\bar{w}(2, 3) < q$. Given Z, we can find the equilibrium for G_1 using backward induction in just the same way as we did for the perfect information setting. In the last round $t = 6$, the proposer, i.e., player 3, will keep the entire pie and the other two players get nothing. In the previous round $t = 5$, the proposer, i.e., player 2, will form a winning coalition by giving δ_3 to player 3 (since he is a Bayesian veto player who gets positive utility in the last time period), nothing to player 1 (since he is a Bayesian veto player but he gets nothing in the last time period), and keeping $1 - \delta_3$ for himself. So the shares will be $(0, (1 - \delta_3) \cdot \delta_2^4, \delta_3^5)$. Continuing backward, we get $(1 - \delta_3^2 + \delta_1 \cdot \delta_3^2 - \delta_1 \cdot \delta_3^4, 0, \delta_3^2 - \delta_1 \cdot \delta_3^2 + \delta_1 \cdot \delta_3^4)$ as the shares for $t = 1$. Here, both Bayesian veto players get a non-zero share and the non-veto player gets nothing. By changing the agenda to $A = (2, 1, 3)$, one can verify that all the three players will get a non-zero share at $t = 1$.

We are now ready to characterize the equilibrium.

Theorem 3 *For a Bayesian voting game G with $0 \leq z < p$ veto players, the bargaining game G_1 with agenda $A = (1, \ldots, p)$ admits the ex-ante subgame perfect equilibrium given in Tables 1 and 2, and results in immediate agreement.*

Proof 3 *The proof is based on backward induction. For the last time period $t = T$, the equilibrium shares will be $(0, \ldots, 0, 1)$. For $t = T - 1$, the shares will be $(0, \ldots, 0, 1 - \delta_p, \delta_p)$ if $p \in Z$ (i.e., p is a Bayesian veto player), and $(0, \ldots, 0, 1, 0)$ otherwise. Note that the* *set Z in Theorem 1 is the set of veto players but here, Z is the set of Bayesian veto players. Given this, the equilibrium for any time period t depends on which players are Bayesian veto players and which ones are not. Thus, in each time period, the proposing player will need to give to every Bayesian veto player his discounted share for the next time period, and nothing to the Bayesian non-veto players. The rest of the proof follows from the proof for Theorem 1.* □

Theorem 3 leads to Observation 2 of Section 3.1 and, in addition, to Observation 3 listed below:

Observation 3 *A player's equilibrium share depends not on his weight but on whether he is a Bayesian veto player or not. For the agenda $A = (1, \ldots, p)$, the first mover (regardless of whether he is a Bayesian veto player or not) will get a non-zero share in the pie. Every Bayesian veto player will get a non-zero share in the pie. None of the Bayesian non-veto players in $\cup_2^p i$ will get any share (indeed, this is what we saw in Example 2: player 2, the only Bayesian non-veto player, got nothing and the pie was split between the two veto players). If there are no Bayesian veto players (i.e., $z = 0$), then the first mover will get the entire pie and the rest will get nothing.*

Theorem 4 *The equilibrium shares for the first time period can be computed in $\mathcal{O}(b^2p^2 + mp^2)$ time.*

Proof 4 *The only difference between this proof and that for Theorem 2 is that, for the latter, it takes $\mathcal{O}(p)$ to time find which of the p players are veto. But, for imperfect information, it takes $\mathcal{O}(b^2p^2)$ time. The rest of the proof follows from Theorem 2.* □

4 Noncooperative Equilibrium and the Core

Following [9], we define the *ex-ante core* of a Bayesian WVG as follows. An allocation x is in the *ex-ante core* of a Bayesian game $G = (P, w, B, q)$ if it is Pareto optimal, individual rational, and for each $C \subseteq P$, $\sum_{i \in C} x_i \geq \bar{v}(C)$. This definition covers both perfect and imperfect information settings (recall that for the former setting $b = 1$ and for the latter $b > 1$).

Let $x(G_1)$ denote the equilibrium allocation for $t = 1$ for the imperfect information case. Then, the conditions for the noncooperative equilibrium to be in the ex-ante core of $G = (P, w, B, q)$ are given in Theorem 5. Those conditions when it is not in the core are given in Theorem 6. In these theorems, $1 \leq L \leq p$ will denote the last Bayesian veto player on the agenda, i.e., all the players $L + 1, \ldots, p$ will be Bayesian non-veto players.

Theorem 5 *The noncooperative equilibrium allocation $x(G_1)$ will be in the ex-ante core of the Bayesian voting game $G = (P, w, B, q)$ with $0 < z < p$ veto players if*

C1 *the first mover is a Bayesian veto player, or*

C2 *the first mover is a Bayesian non-veto player and the discount factor for the last Bayesian veto player on the agenda is $\delta_L = 1$.*

Proof 5 *As per Observation 2 (note that both, Theorem 1 for perfect information and Theorem 3 for imperfect information, lead to Observation 2), the equilibrium solution is IR and PO. Thus, we need to prove that, for each $C \subset P$, $\sum_{i \in C} x_i \geq \bar{v}(C)$. We will do this first for the condition C1. Since $0 < z < p$, we know that there is at least one Bayesian veto player. Given this, as per Observation 3, the pie will be split only between the Bayesian veto players. Also, as per Observation 3, none of the Bayesian non-veto players will get any share. Now, the only Bayesian winning coalitions are those that contain all the Bayesian veto players. Thus, for every $C \in S_W$, $\sum_{i \in C} x_i(G_1) = 1$ and $\bar{v}(C) = 1$. In addition, for every losing coalition $C \in S_L$, $\sum_{i \in C} x_i(G_1) < 1$ and $\bar{v}(C) = 0$. It follows that, for the condition C1, $x(G_1)$ is in the ex-ante core.*

Next, consider the condition C2 for which the first mover is a Bayesian non-veto player and the last veto player's discount factor is $\delta_L = 1$. As per Table 2, only those players that are in $Z \cup \{1\}$ will get a non-zero share while the rest get nothing. Also, as per Table 2, the veto player L will get a 100% of the pie at time $(m - 1)p + L$. Moreover, since $\delta_L = 1$, his share for all the previous time periods will remain the same. Thus, at $t = 1$, L will get the entire pie and the remaining players will get nothing. Since every winning coalition must contain the Bayesian veto player L, for every $C \in S_W$, $\sum_{i \in C} x_i(G_1) = 1$ and $\bar{v}(C) = 1$. In addition, for every losing coalition $C \in S_L$, $\sum_{i \in C} x_i(G_1) < 1$ and $\bar{v}(C) = 0$. It follows that, for the condition C2, $x(G_1)$ is in the ex-ante core.

Theorem 6 *The noncooperative equilibrium $x(G_1)$ will not belong to the ex-ante core of the Bayesian voting game $G = (P, w, B, q)$ if*

C3 *there are no Bayesian veto players in G, or*

C4 *the discount factor for the last Bayesian veto player is $0 < \delta_L < 1$ and the first mover is a Bayesian non-veto player.*

Proof 6 *Consider the condition C3 first. If there are no Bayesian veto players, then, as per Observation 3, the first mover will get the entire pie. Since there are no veto players, the first mover will be a non-veto player. Not every winning coalition will contain the first mover. This means that there will be winning coalitions $C \in S_W$ such that $\sum_{i \in C} x_i(G_1) = 0$ and $\bar{v}(C) = 1$. Clearly, for the condition C3, $x(G_1)$ will not be in the ex-ante core.*

Consider the condition C4. As per Observation 3, the non-veto first mover will get a non-zero share. However, since he is non-veto, there will be winning coalitions that do not include the first mover. In other words, the coalition $P \backslash \{1\}$ will be ex-ante winning, i.e., $\bar{v}(P \backslash \{1\}) = 1$ but $\sum_{i \in P \backslash \{1\}} x_i(G_1) < 1$. Thus, for the condition C4, the noncooperative equilibrium will not be in the ex-ante core.

5 A Non-zero Conflict Point for the Game G_1

As per the game G_1 defined in Section 2.2, all the players get nothing if they fail to reach an agreement within T time periods. That is, the *conflict point* was $\mathbb{O} = \mathbf{0}$. Now, suppose that we change the *conflict point* from $\mathbb{O} = \mathbf{0}$ to $\mathbb{O} = \alpha$: instead of giving nothing, we give a constant share $\alpha_i \geq 0$ (with $\sum_{i=1}^{p} \alpha_i = 1$) to each player $i \in P$ in the case of a conflict. Thus, each player $i \in P$ will receive a share of α_i at time $T + 1$. These shares could, for instance, be determined by a neutral arbitrator and made known to the players at the start of the noncooperative bargaining.

Given this new definition of the conflict point, let us determine the equilibrium offer for Example 1 for the time period T assuming that $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$. At $T = 6$, the proposer (i.e., player 3) will give a non-zero share to those players that he needs to form a winning coalition with and nothing to the others. Since player 3 needs player 1 to form a winning coalition, he will give to player 1 his discounted share (i.e., $\delta_1 \cdot \alpha_1 = \frac{1}{4} \cdot \frac{1}{3}$) and to player 2, he will give nothing (since player 3 does not need player 2 to form a winning coalition). Thus, the equilibrium offer for T will be $(1/12, 0, 11/12)$ and this will be accepted by player 1.

Finding the equilibrium for T was easy for the above example because there were only 3 players. However, in general, this problem will be computationally hard as is demonstrated in Theorem 7.

Theorem 7 *If the player's conflict shares are* α ($\alpha_i \geq 0$ *and,* $\sum_{i=1}^{p} \alpha_i = 1$)*, then the problem of computing an equilibrium offer for* G_1 *is NP-hard even under the perfect information assumption.*

Proof 7 *At* $T+1$*, each player* $i \in P$ *will get a utility of* $\alpha_i \cdot \delta_i^T$*. Consider time* T*. The proposer (i.e., player* p*) will consider all those winning coalitions that he is a member of, and for each* $c \in C_W^T$*, his optimal offer at* T *will be an* x *such that* $x_j^T = \alpha_j \cdot \delta_j$ *for each* $j \in c \backslash \{p\}$*,* $x_j^T = 0$ *for each* $j \notin c$*, and* $x_p^T = 1 - \sum_{j \in c \backslash \{p\}} \alpha_j \cdot \delta_j$*. Thus, we must find a winning coalition that maximizes* x_p^T *(or, equivalently, minimizes* $\sum_{j \in c \backslash \{p\}} \alpha_j \cdot \delta_j$*). That is, we must find a coalition* $c \in B$ *(recall from Section 2.1 that* B *denotes the set of all binary strings of length* p*) that solves the problem* O_T*:*

$$O_T: \quad \textit{Minimize} \quad \sum_{i=1}^{p-1} \alpha_i \cdot \delta_i \cdot c_i$$
$$\textit{s.t.} \quad \sum_{i=1}^{p-1} w_i \cdot c_i + w_p \geq q$$
$$c_i \in \{0,1\} \quad \textit{for} \quad 1 \leq i \leq p-1 \qquad (2)$$

The problem O_T *is NP-hard by reduction from the integer knapsack problem [8]; the weight (profit) of knapsack item* i *is* w_i ($\alpha_i \cdot \delta_i$) *and the knapsack capacity is* $q - w_p$*. Thus, even with perfect information, the equilibrium is hard to compute.* □

Note that, a crucial difference between the optimization problem O_t in Table 1 and the optimization problem O_T in Equation 2 is that, for the former, only the veto players need to be given a non-zero share but, for the latter, one or more non-veto players must be allocated a non-zero share because the conflict utilities are non-zero. Thus, for the latter, we need to find which non-veto players to include in the winning coalition. This difference makes the latter problem hard.

6 Related Research

The existing multilateral noncooperative bargaining games for dividing a pie can be divided into two types: *unanimity games* [4, 15, 11, 10] and *majority games* [1, 14]. [4] studied n-person *unanimity* bargaining with transferable utility, perfect information, discrete time periods, and a *common* discount factor. They showed conditions when *stationary equilibria* are efficient, and when such outcomes converge to core outcomes. [15, 11, 10] conducted similar analysis for a non-transferable utility (NTU) game.

[1] studied *majority bargaining* by assuming perfect information with all players having unit weight and a common discount factor. This work was later extended in [14] to general weighted voting by assuming perfect information, an infinite horizon, and no time discounting. Here, a random proposer is chosen with some probability at the beginning of each time period. The game has a *stationary* equilibrium in which each player's utility is proportional to his voting weight. In contrast, we consider a finite horizon imperfect information setting with heterogeneous discount factors and an exogenously defined agenda (because, generally, the formation of the agenda is not random, but is shaped by political parties). With regard to results, we showed a unique, Pareto optimal, and no-delay Bayesian SPE outcome, and that a player's equilibrium share depends not on his weight but on whether he is a veto player or not.

[2] studied bargaining for formation of multiple coalitions. There is a finite number of discrete rounds and a fixed agenda in which the players are arranged in decreasing order of their weights. The outcome is an offer with majority support. This work assumes perfect information and focuses on comparing the *efficiency* of the equilibrium with the global optimum. In contrast, we focused on non-cooperative equilibrium and its relation to the core in an imperfect information setting, and studied the impact of a player's position on the agenda on his equilibrium share.

[6] analyzed bargaining for resource sharing assuming perfect information, no time discounting, and a fixed deadline equal to the number of players. Here, an offer is implemented just if the subsequent players on the agenda accept it. In contrast, we focus on majority games in an imperfect information setting with heterogeneous discount factors and a flexible deadline. Finally, unlike them, we show how a player's position on the agenda affects his utility.

7 Conclusions and Future Work

We investigated strategic behavior in a finite-horizon noncooperative bargaining game with a deadline and heterogeneous discount factors. The outcome is an allocation with majority support. In the case of a conflict, i.e., there is no agreement within the deadline, all the players get nothing. Majority support is modelled with a Bayesian weighted voting game with uncertainty over the players weights. The ex-ante Bayesian subgame perfect equilibrium for the noncooperative game can be computed in polynomial time and it results in a unique, instant, and Pareto optimal agreement. We gave those conditions when the noncooperative equilibrium is in the core of the Bayesian voting game and those when it is not. We also showed how an individual's share for noncooperative bargaining is influenced by his position on the agenda. Lastly, we showed that, if the conflict point is changed, the problem of computing an equilibrium becomes NP-hard even for the perfect information setting.

In this paper, our focus was on bargaining over a single pie. An obvious extension is to consider multiple pies. Another possibility is to use a *vector voting game* (a generalized version of weighted voting games considered in this paper) for modelling majority support.

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REFERENCES

[1] D. Baron and J. Ferejohn, Bargaining in legislatures, *American Political Science Review*, 83(4): 1181–1206, 1989.
[2] G. Chalkiadakis and E. Elkind and M. Polukarov and N. R. Jennings, The price of democracy in coalition formation, *In Proc. of AAMAS*, 401–408, 2009.
[3] G. Chalkiadakis and E. Elkind and M. Wooldridge, Computational Aspects of Cooperative Game Theory, Morgan-Claypool, 2011.
[4] K. Chatterjee and B. Dutta and D. Ray and K. Sengupta, A noncooperative theory of coalitional bargaining, *Rev. of Economic Studies*, 60:463–477, 1993.
[5] PJ. Duggan and T. Kalandrakis, Dynamic legislative policy making, *Journal of Economic Theory*, 147:1653–1688, 2011.
[6] P. E. Dunne and S. Kraus and E. Manisterski and M. Wooldridge, Solving coalitional resource games, *Artificial Intelligence*, 174:20–50, 2010.
[7] F. Forges and R. Serrano, Cooperative games with incomplete information: Some open problems, *Social Science Research Network*, 2011.
[8] M. Garey and D. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, 2011.
[9] S. Ieong and Y. Shoham, Bayesian coalitional games, *Proc. of AAAI*, 2008.
[10] K. Kultti and H. Vartiainen, Multilateral cooperative bargaining in a general utility space, *International Journal of Game Theory*, 39(4): 677–689, 2010.
[11] B. Moldovanu and E. Winter, Order independent equilibria, *Games and Economic Behavior*, 9: 21–34, 1995.
[12] D. Ray, A Game-Theoretic Perspective on Coalition Formation, Oxford University Press, 2007.
[13] A. Rubinstein, Perfect equilibrium in a bargaining model, *Econometrica*, 50(1): 97–109, 1982.
[14] J. Snyder and M. Ting and S. Ansolabehere, Legislative bargaining under weighted voting, *The American Economic Review*, 95(4): 981–1004, 2005.
[15] J. Vidal-Puga, Forming coalitions and the Shapley NTU value, *European Journal of Operational Research*, 190, 659–671, 2008.