Optimal Agendas for Sequential Auctions for Common and Private Value Objects

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Abstract

This paper analyzes sequential auctions for objects that have both *common* and *private values*. Existing work has studied sequential auctions for objects that are either exclusively common or private value. However, in many cases, an object has both features. Now, in such cases, the common value (which is the same for all bidders) depends on each bidder's valuation of the object. But, generally speaking, a bidder cannot know the true common value, since it does not know how much the other bidders value it. On the other hand, a bidder's private value is independent of the others' private values. Given this, our objective is to study sequential auctions for heterogeneous objects that have both common and private values by treating each bidder's information about the common value as *uncertain*. In so doing, we first determine equilibrium bidding strategies for each auction in a sequence. Then, since the total revenue from all the auctions in a sequence depends on the *auction agenda* (i.e., the order in which the objects are auctioned) we determine the *optimal agenda* (i.e., the one that maximizes total expected revenue across all the auctions in a sequence). Finally, we show that, for a given agenda, the total revenue is the same for first-price, second-price, and English auction rules.

1 Introduction

Market-based mechanisms such as auctions are now being widely studied as a means of buying/selling resources in multiagent systems. This uptake is occurring because auctions are both simple and have a number of desirable properties (typically the most important of which are their ability to generate high revenues to the seller and to allocate resources efficiently) [14, 2]. Now, in many cases the number of objects to be auctioned is greater than one. There are two types of auctions that are used for multiple objects: *combinatorial* [13] and *sequential* [7, 13]. The former are used when the objects for sale are available at the same time, while the latter (which we focus on here) are used when the objects become available at different points in time. Now since the sequential auctions are conducted at different times, a bidder may participate in more than one of them. In such a scenario, [15, 3, 1] show that although there is only one object being auctioned at a time, the bidding behaviour for any individual auction strongly depends on the auctions that are yet to be conducted. For example, in sequential auctions for oil exploration rights the price an oil company will pay for a given area is affected not only by the area that is available in the current round, but also by the areas that will become available in subsequent rounds of leasing. Thus, it would be foolish for a bidder to spend all the money set aside for exploration on the first round of leasing, if potentially even more favourable sites are likely to be auctioned off subsequently. Hence, a bidder must determine effective bidding strategies that specify how much to bid in each individual auction. However, this decision making depends on the *auction agenda* (i.e., the order in which the auctions are conducted) [3]. Given this, our aim in this work is to take the perspective of the auctioneer and determine the optimal agenda (i.e., the one that maximizes total revenue).

A key limitation of existing work on sequential auctions is that it focuses on objects that are either exclusively private value or exclusively common value (see Section 5 for details). Furthermore, some of this work also assumes that agents have complete information. However, in practice, most auctions are neither exclusively private nor common value, but involve an element of both [8]. Again, consider the example of auctioning oil drilling rights. This is, in general, treated as a common value auction. But private value differences may arise, for example, when a superior technology enables some firm to exploit the rights better than others. Also, in such cases, the common value (which is the same for all the bidders) depends on how much each bidder values the object. But, generally speaking, a bidder does not know the true common value, since it does not know how much the other bidders value it. On the other hand, the private value of a bidder is independent of the other bidders' private values.

Against this background, our objective is to study sequential auctions (for heterogeneous objects) that have both common and private value elements. We treat each bidder's information about the common value as *uncertain*. To this end, we first determine equilibrium bidding strategies for each individual auction in a sequence using first-price, second-price, and English auction rules. On the basis of this equilibrium, we show that these three auction forms are equivalent in terms of the selling price of each individual object in a series. Our study also shows that the sum of selling prices of all the objects in a series depends on the order in which the auctions are conducted. Given this, we then determine the auctioneer's optimal agenda (i.e., the agenda that maximizes the sum of the selling prices).

By doing so, our analysis makes the following important contributions to the state of the art in multi-object auctions. First, we analyze for the first time, the incomplete information setting and study the bidding behaviour for sequential auctions that have both private and common value elements. Second, we show revenue equivalence for the above three auction forms. Finally, we determine the auctioneer's optimal agenda for this scenario.

The remainder of the paper is organised as follows. Section 2 describes the auction setting. Section 3 determines equilibrium bidding strategies for English auction, first-price, and second-price rules. In Section 4 we show revenue equivalence and find the auctioneer's optimal agenda. Section 5 discusses related work and Section 6 concludes.

2 The sequential auctions model

Single object auctions that have both private and common value elements have been studied in [5]. We therefore adopt this basic model and extend it to two objects. Before doing so, however, we give a brief overview of the basic model.

Assume there are $n \geq 3$ risk neutral bidders. The *common value* (V_1) of the object to the n bidders is equal, but initially the bidders do not know this value. However, each bidder receives a signal that gives an estimate of this common value. Specifically, bidder $i = 1, \ldots, n$ draws an estimate (v_{i1}) of the objects true value (V_1) from the probability distribution function $Q(v)$ with support $[v_L, v_H]$. Although different bidders may have different estimates, the true value (V_1) is the same for all the bidders and is modelled as the average of the bidders' signals:

$$V_1 \quad = \quad \frac{1}{n}\sum_{i=1}^{n} v_{i1} \tag{1}$$

Furthermore, each bidder has a *cost* which is different for different bidders. Here each bidder's cost is its *private value* part. For $i = 1, \ldots, n$, let c_{i1} denote bidder i's signal for this private value part that is drawn from the distribution function $G(c)$ with support $[c_L, c_H]$ where $c_L \geq 0$ and $v_L \geq c_H$. Cost and value signals are independently and identically distributed across bidders. Henceforth we will use the term *value* for common value and *cost* for private value.

If bidder i wins the object and pays b, it gets a utility of $V_1 - c_{i1} - b$, where $V_1 - c_{i1}$ is i's surplus. Each bidder bids so as to maximize its expected utility. Note that bidder i receives two signals (v_i and c_i) but its bid has to be a single number. Hence, in order to determine their bids, bidders need to combine the two signals into a *summary* statistic. This is done as follows. For i, a one dimensional summary signal called i's surplus[1] is defined as:

$$s_{i1} \quad = \quad v_{i1}/n - c_{i1} \tag{2}$$

which allows i's optimal bids to be determined in terms of s_{i1} (see [5] for more details about the problems with two signals and why a one dimensional surplus signal is required). Note that the probability distribution for the surplus can easily be obtained from functions Q and G. In order to rank bidders from low to high valuations, $Q(v)$ and

[1]Note that i's true surplus is $V_1 - c_{i1}$ which is equal to $v_{i1}/n - c_{i1} + \sum_{j \neq i} v_{j1}/n$. But since $v_{i1}/n - c_{i1}$ depends on i's signals, while $\sum_{j \neq i} v_{j1}/n$ depends on the other bidders' signals, the term 'i's surplus' is also used to mean $v_{i1}/n - c_{i1}$.

$G(c)$ are assumed to be log concave[2]. Under this assumption, the conditional expectations $E(v|s = x)$ and $E(v|s \leq x)$ are non-decreasing in x. Furthermore, $E(c|s = x)$ and $E(c|s \leq x)$ are non-increasing in x. In other words, the bidders can be ranked from low to high values (v_{i1}) on the basis of their surplus.

We extend[3] the above model to $m = 2$ objects. Since the objects we consider are heterogeneous, we have two different value (cost) functions – one for each object. For $j = 1, 2$, let $Q_j(v)$ with support $[v_{Lj}, v_{Hj}]$ denote the probability density function for the value of the jth object. Likewise, let $G_j(c)$ with support $[c_{Lj}, c_{Hj}]$ denote the probability density function for the cost of the jth object where $c_{Lj} > 0$ and $v_{Lj} > cHj$.

Bidders receive the cost and value signals for an auction just before that auction begins. Thus, the signals for the second object are received only after the first auction has been conducted. Consequently, although the bidders know the distribution functions from which the signals are drawn, they do not know the actual signals for the second object until the first auction is over.

These two objects are auctioned one after another. Furthermore, each bidder can win at most one object. Thus, the winner for the first object cannot participate in the second auction. Thus, if n agents participate in the first auction, the number of agents for the second auction is $n - 1$. For the objects $j = 1, 2$ and bidders $i = 1, \ldots, n$, let v_{ij} and c_{ij} denote the common and private value parts respectively. Then the common value of the jth object (denoted V_j) is:

$$V_j = \frac{1}{n - j + 1} \sum_{i=1}^{n-j+1} v_{ij} \qquad (3)$$

For objects $j = 1, 2$ and bidders $i = 1, \ldots, n$, we will denote i's surplus for the jth object as s_{ij}, where

$$s_{ij} = v_{ij}/(n - j + 1) - c_{ij}. \qquad (4)$$

For this model, we determine equilibrium using English auction rules, firts-price rules, and second-price rules (see Section 3).

3 Equilibrium bidding strategies

The two objects are auctioned in two separate auctions that are conducted sequentially. Equilibrium bidding strategies for a single object of the type described in Section 2 have been obtained in [5] for first-price, second-price, and English auction rules. We therefore briefly summarize these strategies and then determine equilibrium for $m = 2$ objects.

[2]Log concavity means that the natural log of the densities is concave. This restriction is met by many commonly used densities that include uniform, normal, chi-square, and exponential, and it ensures that optimal bids are increasing in surplus.

[3]Our model for $m = 2$ objects is a generalisation of [1]. While [1] consider two private value objects, our model considers two objects with both private and common values.

Single object. There is an object with a value say V_1. To begin, consider the English auction rules. These rules are as follows. The auctioneer continuously raises the price, and bidders publicly reveal when they withdraw from the auction. Bidders who drop out from an auction are not allowed to re-enter that auction. A bidder's strategy for the first auction depends on how much profit it expects to get from the second auction which is yet to be conducted. However, since there are two objects and there are no more auctions after the second one, a bidder's strategic behaviour during the second auction is the same as that for a single object English auction. For this case, the equilibrium strategies are as follows.

A bidder's strategy is described by its surplus and indicates how high the bidder should go before dropping out. Since the number of bidders is more than two, the prices at which some bidders drop out convey information (about the common value) to those who remain active. Suppose k bidders have dropped out at bid levels $b_1 \leq \ldots \leq b_k$. Then a bidder's (say i's) strategy is described by functions $B_k(s_i; b_1 \ldots b_k)$, which specify how high it must bid given that k bidders have dropped out at levels $b_1 \ldots b_k$ and given that its surplus is s_i. The n-tuple of strategies $(B(\cdot), \ldots, B(\cdot))$ with $B(\cdot)$ defined in Equation 5 constitute a symmetric equilibrium of the English auction.

$$\begin{aligned} B_0(x) &= E(v_i - c_i | s_i = x) \quad (5) \\ B_k(s_i; b_1 \ldots b_k) &= \frac{n-k}{n} E(v_i | s_i = x) - \\ & E(c_i | s_i = x) + \\ & \frac{1}{n} \sum_{y=0}^{k-1} E(v_i | B_j(s_i; b_1, \ldots, b_j) = b_{j+1}) \end{aligned}$$

The intuition for Equation 5 is as follows. Given its surplus and the information conveyed in others' drop out levels, the highest a bidder is willing to go is given by the expected value of the object, assuming that all other active bidders have the same surplus. For instance, consider the bid function $B_0(s_i)$ which pertains to the case when no bidder has dropped out yet. If all other bidders were to drop out at level $B_0(s_0)$, then i's expected payoff ($ep = V_1 - c_i - B_0(s_0)$) would be:

$$\begin{aligned} ep &= s_i + \frac{n-1}{n} E(v | s = s_0) - B_0(s_0) \\ &= s_i + \frac{n-1}{n} E(v | s = s_0) - E(v - c | s = s_0) \\ &= s_i - s_0 \end{aligned}$$

Using strategy B_0, i remains active until it is indifferent between winning and quitting. Similar interpretations are given to B_k for $k \geq 1$; the only difference is that these functions take into account the information conveyed (about the common value) in the others' drop out levels.

We now turn to the first and second-price rules. Let f_1^n denote the first order statistic of the surplus for the n bidders and let s_1^n denote the second order statistic. Due to symmetry, we consider any bidder (say i). We denote the highest surplus of the $n-1$ bidders (other than bidder i) as f^{n-1}. The n-tuple of strategies $(B(.), \ldots, B(.))$, where

$$B(x) = E(V_1 - c_i | s_i = x, f^n = x) - E(f^n - f^{n-1} | s_i = x, f^n = x) \quad (6)$$

is a symmetric equilibrium for the first price rules. The first term on the right hand side of Equation 6 is what the object is worth (on average) to a bidder assuming that his surplus (x) is the highest. The second term shows how much the bidder shades his bid.

For the second price rules,

$$B(x) = E(V_1 - c_i | s_i = x, f^{n-1} = x) \quad (7)$$

is a symmetric equilibrium.

For the above equilibria (for all three rules), it was shown that the bidder with the highest surplus wins the auction and pays the second highest surplus [5]. Thus, the expected selling price of the object (denoted $E(P_w)$) is:

$$E(P_w) = E(s_1^n) \quad (8)$$

and the winner's expected profit (denoted π_w) is:

$$\pi_w = E(f_1^n) - E(s_1^n) \quad (9)$$

for all the three rules. On the basis of the above equilibrium for a single object, we determine equilibrium for our two objects case.

Two objects. Let V_1 denote the value of the object that is auctioned first and V_2 that of the second. For $i = 1, \ldots, n$ and $j = 1, 2$, let π_j denote a bidder's ex-ante expected profit[4] from the jth auction. Also, let f_1^n denote the first order statistic of the surplus for the first auction (since there are n bidders) and s_1^n the second order statistic. Likewise, let f_2^{n-1} and s_2^{n-1} denote the first and second highest order statistics for the second auction. This is because the number of bidders for the first auction is n and for the second it is $n-1$ (since the winner for the first auction does not participate in the second). For $j = 1, 2$, let B^j denote the bidding strategy during the jth auction. First, we consider the case where the two auctions are conducted using English auction rules.

Theorem 1 *For the first auction, the n-tuple of strategies $(B^1(.), \ldots, B^1(.))$, where*

$$\begin{aligned} B_0^1(x) &= E(v_i - c_i | s_i = x) - \pi_2 \\ B_k^1(x; b_1, \ldots, b_k) &= \frac{n-k}{n} E(v_i | s_i = x) - \\ & E(c_i | s_i = x) + \\ & \frac{1}{n} \sum_{y=0}^{k-1} E(v_i | B_y(s_i; b_1, \ldots, b_y) = b_{y+1}) - \pi_2 \end{aligned} \quad (10)$$

is an equilibrium for the English auction rules. For the second auction, the equilibrium strategies are the same as a single object English auction.

Proof: Since both V_1 and V_2 denote common values, at any stage, a bidder's strategy for the jth auction (for $j = 1, 2$) depends on the drop out levels for that auction at that

[4] As we will show in Equation 11, due to symmetry, the ex-ante expected profit from the jth (for $j = 1, 2$) auction is the same for all the bidders.

stage. Furthermore, since there are no more objects to be auctioned after the second, the bidding behaviour of agents during the second auction is the same as that for a single object auction. However, a bidder's strategy for the first auction depends on its expected profit from the object to be auctioned next. Thus, while a bidder's strategy for the second auction depends only on the others' dropout levels, its strategy for the first auction depends not only the others' drop out levels, but also on its own ex-ante expected profit from the second auction.

Consider the second auction. Equilibrium strategies for this auction $(B^2(\cdot), \ldots, B^2(\cdot))$ are the same as those given in Equation 5 except that the number of bidders now is $n-1$ instead of n. Thus, the bidder with the highest surplus for the second auction wins it and pays the second highest surplus (as per Equation 8 with n replaced with $n-1$).

We now turn to the first auction. Although the bidders know the distribution (from which the cost and value signals are drawn) before the first auction begins, they draw the signals for the second object only after the first auction ends. Given this, each bidder's (i.e., for $i = 1, \ldots, n$) ex ante expected profit for the second auction is obtained from Equation 9 as follows:

$$\pi_2 = \frac{1}{n-1}(E(f_2^{n-1}) - E(s_2^{n-1})) \tag{11}$$

This is because each of the $n-1$ agents that participate in the second auction have ex ante identical chances of winning it. Note that the right hand side of Equation 11 does not depend on i. In other words, since bidders receive their signals for the second auction only after the first auction, the expected profit for the second (prior to the end of the first) is the same for all n bidders.

Now consider a stage where k bidders have dropped out of the first auction at bid levels $b_1, \ldots, b_k$ where $b_1 \leq \ldots \leq b_k$. Also, let b denote the price announced by the auctioneer at this stage. If bidder i makes a bid at b, and wins the auction, it gets a profit of:

$$\begin{aligned} \pi_1 = {} & \frac{n-k}{n}E(v_i|s_{i1} = x) - E(c_i|s_i = x) + \\ & \frac{1}{n}\sum_{y=0}^{k-1} E(v_i|B_y(s_{i1}; b_1, \ldots, b_y) = b_{y+1}) - b \end{aligned} \tag{12}$$

Bidder i bids at b if $\pi_1 \geq \pi_2$, or

$$\begin{aligned} b \leq {} & \frac{n-k}{n}E(v_i|s_{i1} = x) - E(c_i|s_i = x) + \\ & \frac{1}{n}\sum_{y=0}^{k-1} E(v_i|B_y(s_{i1}; b_1, \ldots, b_y) = b_{y+1}) - \pi_2 \end{aligned} \tag{13}$$

In other words, for the first auction, each bidder discounts its surplus by its expected profit for the second auction. Thus, the n-tuple of strategies $(B^1(\cdot), \ldots, B^1(\cdot))$ with $B^1(\cdot)$ defined in Equation 10, constitutes a symmetric equilibrium for the first auction.

Note that in Equation 13, the ex ante expected profit from the second auction (π_2) is the same for all the bidders. As a result, the bidder with the highest surplus for the first auction wins it and pays the second highest surplus. □

It is important to note that the bids in Equation 10 are similar to those in Equation 5 (for the single object case), except that the bids in the former are obtained from the corresponding bids in the latter by shifting it by the constant π_2. In other words, the relative position of bidders remains the same for the single object case and the first auction of the two objects case. Thus, if a bidder makes the ath (for $a = 1, \ldots, n$) highest bid for the single object case, then it makes the ath highest bid if that object is sold first in a series of two auctions. Hence, the equations for the expected selling price (Equation 8) and the winner's expected profit (Equation 9) for the single object case can easily be adapted to the first auction of the two objects case as follows.

$$E(P_{w1}) = E(s_1^n) - \pi_2 \tag{14}$$

$$\begin{aligned} E(P_{w1}) &= (E(f_1^n) - \pi_2) - (E(s_1^n) - \pi_2) \\ &= E(f_1^n) - E(s_1^n) \end{aligned} \tag{15}$$

We now turn to the first and second-price rules. As before, the winner for the first auction does not participate in the second one. Also, due to symmetry, we focus on bidder i whose surplus is $s_{ij} = v_{ij}/n - c_{ij}$ for the jth auction. For the first auction, we denote the highest surplus of the $n-1$ bidders (other than bidder i) as f_1^{n-1}. The highest (second highest) order statistic for all n bidders is denoted f_1^n (s_1^n). Likewise, for the second auction, the highest surplus of the $n-2$ bidders (other than bidder i) is f_2^{n-2}. The highest (second highest) order statistic for all $n-1$ bidders is denoted f_2^{n-1} (s_2^{n-1}).

Theorem 2 (Theorem 3) shows the equilibrium for the case where each of the two auctions is conducted using first-price (second-price) rules.

Theorem 2 *For $j = 1, 2$, the n-tuple of strategies $(B^j(.), \ldots, B^j(.))$, where*

$$\begin{aligned} B^j(x) &= E(V_j - c_{ij} | s_{ij} = x, f_1^n = x) \\ &\quad - E(f_1^n - f_1^{n-1} | s_{ij} = x, f_1^n = x) \\ &\quad - \frac{1}{n-1}(E(f_2^{n-1}) - E(s_2^{n-1})) \end{aligned} \tag{16}$$

for $j = 1$, and

$$\begin{aligned} B^j(x) &= E(V - c_{ij} | s_{ij} = x, f_2^{n-1} = x) \\ &\quad - E(f_2^{n-1} - f_2^{n-2} | s_{ij} = x, f_2^{n-1} = x) \end{aligned} \tag{17}$$

for $j = 2$ is an equilibrium for the first-price rules.

Proof: Consider the second auction. Since there are no more objects after this, the equilibrium for this auction is the same as that for a single object first-price auction with $n-1$ bidders. Using the equilibrium for a single object (from Equation 6) we get Equation 17.

From Equation 9, we know that the winner's expected profit for a single object auction is $(E(f_2^{n-1}) - E(s_2^{n-1}))$. During the first auction, each bidder deducts its ex-ante expected profit (for the second auction) from its equilibrium bid for the first auction. Hence, during the first auction, each bidder's ex-ante expected profit from the second auction is:

$$\pi_2 \quad = \quad \frac{1}{n-1}(E(f_2^{n-1}) - E(s_2^{n-1})) \tag{18}$$

This is because all the bidders have ex-ante identical chances of winning the second auction. Hence Equation 16. ▫

Theorem 3 *For $j = 1, 2$, the n-tuple of strategies $(B^j(.), \ldots, B^j(.))$, where*

$$\begin{aligned} B^j(x) \quad = \quad & E(V_j - c_{ij} | s_{ij} = x, f_1^{n-1} = x) \\ & - \frac{1}{n-1}(E(f_2^{n-1}) - E(s_2^{n-1})) \end{aligned} \tag{19}$$

for $j = 1$, and

$$B^j(x) \quad = \quad E(V_j - c_{ij} | s_{ij} = x, f_2^{n-2} = x) \tag{20}$$

for $j = 2$ is an equilibrium for the second price rules.

Proof: Using the equilibrium for a single object (from Equation 7) we get Equation 20 for the second auction.

During the first auction, each bidder deducts its ex-ante expected profit from the second auction from its equilibrium bid for a corresponding single object auction. This gives the equilibrium of Equation 19. ▫

4 Revenue equivalence and the optimal agenda

We first show that for the model described in Section 2, all the three auction forms we studied are equivalent in terms of the selling prices of individual objects.

Theorem 4 *For all three auction forms, the expected selling price for the first auction (denoted $E(P_1)$) is:*

$$E(P_1) \quad = \quad E(s_1^n) - \frac{1}{n-1}[E(f_2^{n-1}) - E(s_2^{n-1})] \tag{21}$$

and for the second auction it is:

$$E(P_2) \quad = \quad E(s_2^{n-1}). \tag{22}$$

Proof: We know from Equation 8 that the expected selling price for the single object case is the second highest order statistic of the surplus for all the three auction forms. Consequently, the selling price for the second auction of our two objects case is $E(s_2^{n-1})$ for all the three auction forms.

Consider the first auction. We know from Theorems 1, 2, and 3, that during this auction, a bidder discounts its equilibrium bid (for a corresponding single object auction) by its expected profit from the second auction. We also know that the ex-ante expected profit for the second auction is $\frac{1}{n-1}[E(f_2^{n-1}) - E(s_2^{n-1})]$ for all the bidders, for all the three auction forms. Hence, for the first auction, the bidder with the highest surplus is the winner and the selling price is

$$E(P_1) \quad = \quad E(s_1^n) - \frac{1}{n-1}[E(f_2^{n-1}) - E(s_2^{n-1})] \tag{23}$$

for all the three auction forms. □

We know from Theorem 4, that the selling price of an individual object depends on the auction in which it is sold. Consequently, the sum of the selling prices for the two objects depends on the *auction agenda* (i.e., the order in which the objects are auctioned). The agenda for which the sum of the selling prices is the highest among all possible agendas is the seller's *optimal agenda* [10]. Here, we determine this agenda.

For two objects, the two possible agendas are $(1, 2)$ and $(2, 1)$. Let $E(P_{12})$ denote the cumulative expected selling price for the two objects for agenda $(1, 2)$, and $E(P_{21})$ that for agenda $(2, 1)$. If $E(P_{12}) - E(P_{21}) \geq 0$, then the optimal agenda is $(1, 2)$. Otherwise, it is $(2, 1)$. We know from Theorem 4 that $E(P_{12}) - E(P_{21})$ is:

$$\begin{aligned} & E(s_1^n) - \frac{(E(f_2^{n-1}) - E(s_2^{n-1}))}{n-1} + E(s_2^{n-1}) \\ & -[E(s_2^n) - \frac{(E(f_1^{n-1}) - E(s_1^{n-1}))}{n-1} + E(s_1^{n-1})] \\ = \quad & [E(s_1^n) - E(s_1^{n-1})] - [E(s_2^n) - E(s_2^{n-1})] \\ & + \frac{[E(f_1^{n-1}) - E(s_1^{n-1})] - [E(f_2^{n-1}) - E(s_2^{n-1})]}{n-1} \end{aligned} \tag{24}$$

In order to determine the optimal agenda, we use the notion of *dispersion of order statistics* [6] since it helps in determining whether Equation 24 is greater than zero or not. The dispersion of order statistics for the surplus is defined as follows:

Definition 1 *The surplus for object* 1 *is said to have more dispersed order statistics than object* 2 *if the following two conditions are true.*

$$E(f_1^{n-1}) - E(s_1^{n-1}) \quad \geq \quad E(f_2^{n-1}) - E(s_2^{n-1}) \tag{25}$$

$$E(s_1^n) - E(s_1^{n-1}) \quad \geq \quad E(s_2^n) - E(s_2^{n-1}) \tag{26}$$

For example, if the surplus for objects 1 and 2 are distributed uniformly, and the distribution for object 1 has a greater support than that for 2, then 1 has more dispersed order statistics than 2. The following theorem defines the optimal agenda in terms of the dispersion of order statistics for the surplus.

Theorem 5 *For two objects (denoted* 1 *and* 2*), let the surplus for object* 1 *have a more dispersed order statistics than that for object* 2*. Then the seller's optimal agenda is to auction object* 1 *first.*

Proof: The difference between the cumulative expected selling price for agendas $(1, 2)$ and $(2, 1)$ is as given in Equation 24. The first line of this equation is positive from Equation 26 and the second line is positive from Equation 25. □

Intuitively, when the object with a higher dispersion for surplus is auctioned first, there are more bidders around to bid up the price. Also, the expected profits associated with the second object are lower, so the bidders do not discount their first bids by as much. For instance, consider the case where the distribution of the surplus for object 2 is degenerate. If object 2 is auctioned first, each bidder discounts its surplus by its expected profit from the second auction. Furthermore, if the winner of object 2 had one of the two highest surpluses for object 1, then the selling price of object 1 is reduced as well. In contrast, if 1 is auctioned first, then bidders do not discount their bids for 1, since there is no profit from winning object 2 in the second auction.

5 Related work

A key limitation of existing work on sequential auctions is that it focuses on objects that are either exclusively private value or exclusively common[5] value [11, 15, 9, 1, 3]. For instance, Ortega-Reichert [11] determined the equilibrium for sequential auctions for two private value objects using the first price rules. Weber [15] studied the price dynamics for sequential auctions of identical objects with risk neutral bidders who hold independent private values. Milgrom and Weber [9] also studied the price dynamics for sequential auctions in an interdependent values model with affiliated[6] signals. While [15, 9] focus on the study of price dynamics for sequential auctions for identical objects, [1, 3] focus on study of determining the optimal agenda for sequential auctions for heterogeneous objects. Hence the work that is closest to ours is that of [1] and [3]. The former studies sequential auctions for two private value objects in an incomplete information setting. They determine the agenda that maximizes the auctioneer's revenue, for the second price sealed bid form. This optimal agenda is defined in terms of the dispersion of the order statistics for the private values. They show that it is optimal for the auctioneer to auction the object with the higher dispersion first, and then the one with lower dispersion. [3] also studies sequential auctions for two objects in an incomplete information setting. Although this work does not make the complete information assumption, the objects it considers are heterogeneous private-value objects. In more detail, the auctioneer in [3] is a buyer that acquires two services through bidders that sell those services and The main result is that under certain assumptions on the cost function for services, the buyer's optimal agenda is to auction the service that

[5]Single object auctions for both private and common value objects have been studied in [5]. We therefore generalise this to more than one object.

[6]Affiliation is a form of positive correlation. Let $X_1, X_2, \ldots, X_n$ be a set of positively correlated random variables. Positive correlation roughly means that if a subset of X_is are large, then this makes it more likely that the remaining X_js are also large.

has a higher cost first and then the one with lower cost. Furthermore, this work also shows that the optimal agenda is the same as the agenda that maximizes the sum of efficiencies of the two auctions.

Although our work also studies sequential auctions in an incomplete information setting, it differs from [1] and [3] in that we analyze auctions with private and common value elements, while they both study exclusively private value objects. However, in terms of the results of analysis, our result for the two objects case is similar to [1]. In more detail, our work shows that it is optimal for the auctioneer to auction the object with a higher dispersion of order statistics (for the surplus) and then the one with lower dispersion. Likewise, [1] shows that it is optimal to auction the object with a higher dispersion of order statistics (for the private values) and then the one with the lower dispersion. This similarity occurs because of the way in which bidders receive their signals for the private common value elements. Both, for our model and for the model in [1], the bidders receive these signals for an object just before the auction for the object begins. Consequently, during the first auction, all the bidders have the same ex-ante expected profit for the second auction (for both models).

An important issue in the study of auctions with multi-dimensional signals is that of efficiency of auctions. In general, auctions with multi-dimensional signals have been shown to be inefficient [2]. Since our model involves two dimensional signals, we studied the efficiency property in [4]. This study shows that the efficiency of auctions in an agent-based setting is higher than that in an all human setting. This is because of the fact that an agent based setting leads to more competition than an all human setting [12].

6 Conclusions and future work

This paper has analyzed sequential auctions for heterogeneous objects with private and common values in an incomplete information setting. We first determined equilibrium strategies for each auction in a sequence using first-price, second-price, and English auction rules. On the basis of these equilibria, we determined the expected the selling price of each auction in a sequence. We showed that, for a given agenda, the expected selling price for each individual object is the same for all the three auction forms. However, since the sum of selling prices of the two objects depends on the auction agenda, we determined the agenda that maximizes this sum (i.e., the optimal agenda). We showed that it is optimal for the auctioneer to auction objects in the decreasing order of the dispersion of the order statistics of the surpluses.

There are several interesting directions for future work. First, we obtained revenue equivalence results for the case where the bidders are risk neutral, and where the signals for an object are drawn just before the auction for the object. In order to generalise our results, we will extend the analysis to the case where bidders are not risk neutral, and where the signals are affiliated. Second, we also intend to generalise our analysis to more than two objects.

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