Logics for Qualitative Coalitional Games

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Abstract

Qualitative Coalitional Games (QCGs) are a variant of coalitional games in which an agent's desires are represented as goals that are either satisfied or unsatisfied, and each choice available to a coalition is a set of goals, which would be jointly satisfied if the coalition made that choice. A coalition in a QCG will typically form in order to bring about a set of goals that will satisfy all members of the coalition. Our goal in this paper is to develop and study logics for reasoning about QCGs. We begin by introducing a logic for reasoning about "static" QCGs, where participants play a single game, and we then introduce and study *Temporal QCGs* (TQCGs), i.e., games in which a sequence of QCGs is played. In order to represent and reason about such games, we introduce a linear time temporal logic of QCGs, called $\mathcal{L}(TQCG)$. We give a complete axiomatisation of $\mathcal{L}(TQCG)$, use it to investigate the properties of TQCGs, identify its expressive power, establish its complexity, characterise classes of TQGCs with formulas from our logical language, and use it to formulate several (temporal) solution concepts for TQCGs.

Keywords: modal logic, temporal logic, coalitional games, repeated games

1 Introduction

There has recently been much interest in the development of logics for reasoning about game theoretic concepts [18]. One of the key reasons for this interest is that game theory is seen as one of the theoretical underpinnings to the multi-agent systems field [14, 19], and it is therefore very natural to consider the development of knowledge representation formalisms for game-like scenarios.

In this paper we focus on *Qualitative Coalitional Games* (QCGs) [20], a variation of coalitional games in which an agent's desires are represented as goals that are either satisfied or unsatisfied. Every coalition in a QCG has available to it a set of choices, where each choice is the set of goals that would be jointly satisfied if the coalition made that choice. A coalition in a QCG will typically form in order to bring about a set of goals that will satisfy all members of the coalition. The overall aim of this paper is to develop and study logics for reasoning about QCGs, in much the same way that logics for conventional coalitional games were studied in [1].

We begin, in the following section, with a short introduction to QCGs. In section 2.1 we define a logic for expressing properties of QCGs. We investigate the expressive power of this logic, defining a notion of simulation between QCGs and proving that the satisfaction of

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formulae of QCG logic is invariant under simulation. We give a complete axiomatization of the logic, and study the relationship between the logic and conventional modal logic.

We then go on to study *iterated* QCGs. The study of *repeated* games now forms a major component of the game theory literature [12, pp.133–161]. Perhaps the best-known example of such a repeated game is the iterated prisoners' dilemma, which has for example been studied both analytically [5, pp.353-358] and by means of competitions [4]. A standard distinction is made between iterated games with a *finite* horizon (which are repeated a fixed, pre-determined, commonly known number of times), and those with an *infinite* horizon (which are repeated infinitely often). These two types of iterated games tend to have rather different properties: repeated games with a finite horizon can often be treated as "large one shot games", while infinite horizon games often cannot be treated in this way. For example, in the finite horizon version of the prisoner's dilemma, a standard backward induction argument tells us that the dominant strategy is to defect at every round, and hence mutual cooperation seems as unpromising in such repeated games as it does in the one-shot prisoner's dilemma; in contrast, if the game has an infinite horizon, then mutual cooperation becomes a Nash equilibrium [5, p.358].

Given the role of game theory as a theoretical foundation of multi-agent systems, it seems that repeated games are of particular importance to the field. By-and-large, we are not interested in building multi-agent systems that will operate in a "one-shot" fashion: we typically want them to operate over time, often without a pre-defined termination time. Moreover, given the important role that *coalitional games* play in multi-agent systems [15, 16], it seems that repeated coalitional games are also likely to be of significance. However, comparatively little research has considered repeated coalitional games, or coalitional games played over time [10]. In particular, while there has been some work on formalising logical reasoning about coalitional games [1], little work focus on formalising reasoning about *repeated* coalitional games.

We therefore introduce *Temporal QCGs* (TQCGs): games in which QCGs are played repeatedly. In order to represent and reason about such games, we introduce $\mathcal{L}(TQCG)$, a linear time temporal logic of QCGs. We give a complete axiomatisation of $\mathcal{L}(TQCG)$, characterise its expressive power with respect to a type of simulation between TQCGs, establish the computational complexity of satisfiability for TQCGs, investigate the properties of TQCGs by characterising them as formulae in $\mathcal{L}(TQCG)$ and finally characterise some solution concepts of TQCGs in $\mathcal{L}(TQCG)$.

2 Qualitative Coalitional Games

We give a brief introduction to Qualitative Coalitional Games (QCGs): details may be found in [20]. A QCG contains a (non-empty, finite) set $\mathcal{A}=\{1,\ldots,n\}$ of *agents*. Each agent $i\in\mathcal{A}$ is assumed to have associated with it a (finite) set $\mathcal{G}_i$ of *goals*, drawn from a set of overall possible goals $\mathcal{G}$. The intended interpretation is that the members of $\mathcal{G}_i$ represent all the individual rational outcomes for i – intuitively, the outcomes that give it "better than zero utility". That is, agent i would be happy if *any* member of $\mathcal{G}_i$ were achieved – then it has "gained something". But, in QCGs, we are not concerned with preferences over individual goals. Thus, at this level of modelling, i is *indifferent* among the members of $\mathcal{G}_i$: it will be *satisfied* if *at least one* member of $\mathcal{G}_i$ is achieved, and *unsatisfied* otherwise. Note that cases where more than one of an agent's goals are satisfied are not an issue – an agent's aim will simply be to ensure that at least one of its goals is achieved, and there is no sense of an agent i attempting to satisfy as many members of $\mathcal{G}_i$ as possible.

A *coalition*, typically denoted by C, is simply a set of agents, i.e., a subset of $\mathcal{A}$. The *grand coalition* is the set of all agents, $\mathcal{A}$. We assume that each possible coalition has available to it a set of possible *choices*, where each choice intuitively characterises the outcome of one way that the coalition could cooperate. We model the choices available to coalitions via a *characteristic function* with the signature $\mathcal{V}: 2^{\mathcal{A}} \rightarrow 2^{2^{\mathcal{G}}}$. Thus, in saying that $G \in \mathcal{V}(C)$ for some coalition $C \subseteq \mathcal{A}$, we are saying that one choice available to the coalition C is to bring about *exactly* the goals in G. At this point, the reader might expect to see some constraints placed on characteristic functions. For example, at first sight the following *monotonicity* constraint might seem natural: $C \subseteq C'$ implies $\mathcal{V}(C) \subseteq \mathcal{V}(C')$. Although such a constraint is entirely appropriate for many scenarios, there are cases where such a constraint is not appropriate[1].

Bringing these components together, a qualitative coalitional game (QCG) is a tuple:

$$\Gamma = \langle \mathcal{A}, \mathcal{G}, \mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{V} \rangle$$

where:

- $\mathcal{A} = \{1, \ldots, n\}$ is a finite, non-empty set of *agents*;
- $\mathcal{G}$ is a finite, non-empty set of possible *goals*;
- $\mathcal{G}_i \subseteq \mathcal{G}$ is the set of goals for agent $i \in \mathcal{A}$; and
- $\mathcal{V}: 2^{\mathcal{A}} \rightarrow 2^{2^{\mathcal{G}}}$ is the characteristic function of the game.

Let $\mathbf{Q}$ be the class of all QCGs.

Example 1 *Let Γ_1 be the following QCG for a collection of agents and a collection of goals $\{g_1, \ldots\}$. Agent 1 is satisfied with g_1 and g_4, and agent 2 is satisfied with g_2 and g_3. The characteristic function, where C_1, C_2, C_3, C_4 are different coalitions:*

$$\begin{array}{ll} \mathcal{V}(C_1) = \{\ \{g_1, g_2\}\ \} & \mathcal{V}(C_2) = \{\ \{g_2, g_3\}, \{g_1\}\ \} \\ \mathcal{V}(C_3) = \{\ \{g_5, g_6\}\ \} & \mathcal{V}(C_4) = \{\ \{g_2, g_3\}, \{g_1\}, \{g_4\}\ \} \end{array}$$

We will make use of Γ_1 in later examples.

2.1 A Logic for QCGs

A logic tailor made for expressing properties of individual QCGs has not been formalised before. We now introduce such a logic. This logic will later be used as the assertion language, or state language, for the temporal logic we develop in section 3. The language is defined in two parts: $\mathcal{L}_c$ is the *satisfaction language*, and is used to express properties of choices made by agents. The basic constructs in this language are of the form sat_i, meaning "agent i is satisfied". The overall language $\mathcal{L}(QCG)$ is used for expressing properties of QCGs themselves. The main construct in this language is of the form $\langle C \rangle \varphi$, where φ is a formula of the satisfaction language, and means that C have a choice such that this choice makes φ true. For example, $\langle 3 \rangle (sat_1 \wedge sat_4)$ will mean that 3 has a choice that simultaneously satisfies agents 1 and 4.

[1]For example, consider a legal scenario in which certain coalitions are forbidden by monopoly or anti-trust laws.

Formally, the grammar φ_c defines the *satisfaction language* $\mathcal{L}_c$, while φ_q defines the *QCG language* $\mathcal{L}(QCG)$.

$$\begin{array}{lll}\varphi_c & ::= & sat_i \,|\, \neg\varphi_c \,|\, \varphi_c \vee \varphi_c \\ \varphi_q & ::= & \langle C\rangle\varphi_c \,|\, \neg\varphi_q \,|\, \varphi_q \vee \varphi_q\end{array}$$

where $i \in \mathcal{A}$ and $C \subseteq \mathcal{A}$. (We note some similarities between our logical language $\mathcal{L}(QCG)$ and Pauly's language for axiomatizing judgement aggregation procedures [13], although the motivation and use of the languages are quite different.)

We use the usual derived propositional connectives ($\wedge$, $\rightarrow$, $\leftrightarrow$) for both languages $\mathcal{L}_c$ and $\mathcal{L}(QCG)$, and in addition write $[C]\varphi$ to abbreviate $\neg\langle C\rangle\neg\varphi$. The formula $[C]\varphi$ will be defined to be true exactly when φ is a *necessary* consequence of the coalition C making a choice; φ will be true no matter *which* choice the coalition makes. When $C = \{a\}$ is a singleton, we sometimes write $\langle a\rangle$ and $[a]$ for $\langle C\rangle$ and $[C]$.

When $\Gamma = \langle \mathcal{A}, \mathcal{G}, \mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{V}\rangle$ is a QCG, $G \subseteq \mathcal{G}$ and $\varphi \in \mathcal{L}_c$, the satisfaction relation $\Gamma, G \models_Q \varphi$ is defined as follows:

$\Gamma, G \models_Q sat_i$ iff $\mathcal{G}_i \cap G \neq \emptyset$
$\Gamma, G \models_Q \neg\psi$ iff not $\Gamma, G \models_Q \psi$
$\Gamma, G \models_Q \psi_1 \vee \psi_2$ iff $\Gamma, G \models_Q \psi_1$ or $\Gamma, G \models_Q \psi_2$

When $\Gamma = \langle \mathcal{A}, \mathcal{G}, \mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{V}\rangle$ is a QCG and φ is a $\mathcal{L}(QCG)$ formula, $\Gamma \models_Q \varphi$ is defined as follows:

$\Gamma \models_Q \langle C\rangle\psi$ iff there is a $G \in \mathcal{V}(C)$ such that $\Gamma, G \models_Q \psi$
$\Gamma \models_Q \neg\psi$ iff not $\Gamma \models_Q \psi$
$\Gamma \models_Q \psi_1 \vee \psi_2$ iff $\Gamma \models_Q \psi_1$ or $\Gamma \models_Q \psi_2$

Example 2 *Let* Γ_1 *be as in Example* 1*. Then:*

$\Gamma_1 \models_Q \langle C_1\rangle(sat_1 \wedge sat_2)$
$\Gamma_1 \models_Q (\langle C_2\rangle sat_1 \wedge \langle C_2\rangle sat_2) \wedge \neg(\langle C_2\rangle(sat_1 \wedge sat_2))$
$\Gamma_1 \models_Q \neg(\langle C_3\rangle sat_1 \vee \langle C_3\rangle sat_2)$

Summarising, the satisfaction of agents is evaluated against a set of goals, while Boolean combinations of expressions referring to choices of coalitions are evaluated on a QCG Game Γ. The latter combinations will be the atomic assertions in our temporal framework of Section 3.

2.2 Expressive Power of L(QCG)

We now address the question of which properties of QCGs are definable in our language. It is clear from our language definition that what $\mathcal{L}(QCG)$ can express is which coalition can satisfy which set of agents concurrently. Note that we are not interested in *how* the coalitions make certain sets of agents satisfied, nor *why* an agent is satisfied (i.e., which goal satisfied him). We will now demonstrate that the properties of QCGs we can express in the language $\mathcal{L}(QCG)$ are exactly the properties closed under a notion of *QCG-simulation*. In other words, the language cannot differentiate two games Γ and Γ' iff they *QCG-simulate* each other.

Obviously, equivalence of models transcends mere isomorphism. In particular, the semantics of performing a *choice* seem to depend only on which agents are satisfied by the choice. For example, one could imagine a *mapping* from goals in one model to "equivalent" goals of the other, maybe collapsing two goals of the former model into one goal of the latter. However, such a relation between models does not capture all instances of equivalent models. What is needed is a relation between *sets* of goals. This motivates the following definition of a QCG-simulation as a relation between two models. It is only necessary to relate goals that can actually be chosen by some coalition. Furthermore, it only makes sense to relate models that are defined over the same set of agents.

A relation

$$Z \subseteq \bigcup_{C \subseteq \mathcal{A}} (\mathcal{V}(C) \times \mathcal{V}'(C))$$

is a *QCG-simulation* between two QCGs $\Gamma = \langle \mathcal{A}, \mathcal{G}, \mathcal{G}_1, \dots, \mathcal{G}_n, \mathcal{V} \rangle$ and $\Gamma' = \langle \mathcal{A}, \mathcal{G}', \mathcal{G}'_1, \dots, \mathcal{G}'_n, \mathcal{V}' \rangle$ iff the following conditions hold for all coalitions C.

1. If GZG' then $G \cap \mathcal{G}_i = \emptyset$ iff $G' \cap \mathcal{G}'_i = \emptyset$, for all i (the *satisfaction condition*)
2. For every $G \in \mathcal{V}(C)$ there is a $G' \in \mathcal{V}'(C)$ such that GZG' (Z is *total*)
3. For every $G' \in \mathcal{V}'(C)$ there is a $G \in V(C)$ such that GZG' (Z is *surjective*)

If there exists a QCG-simulation between two games Γ and Γ', we write $\Gamma \rightleftharpoons \Gamma'$. If $\Gamma \rightleftharpoons \Gamma'$, we can simulate any choice in one model with a choice in the other, and vice versa. This notion of simulation is somewhat similar to the notion of "alternating simulation" between alternating transition systems in [3].

Example 3 *Let Γ_2 be the QCG with the same agents as in Γ_1 (Example 1), goals $f_1, f_2, \dots$ such that agent 1 is satisfied in f_1 and f_3 and agent 2 is satisfied in f_2, f_3 and f_4, and the following characteristic function:*

$$\mathcal{V}(C_1) = \{\ \{f_3\}\ \} \quad \mathcal{V}(C_2) = \{\ \{f_2\}, \{f_1\}\ \}$$
$$\mathcal{V}(C_3) = \{\ \{f_5\}\ \} \quad \mathcal{V}(C_4) = \{\ \{f_1\}, \{f_2\}, \{f_4\}\ \}$$

Then $\Gamma_1 \rightleftharpoons \Gamma_2$. The relation Z consisting of the following pairs is a QCG-simulation between Γ_1 and Γ_2.

$$\langle \{g_1, g_2\}, \{f_3\} \rangle \quad \langle \{g_2, g_3\}, \{f_2\} \rangle \quad \langle \{g_1\}, \{f_1\} \rangle$$
$$\langle \{g_5, g_6\}, \{f_5\} \rangle \quad \langle \{g_2, g_3\}, \{f_4\} \rangle \quad \langle \{g_4\}, \{f_1\} \rangle$$

Note that Z is not a function, nor the inverse of a function.

We write $\Gamma \equiv \Gamma'$ iff $\forall_{\varphi \in \mathcal{L}(QCG)} [\Gamma \models_Q \varphi \Leftrightarrow \Gamma' \models_Q \varphi]$.

Theorem 1 *Satisfaction is invariant under QCG-simulation:*

$$\Gamma \rightleftharpoons \Gamma' \quad \Rightarrow \quad \Gamma \equiv \Gamma'$$

Proof. Let $\Gamma = \langle \mathcal{A}, \mathcal{G}, \mathcal{G}_1, \dots, \mathcal{G}_n, \mathcal{V} \rangle$ and $\Gamma' = \langle \mathcal{A}, \mathcal{G}', \mathcal{G}'_1, \dots, \mathcal{G}'_n, \mathcal{V}' \rangle$ with $\Gamma \rightleftharpoons \Gamma'$. First, we show that

$$GZG' \Rightarrow (\Gamma, G \models_Q \psi \Leftrightarrow \Gamma', G' \models_Q \psi) \tag{1}$$

for any ψ by induction over ψ. For the base case, let $\psi = sat_i$. $\Gamma, G \models_Q \psi$ iff $\mathcal{G}_i \cap G \neq \emptyset$ iff, by the satisfaction condition, $\mathcal{G}'_i \cap G' \neq \emptyset$ iff $\Gamma', G' \models_Q \psi$. The inductive step (negation and disjunction) is straightforward. We now show that

$$\Gamma \models_Q \varphi \Leftrightarrow \Gamma' \models_Q \varphi$$

for any φ by induction on φ. For the base case, let $\varphi = \langle C \rangle \psi$. For the direction to the right, if $\Gamma \models_Q \varphi$ then there is a $G \in \mathcal{V}(C)$ such that $\Gamma, G \models_Q \psi$. By totality of Z, there is a $G' \in \mathcal{V}'(C)$ such that GZG'. By (1), $\Gamma', G' \models_Q \psi$, and thus $\Gamma' \models_Q \varphi$. The direction to the left is symmetric: if $\Gamma' \models_Q \varphi$ there is a $G' \in \mathcal{V}'(C)$ such that $\Gamma', G' \models_Q \psi$; by surjectivity of Z there is a $G \in \mathcal{V}(C)$ such that GZG'; and by (1) $\Gamma, G \models_Q \psi$ and thus $\Gamma \models_Q \varphi$. The inductive step (negation and disjunction) is straightforward. ∎

The obvious question now is whether every pair of equivalent models are connected by a QCG-simulation. The answer is "yes".

Theorem 2 *If Γ, Γ' are defined over the same set of agents, then:*

$$\Gamma \rightleftharpoons \Gamma' \quad \Leftarrow \quad \Gamma \equiv \Gamma'$$

Proof. Let $\Gamma = \langle \mathcal{A}, \mathcal{G}, \mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{V} \rangle$ and $\Gamma' = \langle \mathcal{A}, \mathcal{G}', \mathcal{G}'_1, \ldots, \mathcal{G}'_n, \mathcal{V}' \rangle$ with $\Gamma \equiv \Gamma'$. With any coalition C and any choice $G \in \mathcal{V}(C)$, associate the set $S^C_G = \{i : G \cap \mathcal{G}_i \neq \emptyset\}$ of agents satisfied if C chooses G. Similarly for Γ': $T^C_{G'} = \{i : G' \cap \mathcal{G}'_i \neq \emptyset\}$ for any $G' \in \mathcal{V}'(C)$.

We define a QCG-simulation $Z : \Gamma \rightleftharpoons \Gamma'$ as follows: for every coalition C and pair of choices $G \in \mathcal{V}(C)$, $H \in \mathcal{V}'(C)$,

$$GZH \Leftrightarrow S^C_G = T^C_H$$

We must show that Z is total, i.e., that if $G \in \mathcal{V}(C)$, then there is a $H \in \mathcal{V}'(C)$ such that $S^C_G = T^C_H$. Suppose not: assume that $i \in S^C_G$ and $i \notin T^C_H$ for all $H \in \mathcal{V}'(C)$ (the argument is similar when $i \notin S^C_G$ and $i \in T^C_H$ for some $H \in \mathcal{V}'(C)$). Then $\Gamma \models_Q \langle C \rangle sat_i$ and $\Gamma' \models_Q \neg \langle C \rangle sat_i$, which contradicts the fact that $\Gamma \equiv \Gamma'$.

Similarly, we must show that Z is surjective, i.e., that if $H \in \mathcal{V}'(C)$, then there is a $G \in \mathcal{V}(C)$ such that $S^C_G = T^C_H$. Suppose not: assume that $i \in T^C_H$ and $i \notin S^C_G$ for all $G \in \mathcal{V}(C)$ (the argument is similar when $i \notin T^C_H$ and $i \in S^C_G$ for some $G \in \mathcal{V}(C)$). Then $\Gamma' \models_Q \langle C \rangle sat_i$ and $\Gamma \models_Q \neg \langle C \rangle sat_i$, which contradicts the fact that $\Gamma \equiv \Gamma'$.

Finally, we show that the satisfaction condition holds. If GZH, then $G \cap \mathcal{G}_i \neq \emptyset$ iff $i \in S^C_G$ iff, by the definition of Z, $i \in T^C_H$ iff $H \cap \mathcal{G}'_i \neq \emptyset$. ∎

2.3 Axiomatisation for QCGs

We define a Hilbert style axiomatisation of qualitative coalitional games, and prove its soundness and completeness. In the next section we then relate the logic to modal logic. We name our axiomatisation for QCGs $\mathbf{K}(QCG)$. This name emphasises the close resemblance to the modal system $\mathbf{K}$, which also indicates that our logic, is in a sense, a weakest basic system for QCGs, to which more sophisticated constraints can easily be added — such extensions are the topic of Section 4. The system $\mathbf{K}(QCG)$ over the language $\mathcal{L}(QCG)$ is defined as follows, where φ, ψ are arbitrary $\mathcal{L}(QCG)$ formulae, α, β are arbitrary $\mathcal{L}_c$ formulae and C an arbitrary coalition:

Prop$^-$	If φ is an $\mathcal{L}(QCG)$-instance of a propositional tautology, then φ is provable
K$^-$	$[C](\alpha \to \beta) \to ([C]\alpha \to [C]\beta)$ is provable
MP$^-$	If $\varphi, \varphi \to \psi$ are provable, then ψ is provable
Nec$^-$	If α is an $(\mathcal{L}_c)$ instance of a propositional tautology, then $[C]\alpha$ is provable

It is easy to see that the deduction theorem holds for $\mathbf{K}(QCG)$.

We will need the following properties of $\mathbf{K}(QCG)$. The proofs are straightforward for readers familiar with modal logic.

Lemma 1 *Let* $\alpha, \beta \in \mathcal{L}_c$*:*

1. $\vdash_{\mathbf{K}(QCG)} \langle C \rangle (\alpha \wedge \beta) \to \langle C \rangle \alpha$
2. $\vdash_{\mathbf{K}(QCG)} \langle C \rangle (\alpha \vee \beta) \to (\langle C \rangle \alpha \vee \langle C \rangle \beta)$
3. $\vdash_{\mathbf{K}(QCG)} (\langle C \rangle \alpha \wedge [C](\alpha \to \beta)) \to \langle C \rangle \beta$

Theorem 3 [*Soundness & Strong Completeness*] *For all* $\Phi \subseteq \mathcal{L}(QCG)$*,* $\varphi \in \mathcal{L}(QCG)$*:* $\Phi \models_Q \varphi \Leftrightarrow \Phi \vdash_{\mathbf{K}(QCG)} \varphi$

Proof. For soundness (the direction to the left), it is easy to see that the axioms are valid, and that the rules preserve logical consequence.

For completeness, let $\Psi \subseteq \mathcal{L}(QCG)$ be $\mathbf{K}(QCG)$ consistent. We show that Ψ is satisfied by some QCG. Let $\mathcal{A}$ be the set of agents and let $n = |\mathcal{A}|$. Let Δ be a $\mathcal{L}(QCG)$ maximal and $\mathbf{K}(QCG)$ consistent set containing Ψ (the proof of existence of such a set is the standard proof of Lindenbaum's lemma). We now construct $\Gamma = \langle \mathcal{A}, \mathcal{G}, \mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{V} \rangle$, intended to satisfy Ψ, as follows:

- $\mathcal{G} = \{sat_1, \ldots, sat_n\}$
- $\mathcal{G}_i = \{sat_i\}$, for each i
- $X \in \mathcal{V}(C) \Leftrightarrow \langle C \rangle \xi_X \in \Delta$, for any $X \subseteq \mathcal{G}$, where

$$\xi_X \equiv \bigwedge_{sat_i \in X} sat_i \wedge \bigwedge_{i \in \mathcal{A},\, sat_i \notin X} \neg sat_i$$

We show that

$$\Gamma \models_Q \gamma \Leftrightarrow \gamma \in \Delta$$

for any γ by structural induction over γ. For the base case, $\gamma = \langle C \rangle \alpha$ for some $\alpha \in \mathcal{L}_c$. Again, we use induction on the structure of α. For the (nested) base case, let $\alpha = sat_i$. For the direction to the right, if $\Gamma \models_Q \gamma$ then there is an $X \in \mathcal{V}(C)$ such that $\Gamma, X \models_Q \alpha$, i.e., there is an $X \subseteq \mathcal{G}$ such that $\langle C \rangle \xi_X \in \Delta$ and $X \cap \{sat_i\} \neq \emptyset$. Thus, $sat_i \in X$, and by Lemma 1.1, $\gamma = \langle C \rangle sat_i \in \Delta$. For the direction to the left, let $\langle C \rangle sat_i \in \Delta$. Let

$$\chi_i = \bigvee_{S \subseteq \mathcal{A}} \xi_{(S \cup \{sat_i\})}$$

$sat_i \rightarrow \chi_i$ is a $\mathcal{L}_c$ instance of a propositional tautology, so $[C](sat_i \rightarrow \chi_i) \in \Delta$ by *Nec*. By Lemma 1.3, $\langle C \rangle \chi_i \in \Delta$. By Lemma 1.2,

$$\bigvee_{S \subseteq \mathcal{A}} \langle C \rangle \xi_{(S \cup \{sat_i\})} \in \Delta$$

and thus $\langle C \rangle \xi_{S \cup \{sat_i\}} \in \Delta$ for some $S \subseteq \mathcal{A}$. It follows that $S \cup \{sat_i\} \in V(C)$, and since Γ, $(S \cup \{sat_i\}) \models_Q sat_i$ we get that $\Gamma \models_Q \langle C \rangle sat_i$ which concludes the proof of the direction to the left in the innermost induction proof. Both the inner and the outer induction steps (negation and disjunction) are straightforward. ∎

Note that the completeness proofs demonstrate that we do not need to represent multiple ways of satisfying an agent: one "satisfaction symbol" for each agent is enough.

2.4 The Non-Normal Modal Logic K(*QCG*)

Formulae of our language can be seen as formulae of modal logic. In this view, the logic **K**(QCG) is a modal logic. Particularly, **K**(QCG) is a *non-normal* modal logic (cf., e.g., [6]): it is not closed, with respect to the language of modal logic, under the syntactic closure conditions of normal modal logics – for example it does not contain all instances of the K axiom, or even all instances of propositional tautologies. In this section we make the relationship between our logic and modal logic precise.

Let $\mathcal{L}$ be the (multi-)modal language over propositions $\Theta = \{sat_i : i \in \mathcal{A}\}$ with one diamond $\langle C \rangle$ for each coalition C, defined in the usual way [6]. As a modal language, $\mathcal{L}$ allows, e.g., arbitrary nesting of diamonds. Clearly, $\mathcal{L}(QCG) \subset \mathcal{L}$.

Let $n = |\mathcal{A}|$ be the number of agents and $m = 2^n$ the number of coalitions. The modal language $\mathcal{L}$, and corresponding logical systems and semantic structures discussed below, are parameterised by m and Θ, which are henceforth taken as implicit.

K(QCG) is the modal system **K** (again, we omit the usual suffix m) with axioms and rules restricted to formulae from $\mathcal{L}(QCG)$: essentially formulae with modal rank equal to one and no occurrence of an atomic proposition outside the scope of a modal operator. So *is* the *logic* **K**(QCG) the logic **K** restricted to the language $\mathcal{L}(QCG)$, i.e., is a theorem of **K** in this restricted language also a theorem of **K**(QCG)? Although the theorem (of **K**) itself is in $\mathcal{L}(QCG)$, it does not automatically mean that every formula in the **K**-proof of the theorem is in $\mathcal{L}(QCG)$ and thus that the theorem is also a theorem of **K**(QCG). Intuitively though, the answer to the question might seem to be positive, and we will return to it shortly. First we go on to compare semantics of the two logics.

We can interpret our language $\mathcal{L}(QCG)$ in a Kripke structure $M = (S, R_{C_1}, \ldots, R_{C_m}, \pi)$ where $\pi : S \rightarrow 2^{\Theta}$. To avoid confusion, we use the symbol $\models_K$ for satisfaction/validity wrt. Kripke structures and use $\models_Q$ for QCGs.

We first define a mapping f from the class of QCGs to the class of Kripke structures. Let $\Gamma = \langle \mathcal{A}, \mathcal{G}, \mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{V} \rangle$. Define $f(\Gamma) = (M, t)$, where $M = (S, R_{C_1}, \ldots, R_{C_m}, \pi)$ is the smallest structure satisfying the following conditions:

- S contains a state t, called *the initial state*
- $\pi(t) = \emptyset$

- For every coalition C and $G \in \mathcal{V}(C)$,
  - S contains a state s_G
  - $\langle t, s_G \rangle \in R_C$
  - If $G \cap \mathcal{G}_i \neq \emptyset$, then $sat_i \in \pi(s_G)$

Lemma 2 *For all $\varphi \in \mathcal{L}(QCG)$ and any QCG Γ,*

$$\Gamma \models_Q \varphi \quad \Leftrightarrow \quad f(\Gamma) \models_K \varphi$$

Proof. We first show that for any $G \subseteq \mathcal{G}$ and any $\psi \in \mathcal{L}_c$

$$\Gamma, G \models_Q \psi \Leftrightarrow M, s_G \models_K \psi$$

by induction over ψ. For the base case, $\Gamma, G \models_Q sat_i$ iff $G \cap \mathcal{G}_i \neq \emptyset$ iff $sat_i \in \pi(s_G)$ iff $M, s_G \models_K sat_i$. The inductive step (negation and disjunction) is straightforward.

The main proof is by induction on φ. Let $f(\Gamma) = (M, t)$. For the base case, $\Gamma \models_Q \langle C \rangle \psi$ iff there is a $G \in \mathcal{V}(C)$ such that $\Gamma, G \models_Q \psi$ iff there is a $G \in \mathcal{V}(C)$ such that $M, s_G \models_K \psi$ iff there is a $s_G \in S$ such that $\langle t, s_G \rangle \in R_C$ and $M, s_G \models_K \psi$ iff $M, t \models_K \langle C \rangle \psi$. The inductive step (negation and disjunction) is straightforward. ∎

We next define a mapping g from the class of pointed Kripke structures for n agents over Θ to the class of QCGs. Let $M = (S, R_{C_1}, \ldots, R_{C_m}, \pi)$ and $s \in S$. $g(M, s) = \Gamma$, where $\Gamma = \langle \mathcal{A}, \mathcal{G}, \mathcal{G}_1, \ldots, \mathcal{G}_n, \mathcal{V} \rangle$ is defined as follows:

- $\mathcal{A} = \{1, \ldots, n\}$
- $\mathcal{G} = \{sat_1, \ldots, sat_n\}$
- $\mathcal{G}_i = \{sat_i\}$
- $(s, s') \in R_C \Leftrightarrow \pi(s') \in \mathcal{V}(C)$

Lemma 3 *For all Kripke structures M, states s in M and formulae $\varphi \in \mathcal{L}(QCG)$,*

$$g(M, s) \models_Q \varphi \Leftrightarrow M, s \models_K \varphi$$

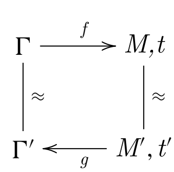
Proof. Let $\Gamma = g(M, s)$. First, we show that for any state s' of M and formula $\psi \in \mathcal{L}_c$,

$$(M, s') \models_K \psi \Leftrightarrow \Gamma, \pi(s') \models_Q \psi$$

The proof is by induction on ψ. For the base case, $(M, s') \models_K sat_i$ iff $sat_i \in \pi(s')$ iff $\pi(s') \cap \{sat_i\} \neq \emptyset$ iff $\Gamma, \pi(s') \models_Q sat_i$. The inductive step (negation and disjunction) is straightforward.

The main proof is by induction on φ. For the base case, $(M, s) \models_K \langle C \rangle \psi$ iff there is a $(s, s') \in R_C$ such that $(M, s') \models_K \psi$ iff there is a $(s, s') \in R_C$ such that $\Gamma, \pi(s') \models_Q \psi$ iff there is a $\pi(s') \in \mathcal{V}(C)$ such that $\Gamma, \pi(s') \models_Q \psi$ iff $\Gamma \models_Q \langle C \rangle \psi$. The inductive step (negation and disjunction) is straightforward. ∎

Lemmas 3 and 2 can be illustrated in the following diagram, where $\approx$ denotes logical equivalence between structures:



Thus, QCGs can (in the context of our logic) be seen as a certain type of Kripke structures, and the other way around. This also immediately answers the conjecture that $\mathbf{K}(QCG)$ and $\mathbf{K}$ coincides for our language $\mathcal{L}(QCG)$.

Theorem 4 *For all formulae* $\varphi \in \mathcal{L}(QCG)$

$$\vdash_{\mathbf{K}(QCG)} \varphi \quad \Leftrightarrow \quad \vdash_{\mathbf{K}} \varphi$$

Proof. Let $\varphi \in \mathcal{L}(QCG)$. The direction to the right is immediate: the theorems of $\mathbf{K}(QCG)$ are theorems of $\mathbf{K}$ since the axioms of $\mathbf{K}(QCG)$ are strictly included in the axioms of $\mathbf{K}$ and every rule application admissible in $\mathbf{K}(QCG)$ is also admissible in $\mathbf{K}$.

For the direction to the left, let $\vdash_{\mathbf{K}} \varphi$. By completeness of $\mathbf{K}$, $\models_K \varphi$. Let Γ be an arbitrary QCG. $f(\Gamma) \models_K \varphi$, and by Lemma 2 $\Gamma \models_Q \varphi$. Thus, $\models_Q \varphi$. By completeness of $\mathbf{K}(QCG)$ (Theorem 3), $\vdash_{\mathbf{K}(QCG)} \varphi$. ∎

While these results show a strong relationship between the logic of qualitative coalitional games and modal logic, there are important differences. Particularly, $\mathbf{K}(QCG)$ has a qualitative coalitional game semantics, but the *modal logic* $\mathbf{K}(QCG)$ (over the modal language $\mathcal{L}$) does not have a Kripke semantics. To be more precise: $\mathbf{K}(QCG)$ is not a complete *modal* logic with respect to any class of Kripke models or frames. As a modal logic, completeness of $\mathbf{K}(QCG)$ is defined in terms of the full language $\mathcal{L}$ of modal logic, rather than just the restricted language $\mathcal{L}(QCG)$. For example, for any $p \in \Theta$ the formula $p \vee \neg p$ (a formula in the language $\mathcal{L}$) is valid on all such mentioned classes, but cannot be derived using the axioms and rules of $\mathbf{K}(QCG)$.

2.5 Coalitional Games and Coalitional Game Logic

Let us briefly comment on the relationship between QCGs and standard coalitional games. A *coalitional game* (without transferable payoff) is an $(m+3)$-tuple [12, p.268]:

$$\Gamma = \langle N, \Omega, V, \sqsupseteq_1, \ldots, \sqsupseteq_m \rangle$$

where:

- $N = \{1, \ldots, m\}$ is a non-empty set of *players* (or *agents*);
- Ω is a non-empty set of *outcomes*;
- $V : (2^N \setminus \emptyset) \rightarrow 2^\Omega$ is the *characteristic function* of Γ, which for every non-empty coalition C defines the choices $V(C)$ available to C; and
- $\sqsupseteq_i \subseteq \Omega \times \Omega$ is a complete, reflexive, and transitive *preference relation*, for each agent $i \in N$.

Overlooking the fact that the characteristic function for coalitional games is not defined for the empty coalition, we can view QCGs as CGs: take $\Omega = 2^{\mathcal{G}}$ (possible outcomes equals possible combinations of goal satisfaction) and $X_1 \sqsupseteq_i X_2$ iff $X_1 \cap \mathcal{G}_i = \emptyset \Rightarrow X_2 \cap \mathcal{G}_i = \emptyset$ (X_1 is as least as good as X_2 for agent i iff satisfaction in X_2 implies satisfaction in X_1). In [1], *Coalitional Game Logic (CGL)* is introduced, for reasoning about (general) coalitional games. The syntax and semantics of CGL is quite similar to the logic introduced above; in particular the language is defined in two stages and only one level of modal nesting is allowed. A difference is that CGL has explicit references to outcomes in the language. The semantics, and implicitly the language, of CGL is defined as follows, where Γ is a coalitional game and ω an outcome in Γ:

$\Gamma, \omega \models \omega'$ iff $\omega = \omega'$
$\Gamma, \omega \models \neg\varphi$ iff not $\Gamma, \omega \models \varphi$
$\Gamma, \omega \models \varphi \vee \psi$ iff $\Gamma, \omega \models \varphi$ or $\Gamma, \omega \models \psi$
$\Gamma \models (\omega_1 \succeq_i \omega_2)$ iff $(\omega_1 \sqsupseteq_i \omega_2)$
$\Gamma \models \langle C \rangle \varphi$ iff $\exists \omega \in V(C)$ such that $\Gamma, \omega \models \varphi$
$\Gamma \models \neg\varphi$ iff not $\Gamma \models \varphi$
$\Gamma \models \varphi \vee \psi$ iff $\Gamma \models \varphi$ or $\Gamma \models \psi$

CGL is very expressive (at least when it comes to *finite* games such as QCGs). We could certainly have used it for the purpose of expressing properties of QCGs. However, CGL is a very general language for arbitrary coalitional games, and not very well suited for the special case of QCGs. For example, to express the sat_i proposition, we would have to write something like

$$\bigvee_{\omega \in \Omega, \mathcal{G}_i \cap \omega \neq \emptyset} \omega$$

Alternatively, a formula such as $\langle C \rangle sat_i$ can be expressed as

$$\bigvee_{\omega \in \Omega} (\omega \succeq_i \mathcal{G}_i \wedge \langle C \rangle \omega).$$

However, expressing properties of QCGs in CGL has two big disadvantages. First, formulae such as those above are defined relative to a given QCG. Different QCGs would give different formulae. Thus, the formulae above illustrate only a very weak form of logical characterisations of game properties. Second, the formulae above are exponentially long in the number of goals in the QCG. The language $\mathcal{L}(QCG)$, on the other hand, is tailor made for QCGs.

3 Temporal QCGs

In principle there are many ways to temporalise QCGs. As a first investigation, we assume a linear time model, in which, at each time point, a (possibly different) QCG Γ is played. A *temporal qualitative coalitional game* ($TQCG$) is then a triple

$$M = \langle S, \sigma, Q \rangle$$

where:

- S is a set of *states*;
- $\sigma : \mathbb{N} \to S$ associates a state $\sigma(u)$ with every natural number time point $u \in \mathbb{N}$; and
- $Q : S \to \mathbf{Q}$, where $\mathbf{Q}$ is the class of all QCGS, is a function associating a qualitative coalitional game $Q(s) = \langle \mathcal{A}^s, \mathcal{G}^s, \mathcal{G}_1^s, \ldots, \mathcal{G}_n^s, \mathcal{V}^s \rangle$ with every state s.

We will make just an additional restriction: that the set of agents and overall goals remains the same in all states. Formally, $\forall s, t \in S$: $\mathcal{A}^s = \mathcal{A}^t$ and $\mathcal{G}^s = \mathcal{G}^t$. This does not mean that an agent's goals must remain fixed, however: we allow for the possibility that an agent has different goals in different states. We also admit the possibility of a coalition having different choices in different states. Since the sets of agents and overall goals are fixed across all states, we will simply denote these by $\mathcal{A}$ and $\mathcal{G}$ respectively, omitting the state index.

3.1 A Logic for TQCGs

To express properties of TQCGs, we extend the QCG language $\mathcal{L}(QCG)$ with the standard temporal operators of linear-time temporal logic: $\bigcirc$ – "next", $\Diamond$ – "eventually", $\Box$ – "always in the future", and $\mathcal{U}$ – "until" [11]. Formally, the formulae φ_t of the language $\mathcal{L}(TQCG)$ are defined as follows.

$$\varphi_t \quad ::= \quad \langle C \rangle \varphi_c \,|\, \neg\varphi_t \,|\, \varphi_t \vee \varphi_t \,|\, \varphi_t \mathcal{U} \varphi_t \,|\, \bigcirc\varphi_t$$

where the formulae φ_c of the satisfaction language are defined as before. We again assume the usual derived propositional connectives, in addition to $\Diamond\varphi$ for $\top\mathcal{U}\varphi$ and $\Box\varphi$ for $\neg\Diamond\neg\varphi$. Moreover, we define $\Box^*\varphi$ as $(\varphi \wedge \Box\varphi)$ (φ is true now and always in the future), and $\Diamond^*\varphi = \neg\Box^*\neg\varphi$ (φ is true now or sometime in the future).

When $M = (S, \sigma, Q)$ is a TQCG, $u \in \mathbb{N}$, and φ is a $\mathcal{L}(TQCG)$ formula, the satisfaction relation $M, u \models_T \varphi$ is defined as follows:

$M,u \models_T \varphi$ iff $Q(\sigma(u)) \models_Q \varphi$, when $\varphi \in \mathcal{L}(QCG)$
$M,u \models_T \neg\psi$ iff not $M,u \models_T \psi$
$M,u \models_T \psi_1 \vee \psi_2$ iff $M,u \models_T \psi_1$ or $M,u \models_T \psi_2$
$M,u \models_T \bigcirc\psi$ iff $M,u+1 \models_T \psi$
$M,u \models_T \psi_1 \mathcal{U} \psi_2$ iff there is some i such that $M,u+i \models_T \psi_2$ and for all $0 < j < i$ $M,u+j \models_T \psi_1$

For instance, the following formula of $\mathcal{L}(TQCG)$ means that eventually, agent 3 can always choose to satisfy agents 1 and 4 simultaneously:

$$\Diamond\,\Box\langle 3 \rangle(sat_1 \wedge sat_4).$$

We will henceforth use $\mathcal{L}(TQCG)$ to refer to both the language, and the logic we have defined over this language.

3.2 An Example

We illustrate the logic by a small example. We focus here on temporal properties of goal satisfaction, rather than on contrasting the power of different coalitions (i.e., on which coalitions are likely to form). The latter type of properties are discussed in detail in Section 4.

We model the following situation by a temporal qualitative coalitional game. Two agents 1 and 2 both need to use the same resource, say a web service, from time to time. Sometimes an agent needs *read* access, and sometimes it needs *write* access. The integrity of the web service is violated if at the same time either i) both read and write accesses are granted (inconsistent reads), ii) two write accesses are granted (inconsistent writes) or iii) no read access and no write access are granted (inefficiency).

Let $M = (S, \sigma, Q)$ be a TQCG where S is some infinite set of states, and σ and Q are such that the following holds for $Q(\sigma(k)) = \langle \mathcal{A}, \mathcal{G}, \mathcal{G}_1^{\sigma(k)}, \mathcal{G}_2^{\sigma(k)}, \mathcal{G}_{sys}^{\sigma(k)}, \mathcal{V}^{\sigma(k)} \rangle$ for any $k \geq 0$:

- $\mathcal{A} = \{1, 2, sys\}$. We model the agents as players 1 and 2, and the web service as player sys ("the system").
- $\mathcal{G} = \{r, w_1, w_2, ok\}$. That each of these goals are achieved means that right now:

  r : every client is granted read access
  w_i : agent i is granted write access
  ok : the integrity of the system is not violated

- $\mathcal{G}_1^{\sigma(k)} = \begin{cases} \{w_1\} & \text{if } k \mod 5 = 0 \\ \{r, w_1\} & \text{otherwise} \end{cases}$
  Agent 1 needs to have write access at every fifth point in time. At any other point in time, it is happy as long as it is not left idle, i.e., if it has either read or write access.
- $\mathcal{G}_2^{\sigma(k)} = \begin{cases} \{w_2\} & \text{if } k \mod 3 = 0 \\ \{r, w_2\} & \text{otherwise} \end{cases}$
  Agent 2's goals are similar to agent 1's, except that it needs write access at every third instead of fifth time point.
- $\mathcal{G}_{sys}^{\sigma(k)} = \{ok\}$. The system is satisfied if the integrity is not violated. Note that $\mathcal{G}_{sys}^{\sigma(k)}$ does not depend on k; the system's goal does not vary over time.
- $\mathcal{V}^{\sigma(k)}(\{sys\}) = \left\{ \begin{array}{c} \emptyset, \{w_1, ok\}, \{w_2, ok\}, \{r, ok\}, \\ \{w_1, w_2\}, \{w_1, r\}, \{w_2, r\}, \{w_1, w_2, r\} \end{array} \right\}$. The web service can satisfy certain sets of goals. These sets does not necessarily include the goal that the integrity is not violated. We have implicitly defined what the desired behaviour of the system is: each choice involving *ok* implements a choice in which the integrity invariant is not violated. Note that the choices available to the system do not vary over time. In this example we don't care about $\mathcal{V}^s(C)$ when C is a coalition different from $\{sys\}$ (proper *coalitional* ability will be studied in Section 4).

The following properties hold in $M, 1$.

1. $\Box\langle sys \rangle sat_{sys}$. The system can maintain integrity.
2. $\Box(\langle sys \rangle sat_1 \wedge \langle sys \rangle sat_2)$. Agent 1 can always be satisfied by the system, and the same for agent 2.
3. $\Box\langle sys \rangle (sat_1 \wedge sat_2)$. Agents 1 and 2 can always be *simultaneously* satisfied by the system.
4. $\Diamond\neg\langle sys \rangle (sat_1 \wedge sat_2 \wedge sat_{sys})$. The system cannot always satisfy agents 1 and 2 simultaneously without violating the integrity of the system.
5. $\Box\langle sys \rangle \neg sat_1$. The system can keep agent 1 unsatisfied forever.
6. $\Box\Diamond\langle sys \rangle (\neg sat_1 \wedge \neg sat_2 \wedge sat_{sys})$. It is infinitely often the case that the system can make agents 1 and 2 unsatisfied at the same time without violating integrity (this happens at multiples of fifteen).
7. $\langle sys \rangle (\neg sat_1 \wedge \neg sat_2 \wedge sat_{sys}) \mathcal{U} \neg\langle sys \rangle (sat_1 \wedge sat_2 \wedge sat_{sys})$. At some point in the future (i.e., $u = 15$), the system is unable to jointly satisfy agents 1 and 2 without violating integrity. Up until that time, *sys* is always able to make agents 1 and 2 jointly *un*satisfied (note that we evaluate the formula in $M, 1$).
8. $\psi \wedge \Box(\psi \rightarrow \overbrace{\bigcirc(\neg\psi \wedge \cdots \bigcirc(\neg\psi \wedge}^{\text{"}\bigcirc(\neg\psi\wedge\text{" 14 times}} \bigcirc\psi)\cdots))$ where $\psi = \langle sys \rangle (\neg sat_1 \wedge \neg sat_2 \wedge sat_{sys})$. The system can make agents 1 and 2 jointly unsatisfied without violating integrity at time points that are multiples of fifteen, and at no other time points.

As a final point, observe that from a logical point of view, the situations at time points 3 and 5 are indistinguishable:

$$Q(\sigma(3)) \rightleftharpoons Q(\sigma(5))$$

This once again demonstrates that our logic abstracts away from *how* a coalition satisfies individuals: obviously, to satisfy agent 1 for instance, *sys* has to make different choices in $\sigma(3)$ from those in $\sigma(5)$.

3.3 Expressive Power of TQCGs

The notion of simulation for QCGs (Section 2.2) can be naturally lifted to the temporal case. When $M = (S, \sigma, Q)$ and $M' = (S', \sigma', Q')$ are TQCGS and $k \geq 0$, we define

$$\begin{array}{lcl} M, k \rightleftharpoons_T M', k & \Leftrightarrow & Q(\sigma(k)) \rightleftharpoons Q'(\sigma'(k)) \\ M \rightleftharpoons_T M' & \Leftrightarrow & \forall_{n \geq 0} M, n \rightleftharpoons_T M', n \end{array}$$

The notion of elementary equivalence for TQCGS over the language $\mathcal{L}(TQCG)$ can be defined as follows. $M, k \equiv M', k$ iff, for every $\varphi \in \mathcal{L}(TQCG)$, $M, k \models_T \varphi$ iff $M', k \models_T \varphi$. $M \equiv M'$ iff $M, k \equiv M', k$ for every $k \geq 0$.

Theorem 5 *For all TQCGs* M, M'*:* $M \rightleftharpoons_T M' \Leftrightarrow M \equiv M'$

Note that in the temporal case, the fact that $M, k \rightleftharpoons_T M', k$ is not sufficient for $M, k \equiv M', k$ to hold.

3.4 Satisfiability

The *satisfiability* problem for $\mathcal{L}(TQCG)$ is as follows: given a formula $\varphi \in \mathcal{L}(TQCG)$, does there exist a TQCG M and $u \in \mathbb{N}$ such that $M, u \models \varphi$?

Theorem 6 *The satisfiability problem for* $\mathcal{L}(TQCG)$ *is* PSPACE*-complete.*

Proof. Membership of PSPACE follows from the fact that satisfiability for LTL+K_n (the fusion of LTL and multi-modal K) is PSPACE-complete [8]. Any $\mathcal{L}(TQCG)$ formula is also a formula of LTL+K_n, interpreting sat_i as Boolean variable. (The reverse is not the case, of course.) But the relationship is more than merely syntactic: for all $\varphi \in \mathcal{L}$(TQCG):

$$\varphi \text{ is } \mathcal{L}(\text{TQCG})\text{-satisfiable iff } \varphi \text{ is LTL+}K_n \text{ satisfiable}$$

(Notice that we are here quantifying over $\mathcal{L}$(TQCG), formulae, not LTL+K_n formulae.) Given an LTL+K_n interpretation that satisfies $\varphi \in \mathcal{L}$(TQCG), it is straightforward to extract from this a TQCG that satisfies φ.

For PSPACE-hardness, we reduce LTL satisfiability [17]. First, let $\varphi^\dagger$ denote the result of systematically replacing each Boolean variable p that occurs in LTL formula φ with a symbol sat_p. Next, we define a transformation τ, from LTL formulae to $\mathcal{L}$(TQCG), as follows:

$$\tau(\varphi) = \begin{cases} [1](\varphi^\dagger) & \text{where } \varphi \text{ is propositional} \\ \#\tau(\psi) & \text{where } \varphi = \#\psi \text{ and } \# \in \{\neg, \bigcirc\} \\ \tau(\psi)\#\tau(\chi) & \text{where } \varphi = \psi\#\chi \text{ and } \# \in \{\vee, \mathcal{U}\} \end{cases}$$

Finally, given an LTL formula φ, the $\mathcal{L}$(TQCG) instance φ^τ we create is:

$$\varphi^\tau = (\langle 1 \rangle \top) \wedge (\Box \langle 1 \rangle \top) \wedge \tau(\varphi)$$

We claim that φ is LTL satisfiable iff φ^τ is $\mathcal{L}$(TQCG) satisfiable; the proof is an easy induction. The key point is that the choice sets of agent 1 in any TQCG satisfying φ^τ define an appropriate valuation for propositional variables in a corresponding LTL interpretation satisfying φ, and *vice versa* (remember that $[1]\varphi$ iff φ holds for all of 1's choices). The first two conjuncts in the definition of φ^τ ensure that such a choice set always exists. ∎

3.5 Axiomatisation for TQGCs

The system **KLin**($TQCG$) over the language $\mathcal{L}(TQCG)$ is defined as follows, where φ, ψ are arbitrary $\mathcal{L}(TQCG)$ formulae, A,B are arbitrary $\mathcal{L}(QCG)$ formulae, α, β are arbitrary $\mathcal{L}_c$ formulae and C an arbitrary coalition. For simplicity, we write $\vdash_T$ instead of $\vdash_{\mathbf{KLin}(TQCG)}$ for derivability in **KLin**($TQCG$).

Prop$^-$	If A is an ($\mathcal{L}(QCG)$) instance of a propositional tautology, then $\vdash_T A$
K$^-$	$\vdash_T [C](\alpha \to \beta) \to ([C]\alpha \to [C]\beta)$
MP$^-$	If $\vdash_T A$ and $\vdash_T A \to B$, then $\vdash_T B$
Nec$^-$	If α is an ($\mathcal{L}_c$) instance of a propositional tautology, then $\vdash_T [C]\alpha$
*A*1	$\vdash_T \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
*A*2	$\vdash_T \bigcirc\neg\varphi \leftrightarrow \neg\bigcirc\varphi$
*A*3	$\vdash_T \bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$
*A*4	$\vdash_T \Box\varphi \to (\bigcirc\varphi \wedge \bigcirc\Box\varphi)$
*A*5	$\vdash_T \Box(\varphi \to \bigcirc\varphi) \to (\bigcirc\varphi \to \Box\varphi)$
*U*1	$\vdash_T \varphi\mathcal{U}\psi \to \Diamond\psi$
*U*2	$\vdash_T \varphi\mathcal{U}\psi \leftrightarrow \bigcirc\psi \vee (\bigcirc\varphi \wedge \bigcirc(\varphi\mathcal{U}\psi))$
Prop	If φ is an ($\mathcal{L}(TQCG)$) instance of a propositional tautology, then $\vdash_T \varphi$
MP	If $\vdash_T \varphi$ and $\vdash_T \varphi \to \psi$, then $\vdash_T \psi$
Nec	If $\vdash_T \varphi$ then $\vdash_T \Box\varphi$

Axioms *Prop*$^-$ and *K*$^-$ and rules *MP*$^-$ and *Nec*$^-$ say that every **K**(QCG)-theorem is also a **KLin**($TQCG$)-theorem. The sub-system consisting of axioms *A*1–*U*2 and rules *Prop*–*Nec* is a version (with $\mathcal{L}(QCG)$ formulae in place of atomic propositions) of an axiomatisation of linear time logic proved to be be sound and complete in [9].

Theorem 7 [*Soundness & Completeness*] *For all* $\varphi \in \mathcal{L}(TQCG)$: $\vdash_T \varphi \quad \Leftrightarrow \quad \models_T \varphi$

Proof. The logic **KLin**($TQCG$) is what Finger and Gabbay [7] calls a *temporalisation* of **K**(QCG): the language of **KLin**($TQCG$) has atomic **K**(QCG) formulae in place of atomic propositions; the semantic structures of **KLin**($TQCG$) identify a semantic structure for **K**(QCG) at each time point used to interpret **K**(QCG) formulae; and the axioms/rules of **KLin**($TQCG$) are the axioms/rules of the temporal logic for temporal formulae in addition to axioms/rules of **K**(QCG) for **K**(QCG) formulae.

Finger and Gabbay show that the temporalisation of a sound and complete system is sound and complete. It should be noted that our definition of **KLin**($TQCG$) differs from the definition of a temporalisation in [7] by the following. First, we do not have past-time operators in our language. The expressive power is nevertheless the same [9]. Second, we use a slightly different temporal axiomatisation. Neither of these differences change the soundness and completeness proof in [7] in any significant degree. The theorem thus follows immediately from Theorem 3. ∎

4 Characterizing TQCGs

In this section, we investigate the axiomatic characterisation of various classes of TQCGs. As usual, in saying that a formula scheme φ characterises a property P of models, we mean that φ is valid in a model M iff M has property P; if only the right-to-left part of this biconditional holds, then we say property P implies φ. Also note that for an $\mathcal{L}(TQCG)$ formula φ, to say that φ is valid in a class of models, is the same as saying that $\Box^*\varphi$ is valid in that class.

4.1 Basic Correspondences

Let $h^s(C)$ denote the set of all agents that could possibly be satisfied (not necessarily jointly) by coalition C in state s:

$$h^s(C) = \{i : i \in \mathcal{A} \text{ \& } \exists G \in \mathcal{V}^s(C), \mathcal{G}_i^s \cap G \neq \emptyset\}$$

The "h" here is for "happiness": we think of $h^s(C)$ as all the agents that C could possibly make happy in s. Thus the semantic property $i \in h^s(C)$ is a counterpart to the syntactic expression $\langle C \rangle sat_i$.

The first property on models that we consider is the *persistence of happiness* (PH): if coalition C can make i happy in a state s, they can make i happy in the state immediately following s.

$$\forall u \in \mathbb{N}, (i \in h^{\sigma(u)}(C)) \rightarrow (i \in h^{\sigma(u+1)}(C)) \qquad (PH)$$

We have the following characterisation.

Lemma 4 *$\langle C \rangle sat_i \rightarrow \bigcirc \langle C \rangle sat_i$ characterises PH.*

In the same way, we can characterise the persistence of *un*happiness: property PU says that if C cannot make i happy in a state s, then they cannot make i happy in the state t that immediately follows s.

$$\forall u \in \mathbb{N}, (i \notin h^{\sigma(u)}(C)) \rightarrow (i \notin h^{\sigma(u+1)}(C)) \qquad (PU)$$

Lemma 5 *$\neg \langle C \rangle sat_i \rightarrow \bigcirc \neg \langle C \rangle sat_i$ characterises PU.*

Now consider the following two constraints. The first, EH, says that eventually, C will be able to make i happy.

$$\exists u \in \mathbb{N}, (i \in h^{\sigma(u)}(C)) \qquad (EH)$$

Notice that in the terminology of reactive systems, this is a *fairness* or *response* property [11, p.288].

Lemma 6 *$\Diamond^* \langle C \rangle sat_i$ characterises EH.*

The obvious counterpart to *EH* is of course the property *EU*, which states that, eventually, C will be unable to satisfy i.

$$\exists u \in \mathbb{N}, (i \notin h^{\sigma(u)}(C)) \tag{EU}$$

Lemma 7 *$\Diamond^* \neg\langle C\rangle sat_i$ characterises EU.*

Combining these properties, we get the following.

Lemma 8 *PH and EH together imply $\Diamond^* \Box^* \langle C\rangle sat_i$, while properties PU and EU together imply $\Diamond^* \Box^* \neg\langle C\rangle sat_i$.*

Finally, we consider *safety* properties. The constraint AH says that C can always make i happy, while the constraint AU says that C can never make i happy.

$$\forall s \in S, (i \in h^s(C)) \tag{AH}$$

$$\forall s \in S, (i \notin h^s(C)) \tag{AU}$$

The characterizations are as follows. (Note that there are some obvious implications between these and other properties that we do not list explicitly – e.g., *AH* implies both *EH* and *PH*.)

Lemma 9 *$\langle C\rangle sat_i$ characterises AH, and $\neg\langle C\rangle sat_i$ characterises AU.*

4.2 Basic Properties of Choice Sets

Three obvious constraints that we might consider relate to whether or not a particular coalition C has any "real" choices. The first, *ECS*, says that C never has any choices.

$$\forall s \in S, \mathcal{V}^s(C) = \emptyset \tag{ECS}$$

The second says that C always has a meaningful choice.

$$\forall s \in S, \exists G \in \mathcal{V}^s(C), G \neq \emptyset \tag{NECS}$$

The third says that C can choose everything.

$$\forall s \in S, \mathcal{G} \in \mathcal{V}^s(C) \tag{CCS}$$

Lemma 10 *Any model that satisfies ECS also satisfies AU, and so ECS implies $\neg\langle C\rangle sat_i$, while any model that satisfies CCS also satisfies AH, and so CCS implies $\langle C\rangle sat_i$.*

Note that *NECS* alone does not have any characterization: however, when combined with other properties, below, we will see that it has a role.

4.3 Static Goal Sets and Choices

Two other simple properties are that the goal sets for each agent and the choice sets for each coalition are guaranteed to remain unchanged. We get the following two constraints, stating that agent i's goal sets are static (constraint *SGS*) and that coalition C's choices remain static (*SC*).

$$\forall s, s' \in S, (\mathcal{G}_i^s = \mathcal{G}_i^{s'}) \qquad (SGS)$$

$$\forall s, s' \in S, (\mathcal{V}^s(C) = \mathcal{V}^{s'}(C)) \qquad (SC)$$

Taken separately, there does not seem too much we can say about static goal sets and static choice sets. However, taken together, we get the following.

Lemma 11 *Any model satisfying both SGS and SC also satisfies PH and PU, and as a consequence, SGS and SC together imply* $\langle C\rangle sat_i \leftrightarrow \bigcirc\langle C\rangle sat_i$.

Note that we do not immediately derive a characterisation here. It is perfectly well possible that $\langle C\rangle sat_i \leftrightarrow \bigcirc\langle C\rangle sat_i$ is true in a model M not just because all agents' goals and all coalitions' choices stay fixed, but because there is an intricate interplay going on between for instance an agent changing some of his goals, while at the same time, the coalition C 'synchronously' changing its options. Note that in our example of Section 3.2 for instance, both (*SGS*) and (*SC*) are true for $C = \{sys\}$ and $i = sys$, so that, indeed, $\langle\{sys\}\rangle sat_{sys} \leftrightarrow \bigcirc\langle\{sys\}\rangle sat_{sys}$. On the other hand, taking $C = \{sys\}$ and $i = 1$, we don't have (*SGS*) and (*SC*), although we still have $\langle\{sys\}\rangle sat_1 \leftrightarrow \bigcirc\langle\{sys\}\rangle sat_1$.

4.4 Dynamic Goal Sets

There are several properties we can investigate with respect to goal sets. First, suppose that agent i's goal set is guaranteed to *monotonically decrease* over time. Roughly, this condition means that every agent is guaranteed to get *no easier* to satisfy over time. Formally, this condition on a model M is defined by the following property.

$$\forall u \in \mathbb{N}\ (\mathcal{G}_i^{\sigma(u+1)} \subseteq \mathcal{G}_i^{\sigma(u)}) \qquad (MDGS)$$

Lemma 12 *Any model satisfying SC and MDGS will satisfy PU, and hence SC and MDGS together imply* $\neg\langle C\rangle sat_i \rightarrow \bigcirc\neg\langle C\rangle sat_i$.

Suppose we make this condition is *strict*, so that an agent i is guaranteed to get strictly harder to satisfy over time. This condition is defined by the following further constraint, in addition to *MDGS*.

$$\forall u \in \mathbb{N} \quad \begin{array}{l}(\mathcal{G}_i^{\sigma(u)} = \emptyset) \vee \\ (\exists v \in \mathbb{N} : (v > u) \wedge (\mathcal{G}_i^{\sigma(v)} \subset \mathcal{G}_i^{\sigma(u)}))\end{array} \qquad (SMDGS)$$

We get the following.

Lemma 13 *Any model satisfying SC, MDGS, and SMDGS will also satisfy PU and EU, and so SC, MDGS, and SMDGS together imply* $\Diamond^* \Box^* \neg\langle C\rangle sat_i$.

Now suppose agent i has monotonically *increasing* goal sets: that is, agent i gets *no harder* to satisfy over time.

$$\forall u \in \mathbb{N}, (\mathcal{G}_i^{\sigma(u)} \subseteq \mathcal{G}_i^{\sigma(u+1)}) \qquad (MIGS)$$

We get the following.

Lemma 14 *Any model satisfying both SC and MIGS will satisfy constraint PH, and hence SC and MIGS together imply* $\langle C\rangle sat_i \rightarrow \bigcirc\langle C\rangle sat_i$.

The associated strictness constraint is as follows.

$$\forall u \in \mathbb{N} \quad \begin{array}{l} (\mathcal{G}_i^{\sigma(u)} = \mathcal{G}) \vee \\ (\exists v \in \mathbb{N} : (v > u) \wedge (\mathcal{G}_i^{\sigma(u)} \subset \mathcal{G}_i^{\sigma(v)})) \end{array} \qquad (SMIGS)$$

We might expect that *SC*, *MIGS*, and *SMIGS* together imply the validity of the formula scheme $\Diamond^* \Box^* \langle C\rangle sat_i$, but this is not the case. A counter example is given by a model that satisfies the empty choice set property (*ECS*) for coalition *C*, as described above. If we add the constraint that the choices for *C* are non-empty (*NECS*), however, then we get the following.

Lemma 15 *Any model that satisfies NECS, SC, MIGS, and SMIGS also satisfies PH and EH, and hence the following formula scheme will be valid in any model satisfying NECS, SC, MIGS, and SMIGS:* $\Diamond^* \Box^* \langle C\rangle sat_i$.

4.5 Dynamic Choices

We can also consider the ways in which the choices available to coalitions may change over time. Analogously to *MIGS* and *MDGS*, we can define properties *MICS* and *MDCS*, which say that the sets of choices available to coalition *C* monotonically increase and decrease respectively.

$$\forall u \in \mathbb{N}, (\mathcal{V}^{\sigma(u)}(C) \subseteq \mathcal{V}^{\sigma(u+1)}(C)) \qquad (MICS)$$

$$\forall u \in \mathbb{N}, (\mathcal{V}^{\sigma(u+1)}(C) \subseteq \mathcal{V}^{\sigma(u)}(C)) \qquad (MDCS)$$

Notice that taken together, these two conditions imply static choice sets (*SC*). Alone, the properties do not have any characterisation, but axioms emerge when we make assumptions about goal sets.

Lemma 16 *(1) Any model satisfying MICS and SGS will satisfy constraint PH, and hence MICS and SGS together imply* $\langle C\rangle sat_i \rightarrow \bigcirc\langle C\rangle sat_i$.
(2) Any model satisfying MDCS and SGS will satisfy constraint PU, and hence MICS and SGS together imply $\neg\langle C\rangle sat_i \rightarrow \bigcirc\neg\langle C\rangle sat_i$.

The associated strictness condition for increasing choice sets is:

$$\begin{array}{l} \forall u \in \mathbb{N}, \forall G_1 \in \mathcal{V}^{\sigma(u)}(C)(G_1 = \mathcal{G}) \vee \\ \quad (\exists v \in \mathbb{N}, \exists G_2 \in \mathcal{V}^{\sigma(v)}(C), (v > u) \wedge (G_1 \subset G_2)) \end{array} \qquad (SMICS)$$

Lemma 17 *Any model satisfying MICS, SGS, and SMICS or MICS, MIGS, and SMICS will also satisfy constraints PH and EH, and hence MICS, SGS, and SMICS together imply* $\Diamond^* \Box^* \langle C\rangle sat_i$.

The strictness condition for monotonically *de*creasing choice sets is:

$$\begin{array}{l} \forall u \in \mathbb{N}, \forall G_1 \in \mathcal{V}^{\sigma(u)}(C) \\ \quad (G_1 = \emptyset) \vee \\ \quad (\exists v \in \mathbb{N}, \exists G_2 \in \mathcal{V}^{\sigma(v)}(C), (v > u) \wedge (G_2 \subset G_1)) \end{array} \qquad (SMIGS)$$

Lemma 18 *Any model satisfying MDCS, SGS, and SMDCS or MDCS, MDGS, and SMDCS will also satisfy constraints PU and EU, and hence MDCS, SGS, and SMDCS together imply* $\Diamond^* \Box^* \neg\langle C\rangle sat_i$, *as do MDCS, MDGS, and SMDCS.*

4.6 Solution Concepts

In [20], a range of different solution concepts were defined for QCGs. It should be clear that many of the solution concepts of [20] can be characterised via formulae of $\mathcal{L}(QCG)$. For example, a basic solution concept is that of a *successful* coalition – one that has a choice available such that this choice satisfies all its members [20, p.47]. We can characterise this via a predicate $succ(C)$, as follows.

$$succ(C) \equiv \langle C\rangle(\bigwedge_{i\in C} sat_i)$$

Similarly, the notion of a minimal coalition (one such that no subset of the coalition is successful [20, p.51]) may be captured as follows.

$$min(C) \equiv \bigwedge_{C'\subseteq C} \neg succ(C')$$

Thus the core of a coalition being non-empty [20, p.54] may be captured as follows:

$$cne(C) \equiv (succ(C) \wedge min(C))$$

The idea of agent i being a veto player for agent j [20, p.57] is defined by:

$$veto(i,j) \equiv \bigwedge_{C\subseteq \mathcal{A}} (\langle C\rangle sat_j \rightarrow \neg\langle C\setminus\{i\}\rangle sat_j)$$

And finally, the idea of a coalition being mutually dependent [20, p.58] is captured as follows:

$$md(C) \equiv \bigwedge_{i\neq j\in C} veto(i,j)$$

How might these concepts be extended into the temporal dimension of TQCGs and $\mathcal{L}(TQCG)$? It should first be clear that each concept has four different temporal versions, corresponding to prefixing the formula characterising it with one of the following four, increasingly powerful temporal operators:

$$\Diamond \qquad \Box\Diamond \qquad \Diamond\Box \qquad \Box$$

Thus, for example, $\Box\Diamond succ(C)$ means that coalition C are successful *infinitely often* – no matter which time point we pick, there will be a subsequent time point at which C are successful. (Using the terminology of reactive systems [11], we might then say that C are hence *fairly successful*.) Similarly, a *temporally strong* form of coalitional stability is captured by the formula $\Box cne(\mathcal{A})$: if this formula is satisfied in a TQCG, then, it can be argued, the only coalition that will ever form is the grand coalition.

It is potentially more interesting, however, to study a richer interplay between temporal and QCG dimensions. For example, from agent i's point of view, perhaps the only really interesting issue is whether at every time point there is some stable coalition, containing this agent.

$$tstable(i) \equiv \Box \bigvee_{C \subseteq \mathcal{A} : i \in C} cne(C)$$

From the point of view of a coalition C, which seeks to form, the notion of a *stable government* seems relevant: a stable government is a coalition that can always satisfy its “electorate”.

$$sg(C) \equiv \Box \langle C \rangle (\bigwedge_{i \in \mathcal{A}} sat_i)$$

This can of course be strengthened, requiring C to in addition be an internally stable coalition.

$$sg'(C) \equiv \Box (cne(C) \wedge \langle C \rangle (\bigwedge_{i \in \mathcal{A}} sat_i))$$

With respect to mutual dependence, one possibility, captured by the formula $\Box md(C)$, is that a coalition is *always* mutually dependent. However, we can capture a weaker type of mutual dependence as follows:

$$wmd(C) \equiv \bigwedge_{i \neq j \in C} \Diamond veto(i,j)$$

We draw two conclusions. The first is that the language $\mathcal{L}(TQCG)$ is well suited to capturing such solution concepts: it makes it possible to express elegantly concepts that would be difficult to understand were they expressed at the semantic level. The second is that extending QCGs into the temporal dimension adds an entirely new level of richness to their structure, which as these examples suggest, demands further study.

5 Conclusion

Qualitative Coalitional Games were introduced in [20] to model cooperative scenarios in which agents are concerned with achieving goals, rather than maximising utility. Thus rather than associating a utility to every choice, the emphasis is on *satisfaction* of agents, which is triggered or not by the *choices* made by coalitions.

The logical analysis of such games underlines the idea that we have a basic and simple notion of coalitional games with QCGs: a natural language for it gives rise to an axiomatisation that is closely related to the simplest modal logic, **K**. We established several technical results for this language, and were then able to lift them to the domain of temporal QCGs.

There are many possible directions for further research. First, the properties of TQGCs that we characterised in Section 4 are only the most straightforward. Even in static games, there are other interesting conditions to be investigated (see the *monotonicity* property mentioned in Section 2, for example). Second, our way of temporalising QCGs also only reflects a simple case. It would be interesting to add temporal structure to the games themselves, and reason about what agents can achieve over time, by applying suitable *strategies*, rather than making "one-shot choices". In addition, *finite horizon* versions of TQCGs might also be worth investigating: for example, if an agent is only concerned about being satisfied *once*, then it might be prepared to join a coalition that does not satisfy it throughout a game, as long as, in the final state of the game, the coalition *does* satisfy it. Such strategising is not possible or appropriate in infinite horizon games.

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References

[1] Thomas Ågotnes, Wiebe van der Hoek, and Michael Wooldridge. Reasoning about coalitional games. *Artificial Intelligence*, 173(1):45–79, 2009.

[2] Thomas Ågotnes, Michael Wooldridge, and Wiebe van der Hoek. Temporal qualitative coalitional games. In P. Stone and G. Weiss, editors, *Proceedings of the Fifth International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 177–184, Hakodate, Japan, May 2006. ACM Press.

[3] Rajeev Alur, Thomas A. Henzinger, Orna Kupferman, and Moshe Y. Vardi. Alternating refinement relations. In *Proceedings of the 9th International Conference on Concurrency Theory (CONCUR'98)*, volume 1466 of *Lecture Notes in Computer Science*, pages 163–178. Springer-Verlag, Berlin, Germany, 1998.

[4] R. Axelrod. *The Evolution of Cooperation*. Basic Books: New York, 1984.

[5] K. Binmore. *Fun and Games: A Text on Game Theory*. D. C. Heath and Company: Lexington, MA, 1992.

[6] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press: Cambridge, England, 2001.

[7] M. Finger and D. M. Gabbay. Adding a temporal dimension to a logic system. *Journal of Logic, Language, and Information*, 1:203–233, 1992.

[8] D. M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*. Elsevier, 2003.

[9] Dov Gabbay, Amir Pnueli, Saharon Shelah, and Jonathan Stavi. On the temporal analysis of fairness. In *Conference Record of the Seventh ACM Symposium on Principles of Programming Languages (POPL '80)*, pages 163–173, New York, USA, January 1980. ACM Press.

[10] M. Klusch and A. Gerber. Dynamic coalition formation among rational agents. *IEEE Intelligent Systems*, 17(3):42–47, June 2002.

[11] Z. Manna and A. Pnueli. *The Temporal Logic of Reactive and Concurrent Systems*. Springer-Verlag: Berlin, Germany, 1992.

[12] M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. The MIT Press: Cambridge, MA, 1994.

[13] M. Pauly. Axiomatising judgement aggregation procedures in a minimal logical language. Unpublished paper.

[14] T. Sandholm. Distributed rational decision making. In G. Wei , editor, *Multiagent Systems*, pages 201–258. The MIT Press: Cambridge, MA, 1999.

[15] T. Sandholm, K. Larson, M. Andersson, O. Shehory, and F. Tohmé. Coalition structure generation with worst case guarantees. *Artificial Intelligence*, 111(1–2):209–238, 1999.

[16] O. Shehory and S. Kraus. Methods for task allocation via agent coalition formation. *Artificial Intelligence*, 101(1-2):165–200, 1998.

[17] A. P. Sistla and E. M. Clarke. The complexity of propositional linear temporal logics. *Journal of the ACM*, 32(3):733–749, 1985.

[18] W. van der Hoek and M. Pauly. Modal logic for games and information. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, pages 1077–1148. Elsevier Science Publishers B.V.: Amsterdam, The Netherlands, 2006.

[19] M. Wooldridge. *Reasoning about Rational Agents*. The MIT Press: Cambridge, MA, 2000.

[20] M. Wooldridge and P. E. Dunne. On the computational complexity of qualitative coalitional games. *Artificial Intelligence*, 158(1):27–73, 2004.