Completeness and Complexity of Multi-Modal CTL

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Abstract

We define a multi-modal version of Computation Tree Logic (CTL) by extending the language with path quantifiers E^δ and A^δ where δ denotes one of finitely many dimensions, interpreted over Kripke structures with one total relation for each dimension. As expected, the logic is axiomatised by taking a copy of a CTL axiomatisation for each dimension. Completeness is proved by employing the completeness result for CTL to obtain a model along each dimension at a time. We also show that the logic is decidable and that its satisfiability problem is no harder than the corresponding problem for CTL.

Keywords: Temporal logic, computation tree logic, fusion, completeness

1 Introduction

Computation Tree Logic (CTL) is one of the most popular and successful logics in computer science [4]. CTL has been very widely applied, and has received particular prominence through the development of efficient and industrially applicable CTL model checking systems such as SMV [3].

CTL is a branching time temporal logic, and temporal operators in CTL are made by combining a *path quantifier* with a *tense modality.* The possible path quantifiers are E ("for some path"), and A ("for all paths") while the possible tense modalities are $\Diamond$ ("eventually"), $\Box$ ("always"), $\bigcirc$ ("next"), and $\mathcal{U}$ ("until"). Thus, a formula such as $\mathsf{A}\Box\phi$ expresses the fact that ϕ is an invariant, i.e., ϕ is true at every state along every future path. CTL formulae are interpreted in a state in a Kripke structure, with a single next-state relation. The relation is usually required to be total (i.e., every state has a successor), and a state can have more than one possible next state, modelling branching time.

In this paper we generalise CTL to a finite set of *dimensions* Δ. Syntactically,

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we have one version E^δ, A^δ of the path quantifiers for each dimension $\delta \in \Delta$. Semantically, the structures are extended with one total relation for each dimension (over the same state space). Many applications can be envisioned for such a multi-modal variant of CTL. State transition systems are popular as formal models of *multi-agent systems* [17]. If we make the not unreasonable assumptions that agents can act whenever they want and never acts at exactly the same time, we essentially have a structure where the transitions are labeled by agent names (and where there is at least one outgoing transition for each agent in each state), and a formula of the form $E^a\phi$ means that if only agent a acts then she can act in such a way that ϕ is true. A related example is reasoning about interleaving computations of several processes with shared resources. Another application closely related to multi-agent systems is reasoning about *normative systems*: in [1] expressions of the forms $P_\eta T\phi$ and $O_\eta T\phi$, where T is a CTL tense modality, ϕ a formula, and η denotes a normative system, meaning that in the context of the normative system η, $T\phi$ is permitted or obligatory, respectively, are interpreted in the same way as CTL connectives. Finally, multi-modal CTL could find application as a query language over tree-like structures: Gottlob and Kock [14] use different versions of the tense modalities corresponding to the different directions in XPath in order to encode a fragment of XPath. We could, e.g., take $E^{\downarrow}\bigcirc$ to mean "there is a next child" and $E^{\rightarrow}\bigcirc$ to mean "there is a next sibling".

The main concern in this paper is a complete axiomatisation of multi-modal CTL. It should come as no surprise that an axiomatisation is obtained by taking one "copy" of a CTL axiomatisation for each dimension. The main contribution of the paper is a proof of this fact.

Combinations of modal logics, e.g., of epistemic logic and temporal logic, have been studied to some extent both for particular logics and from a more abstract viewpoint [12]. Combinations of temporal logics into multi-dimensional temporal logics have been studied in the non-branching case [9,11], but we are not aware of existing results for similar combinations of branching-time logics such as CTL. Multi-modal CTL can be seen as a *fusion* of several "copies" of CTL. Studies of fusions and other combinations of modal logics have focussed on the transfer of meta-logical properties of the combined logics, such as soundness, completeness, decidability, etc. Many general transfer results exist for the fusion of *normal* modal logics [15,16,7,13,12]. However, CTL is not a normal modal logic [1], and these general results do not apply directly. Moreover, it is known that the common proof strategy of viewing the fusion as the union of *iterated modalisations* cannot always be used for non-normal modal logics [6]. The proof strategy we employ in this paper has similarities with the mentioned common strategy, but is not a direct application of it.

Rather than extending the tableau-based method for proving the completeness of CTL in [4], we use a construction which employs the CTL completeness result directly,

[1] While CTL is interpreted in Kripke structures, the interpretation is not the standard one used in normal modal logics. To see that CTL is indeed not a normal modal logic, first observe that, e.g., $E\Box$ neither distributes over conjunction nor disjunction and is thus neither a "box" nor a "diamond" of a normal modal logic. $E\Box$ is derived, however, but we can make a similar argument for, e.g., the primary operator $A\mathcal{U}$. Note that $A\mathcal{U}$ is a dyadic operator; see [2, p. 195] for definitions of normality and the K axiom (and duals) for arbitrary similarity types. It is easy to see that the K axiom does not hold for the $A\mathcal{U}$ operator (and not for the dual of that operator either).

viewing a multi-modal CTL formula as a CTL formula for one dimension $\delta \in \Delta$ at a time by reading A^δ and E^δ as CTL path quantifiers A and E, respectively, and treating formulae starting with a δ'-operator ($\delta' \neq \delta$) as atomic formulae. In the resulting model, we "expand" each state along each dimension by repeating the process for the formulae labelling the state, and "glue" together the obtained CTL models. The constructed models are finite, ensuring decidability. This general strategy is not new; it is a known model construction technique in the context of fusions of modal logics. The contribution of this paper is to develop the method for application to the CTL case.

The paper is organised as follows. In the next section, CTL is briefly reviewed, before multi-modal CTL is formally defined in the following section. The axiomatisation and completeness proof are found in Section 4, where we also discuss the computational complexity of the logic. We first give an informal outline of the proof and a detailed example, before we describe the proof in detail in Section 4.2.

2 CTL

Given a set of primitive propositions Θ, the language $\mathcal{L}_{CTL}(\Theta)$ of CTL is defined by the following grammar.

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \vee \phi \mid E\bigcirc\phi \mid E(\phi\,\mathcal{U}\,\phi) \mid A\bigcirc\phi \mid A(\phi\,\mathcal{U}\,\phi)$$

where $p \in \Theta$. The usual derived propositional connectives are used, in addition to $E\Diamond\phi$ ($A\Diamond\phi$) for $E(\top\,\mathcal{U}\,\phi)$ ($A(\top\,\mathcal{U}\,\phi)$) and $E\,\Box\phi$ ($A\,\Box\phi$) for $\neg A\Diamond\neg\phi$ ($\neg E\Diamond\neg\phi$).

A CTL *model* over Θ is a tuple $M = (S, R, L)$ where S is a set of states, $R \subseteq S \times S$ is total[2] and $L(s) \subseteq \Theta$ for each $s \in S$. The class of all models over Θ is denoted $\mathcal{M}_{CTL}(\Theta)$. A model is *finite* if the set of states is finite. In general, given a set S and a total relation R over S, we will use $paths(R, s)$ to denote the R-paths starting in s, i.e., the set of sequences $x_0x_1\cdots$ such that $x_0 = s$ and for each $i \geq 0$, $(x_i, x_{i+1}) \in R$. For $x \in paths(R, s)$ and $k \geq 0$, $x[k]$ denotes the the kth element of x (x_k). A *pointed model* is a pair M, s where M is a model and s is a state in M.

[2] For every $s \in S$ there is some $s' \in S$ such that Rss'.

Satisfaction of $\mathcal{L}_{CTL}(\Theta)$ formulae in a pointed $\mathcal{M}_{CTL}(\Theta)$ model $M = (S, R, L)$ is defined as follows. Let $s \in S$.

$$M, s \models_{CTL} \top$$

$$M, s \models_{CTL} p \quad \Leftrightarrow p \in L(s) \quad (p \in \Theta)$$

$$M, s \models_{CTL} \neg\phi \quad \Leftrightarrow M, s \not\models_{CTL} \phi$$

$$M, s \models_{CTL} \phi \vee \psi \quad \Leftrightarrow M, s \models_{CTL} \phi \text{ or } M, s \models_{CTL} \psi$$

$$M, s \models_{CTL} E\bigcirc\phi \quad \Leftrightarrow \exists(x \in paths(R, s))M, x[1] \models_{CTL} \phi$$

$$M, s \models_{CTL} A\bigcirc\phi \quad \Leftrightarrow \forall(x \in paths(R, s))M, x[1] \models_{CTL} \phi$$

$$M, s \models_{CTL} E(\phi\,\mathcal{U}\,\psi) \Leftrightarrow \exists(x \in paths(R, s))\exists(j \geq 0)$$
$$M, x[j] \models_{CTL} \psi \text{ and } \forall(0 \leq k < j)M, x[k] \models_{CTL} \phi$$

$$M, s \models_{CTL} A(\phi\,\mathcal{U}\,\psi) \Leftrightarrow \forall(x \in paths(R, s))\exists(j \geq 0)$$
$$M, x[j] \models_{CTL} \psi \text{ and } \forall(0 \leq k < j)M, x[k] \models_{CTL} \phi$$

Let $\mathcal{S}_{CTL}(\Theta)$ be the logical system over $\mathcal{L}_{CTL}(\Theta)$ defined in Figure 1.

(Ax1) All validities of propositional logic
(Ax4) $E\bigcirc(\phi \vee \psi) \leftrightarrow (E\bigcirc\phi \vee E\bigcirc\psi)$
(Ax5) $A\bigcirc\phi \leftrightarrow \neg E\bigcirc\neg\phi$
(Ax6) $E(\phi\,\mathcal{U}\,\psi) \leftrightarrow (\psi \vee (\phi \wedge E\bigcirc E(\phi\,\mathcal{U}\,\psi)))$
(Ax7) $A(\phi\,\mathcal{U}\,\psi) \leftrightarrow (\psi \vee (\phi \wedge A\bigcirc A(\phi\,\mathcal{U}\,\psi)))$
(Ax8) $E\bigcirc\top \wedge A\bigcirc\top$
(Ax9) $A\,\Box(\phi \rightarrow (\neg\psi \wedge E\bigcirc\phi)) \rightarrow (\phi \rightarrow \neg A(\gamma\,\mathcal{U}\,\psi))$
(Ax9b) $A\,\Box(\phi \rightarrow (\neg\psi \wedge E\bigcirc\phi)) \rightarrow (\phi \rightarrow \neg A\Diamond\psi)$
(Ax10) $A\,\Box(\phi \rightarrow (\neg\psi \wedge (\gamma \rightarrow A\bigcirc\phi))) \rightarrow (\phi \rightarrow \neg E(\gamma\,\mathcal{U}\,\psi))$
(Ax10b) $A\,\Box(\phi \rightarrow (\neg\psi \wedge A\bigcirc\phi)) \rightarrow (\phi \rightarrow \neg E\Diamond\psi)$
(Ax11) $A\,\Box(\phi \rightarrow \psi) \rightarrow (E\bigcirc\phi \rightarrow E\bigcirc\psi)$
(R1) If $\vdash \phi$ then $\vdash A\,\Box\phi$ (generalization)
(R2) If $\vdash \phi$ and $\vdash \phi \rightarrow \psi$ then $\vdash \psi$ (modus ponens)

Fig. 1. $\mathcal{S}_{CTL}(\Theta)$ [5]

The following theorem gives completeness and decidability of CTL.

Theorem 2.1 ([5]) *Any $\mathcal{S}_{CTL}(\Theta)$-consistent $\mathcal{L}_{CTL}(\Theta)$-formula is satisfiable in a finite $\mathcal{M}_{CTL}(\Theta)$ model.*

3 Multi-Modal CTL

We define a multi-modal version of CTL. Let Δ be a finite set of indices and Θ a set of primitive propositions. The language $\mathcal{L}_{MCTL}(\Theta, \Delta)$ of MCTL is defined by the following grammar.

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \vee \phi \mid E^\delta \bigcirc\phi \mid E^\delta(\phi\,\mathcal{U}\,\phi) \mid A^\delta \bigcirc\phi \mid A^\delta(\phi\,\mathcal{U}\,\phi)$$

where $\delta \in \Delta$ and $p \in \Theta$. The usual derived propositional connectives are used, in addition to $E^\delta\Diamond\phi$ ($A^\delta\Diamond\phi$) for $E^\delta(\top\,\mathcal{U}\,\phi)$ ($A^\delta(\top\,\mathcal{U}\,\phi)$) and $E^\delta\,\Box\phi$ ($A^\delta\,\Box\phi$) for

$\neg A^\delta \Diamond \neg\phi$ $(\neg E^\delta \Diamond \neg\phi)$.

We henceforth use the following terminology: a *temporal atom* is a formula starting with a temporal operator; a *temporal δ-atom*, or sometimes just a δ-atom, is a formula starting with a temporal operator marked with δ.

A MCTL *model* over Θ and Δ is a tuple $M = (S, \{R_\delta : \delta \in \Delta\}, L)$ where S is a set of states, $R_\delta \subseteq S \times S$ is total for each δ and $L(s) \subseteq \Theta$ for each $s \in S$. The class of all models over Θ and Δ is denoted $\mathcal{M}_{MCTL}(\Theta, \Delta)$.

The satisfaction relation between pointed $\mathcal{M}_{MCTL}(\Theta, \Delta)$ models and $\mathcal{L}_{MCTL}(\Theta, \Delta)$ formulae is defined exactly as for CTL, only that R_δ is used to interpret temporal operators marked with δ:

$$\begin{aligned}
M, s \models E^\delta \bigcirc \phi \quad &\Leftrightarrow \exists (x \in paths(R_\delta, s)) M, x[1] \models \phi \\
M, s \models A^\delta \bigcirc \phi \quad &\Leftrightarrow \forall (x \in paths(R_\delta, s)) M, x[1] \models \phi \\
M, s \models E^\delta (\phi \mathcal{U} \psi) &\Leftrightarrow \exists (x \in paths(R_\delta, s)) \exists (j \geq 0) \\
&\qquad M, x[j] \models \psi \text{ and } \forall (0 \leq k < j) M, x[k] \models \phi \\
M, s \models A^\delta (\phi \mathcal{U} \psi) &\Leftrightarrow \forall (x \in paths(R_\delta, s)) \exists (j \geq 0) \\
&\qquad M, x[j] \models \psi \text{ and } \forall (0 \leq k < j) M, x[k] \models \phi
\end{aligned}$$

(and as usual for the Booleans).

4 Axiomatisation

Let $\mathcal{S}_{MCTL}(\Theta, \Delta)$ be the logical system over the logical language $\mathcal{L}_{MCTL}(\Theta, \Delta)$ defined in Figure 2, obtained by taking one "copy" of the CTL axiomatisation for each dimension. We will show that it is sound and complete with respect to $\mathcal{M}_{MCTL}(\Theta, \Delta)$.

(Ax1) All validities of propositional logic

(Ax4) $E^\delta \bigcirc (\phi \vee \psi) \leftrightarrow (E^\delta \bigcirc \phi \vee E^\delta \bigcirc \psi)$

(Ax5) $A^\delta \bigcirc \phi \leftrightarrow \neg E^\delta \bigcirc \neg \phi$

(Ax6) $E^\delta (\phi \mathcal{U} \psi) \leftrightarrow (\psi \vee (\phi \wedge E^\delta \bigcirc E^\delta (\phi \mathcal{U} \psi)))$

(Ax7) $A^\delta (\phi \mathcal{U} \psi) \leftrightarrow (\psi \vee (\phi \wedge A^\delta \bigcirc A^\delta (\phi \mathcal{U} \psi)))$

(Ax8) $E^\delta \bigcirc \top \wedge A^\delta \bigcirc \top$

(Ax9) $A^\delta \Box (\phi \rightarrow (\neg\psi \wedge E^\delta \bigcirc \phi)) \rightarrow (\phi \rightarrow \neg A^\delta (\gamma \mathcal{U} \psi))$

(Ax9b) $A^\delta \Box (\phi \rightarrow (\neg\psi \wedge E^\delta \bigcirc \phi)) \rightarrow (\phi \rightarrow \neg A^\delta \Diamond \psi)$

(Ax10) $A^\delta \Box (\phi \rightarrow (\neg\psi \wedge (\gamma \rightarrow A^\delta \bigcirc \phi))) \rightarrow (\phi \rightarrow \neg E^\delta (\gamma \mathcal{U} \psi))$

(Ax10b) $A^\delta \Box (\phi \rightarrow (\neg\psi \wedge A^\delta \bigcirc \phi)) \rightarrow (\phi \rightarrow \neg E^\delta \Diamond \psi)$

(Ax11) $A^\delta \Box (\phi \rightarrow \psi) \rightarrow (E^\delta \bigcirc \phi \rightarrow E^\delta \bigcirc \psi)$

(R1) If $\vdash \phi$ then $\vdash A^\delta \Box \phi$ (generalization)

(R2) If $\vdash \phi$ and $\vdash \phi \rightarrow \psi$ then $\vdash \psi$ (modus ponens)

Fig. 2. $\mathcal{S}_{MCTL}(\Theta, \Delta)$. δ ranges over Δ.

Proposition 4.1 *$\mathcal{S}_{MCTL}(\Theta, \Delta)$ is sound wrt. $\mathcal{M}_{MCTL}(\Theta, \Delta)$.*

Proof. Straightforward: all axioms are valid and all rules preserve validity. □

Theorem 4.2 *Any $\mathcal{S}_{MCTL}(\Theta,\Delta)$-consistent $\mathcal{L}_{MCTL}(\Theta,\Delta)$-formula is satisfiable in a finite $\mathcal{M}_{MCTL}(\Theta,\Delta)$ model.*

The proof of Theorem 4.2 is presented in the following sections. The following corollaries are immediate.

Corollary 4.3 *$\mathcal{S}_{MCTL}(\Theta,\Delta)$ is complete wrt. $\mathcal{M}_{MCTL}(\Theta,\Delta)$.*

Corollary 4.4 *The satisfiability problem for* MCTL *is decidable.*

In fact, we can sharpen this result: we will show that, as a corollary of the construction used in the proof of Theorem 4.2, the satisfiability problem is in fact decidable in exponential time (and is thus EXPTIME-complete — no harder than the corresponding problem for CTL).

Before a detailed proof of Theorem 4.2 in Section 4.2, we give an outline of the proof and an illustrating example.

4.1 Outline of Completeness Proof

Let ϕ_0 be a consistent formula. Rather than extending the tableau-based method for proving the completeness of CTL in [4], we use a construction which employs the CTL completeness result (Theorem 2.1) directly, viewing a formula as a CTL formula for one dimension $\delta \in \Delta$ at a time by reading A^δ and E^δ as CTL path quantifiers A and E, respectively, and treating formulae starting with a δ'-operator ($\delta' \neq \delta$) as atomic formulae. By completeness of CTL, we get a CTL model for the formula (if it is consistent), where the states are labelled with atoms such as $A^{\delta'}\psi$ or $E^{\delta'}\psi$ (for $\delta' \neq \delta$). Then, for each δ' and each state, we expand the state by taking the conjunction of δ'-formulae the state is labelled with, construct a (single-modal) CTL model of that formula, and "glue" the root of the model together with the state. Repeat for all dimensions and all states.

In order to keep the formulae each state is labelled with finite, we consider only subformulae of ϕ_0; by a δ-atom we here mean a subformula of ϕ_0 starting with either E^δ or A^δ. Let $At^{-\delta}$ denote the union of all sets of δ'-atoms for each $\delta' \neq \delta$. Furthermore, we assume that ϕ_0 is such that every occurrence of $E^\delta(\alpha_1 \,\mathcal{U}\, \alpha_2)$ ($A^\delta(\alpha_1 \,\mathcal{U}\, \alpha_2)$) is immediately preceeded by $E^\delta \bigcirc$ ($A^\delta \bigcirc$) – we call this *XU form*. Any formula can be rewritten to XU form by recursive use of the axioms (Ax6) and (Ax7). We start with a model with a single state labelled with the literals in a consistent disjunct of ϕ_0 written in disjunctive normal form. We continue by expanding states labelled with formulae, one dimension δ at a time. In general, let $at(\delta, s)$ be the union of the set of δ-atoms s is labelled with and the set of negated δ-atoms of XU form s is *not* labelled with. We can now view $\bigwedge at(\delta, s)$ as a CTL formula over a language with primitive propositions $\Phi \cup At^{-\delta}$. The following can be shown: any MCTL consistent formula is satisfied by a state s' in some finite CTL model M' viewing $\Phi \cup At^{-\delta}$ as primitive propositions, such that for any $\delta' \neq \delta$ and any state t of M', $\bigwedge at(\delta', t)$ is MCTL-consistent, and s' does not have any ingoing transitions. This ensures that we can "glue" the pointed model M', s' to the state s while labelling the transitions in the model with the dimension δ we expanded – M', s' satisfies the formulae needed to be true there. The fact that s' does not have any ingoing transitions ensures that we can append M', s' to s without changing the truth of δ-atoms at s'. The fact that ϕ_0 is of XU form ensures that all labelled formulae are of XU form, which again ensures that we don't add new labels to a state when we expand it (because all the formulae we expand start with a next-modality). The fact that $\bigwedge at(\delta', t)$ is consistent for states t in the expanded model, ensures that we can repeat the process. Only a finite number of repetitions are needed, depending on the number of nested operators of different dimensions in ϕ_0, after which we can remove the non-Φ labels without affecting the truth of ϕ_0 and obtain a proper model.

4.1.1 Example

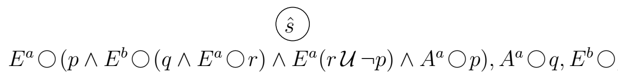
Take $\Delta = \{a, b\}$ and $\Theta = \{p, q, r\}$. We illustrate the method for finding a satisfying MCTL model for the formula

$$\phi_0 = E^a \bigcirc (p \wedge E^b \bigcirc (q \wedge E^a \bigcirc r) \wedge E^a (r \,\mathcal{U} \neg p) \wedge A^a \bigcirc p) \wedge A^a \bigcirc q \wedge E^b \bigcirc p$$

We define the model in steps. Some of the information given here for each step refer to the proof in the following section.

The initial model M_0 consists simply of a single state $\hat{s}$ labelled with the temporal atoms required to be true. In this model *every* temporal atom is viewed as a primitive proposition.

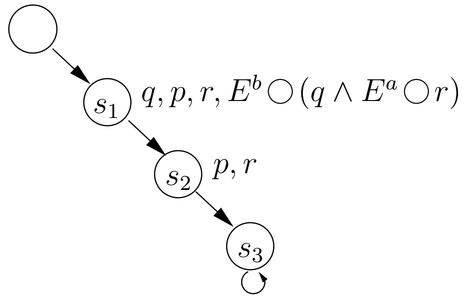
$\mathbf{M^0}$: ($U^0 = \{\hat{s}\}$, $T^0 = \emptyset$, $\tau^0(\hat{s}) = \epsilon$)



In general, the model M_{j+1} is constructed from M_j by expanding each node in U^j by constructing one CTL model for the temporal atoms in that node of each dimension, and then attaching these CTL models to the node we expand.

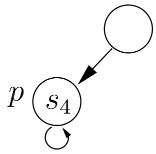
Expanding $\hat{s}$ along dimension a, we treat the temporal atoms of a dimension different from a as primitive propositions, and E^a and A^a as the CTL path quantifiers E and A, respectively. From completeness of CTL we know that there is a model for the formulae $\hat{s}$ is labelled with. There are, of course, *many* CTL models, but we choose one with certain properties. In particular, we choose a model where the labels (temporal atoms of dimensions different from a) are MCTL-*consistent* – which ensures that we can repeat the process and expand the new nodes again by choosing a CTL model – and where there are no ingoing transitions to the root – ensuring that we can glue models of different dimensions together. (The existence of models with these properties is formally ensured by Proposition 4.8 below). We get, e.g., the following (single-modal) CTL-model, satisfying the set of CTL formulae $\{E(p \wedge$

$t \wedge E(r\,\mathcal{U}\,\neg p) \wedge A\bigcirc p), A\bigcirc q\}$, where t is an atom representing $E^b\bigcirc(q \wedge E^a\bigcirc r)$:



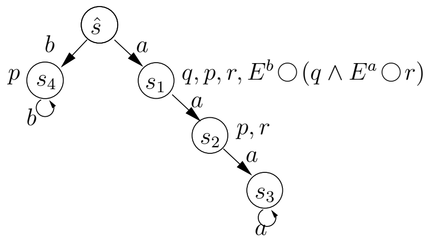
This is a proper CTL-model, with a single, total, relation.

Expanding $\hat{s}$ along dimension b we get the (single-modal) CTL-model:



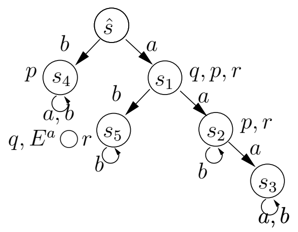
There was only one state in U^0, and two dimensions, so we are done. Gluing these two CTL models together with the state we expanded, $\hat{s}$, we get M_1:

M¹: $(U^1 = \{s_1, s_2, s_3, s_4\},\ T^1 = \{\hat{s}\},\ \tau^1(s_1) = \tau^1(s_2) = \tau^1(s_3) = a; \tau^1(s_4) = b)$



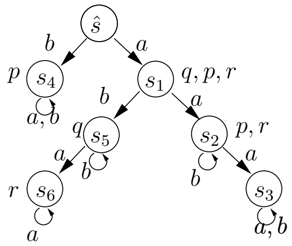
U^1 is the set of nodes added in the previous round, which will be expanded now. It might seem that s_4 does not need to be expanded because it is not labelled by any temporal formulae, but it must be expanded along the a-dimension in a trivial way: a self loop must be added to make sure that the a-relation is total. Similarly for s_2 and s_3 wrt. b. The result is M_2:

M²: $(U^2 = \{s_5\}, T^2 = \{\hat{s}, s_1, s_2, s_3, s_4\}, \tau^2(s_5) = b)$



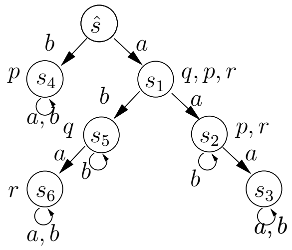
M_3 is as follows:

M³: $(U^3 = \{s_6\}, T^3 = \{\hat{s}, s_1, s_2, s_3, s_4, s_5\}, \tau^3(s_6) = a)$



Finally, M_4 trivially expands s_6 by gluing on a model for each of the formulae in each of the dimensions different from $\tau(s_6)$. There are no such formulae, so the models are trivial (satisfying tautologies) but total:

M⁴: $(U^4 = \emptyset, T^4 = \{\hat{s}, s_1, s_2, s_3, s_4, s_5, s_6\})$



There are no more states to expand, and the construction is finished.

4.2 Completeness Proof

We now formally prove Theorem 4.2.

Let ϕ_0 be a $\mathcal{S}_{MCTL}(\Theta, \Delta)$ consistent $\mathcal{L}_{MCTL}(\Theta, \Delta)$ formula. We will show that ϕ_0 is satisfied by a finite model in $\mathcal{M}_{MCTL}(\Theta, \Delta)$. We repeat the definition of XU form:

Definition 4.5 [XU form] A formula $\phi \in \mathcal{L}_{MCTL}(\Theta, \Delta)$ is of *XU form* if every occurence of a subformula of the form $E^\delta(\psi_1 \mathcal{U} \psi_2)$ $(A^\delta(\psi_1 \mathcal{U} \psi_2))$ in ϕ is immediately preceeded by an $E^\delta \bigcirc$ $(A^\delta \bigcirc)$ operator.

Lemma 4.6 *Any $\mathcal{L}_{MCTL}(\Theta, \Delta)$ formula ϕ is equivalent to a $\mathcal{L}_{MCTL}(\Theta, \Delta)$ formula of XU form.*

Proof. Rewrite the formula using axioms (Ax6) and (Ax7) (which are valid) recursively, until the formula is of the form. □

Thus, we will w.l.o.g. henceforth assume that ϕ_0 is of XU form.

Let $Subf(\phi)$ be the set of all subformulae of a formula ϕ.

We can view the language $\mathcal{L}_{MCTL}(\Theta, \Delta)$ as a CTL language, by fixing some δ and reading $E^\delta X$ as EX, $A^\delta X$ as AX, and so on, and treating the other temporal atoms, such as $E^{\delta'} X\phi$, $\delta' \neq \delta$, as primitive propositions (in addition to Θ). For technical reasons, we only consider temporal atoms occurring in $Subf(\phi_0)$. Let:

$$At^{\delta} = \{E^{\delta} \bigcirc \phi, E^{\delta}(\phi \mathcal{U} \psi), A^{\delta} \bigcirc \phi, A^{\delta}(\phi \mathcal{U} \psi) : \phi, \psi \in Subf(\phi_0)\}$$

– in particular, At^{δ} includes the set of temporal atoms of type δ occuring in ϕ_0 – let

$$At = \bigcup_{\delta \in \Delta} At^{\delta}$$

– in particular, At includes all temporal atoms in ϕ_0 – and let

$$At^{-\delta} = \bigcup_{\delta' \neq \delta} At^{\delta'}$$

– in particular, $At^{-\delta}$ includes the temporal atoms occurring in ϕ_0 which are not of type δ. We can now view any formula in $\mathcal{L}_{MCTL}(\Theta, \Delta) \cap Subf(\phi_0)$ as a $\mathcal{L}_{CTL}(At^{-\delta} \cup \Theta)$ formula by reading any E^{δ}, A^{δ} which is not in the scope of any $E^{\delta'}, A^{\delta'}$ ($\delta' \neq \delta$) as E, A, and treating temporal formulae such as $E^{\delta'} X\phi$ where $\delta' \neq \delta$ as primitive propositions. When Θ and Δ are understood, we will use $\mathcal{L}_{CTL}(\delta)$ as shorthand for the CTL language $\mathcal{L}_{CTL}(At^{-\delta} \cup \Theta)$ and $\mathcal{M}_{CTL}(\delta)$ as a shorthand for the associated CTL model class $\mathcal{M}_{CTL}(At^{-\delta} \cup \Theta)$. A model $M \in \mathcal{M}_{CTL}(\delta)$ has a transition relation for interpreting temporal δ-atoms, and the labelling function interprets the other temporal atoms occurring in ϕ_0 in addition to primitive propositions Θ in the states. Similarly, we use $\mathcal{S}_{CTL}(\delta)$ to denote the CTL axiom system $\mathcal{S}_{CTL}(At^{-\delta} \cup \Theta)$ over the language $\mathcal{L}_{CTL}(\delta)$. Thus, we will henceforth sometimes view a MCTL formula ϕ also as a $\mathcal{L}_{CTL}(\delta)$ formula for some given δ, and write, e.g., $M, s \models_{CTL} \phi$ when $M \in \mathcal{M}_{CTL}(\delta)$ with the meaning defined by reading E^{δ} as E, etc., as explained above. Similarly, we sometimes implicitly view a $\mathcal{L}_{CTL}(\delta)$ formula as a MCTL formula (i.e., the MCTL formula obtained by replacing every E with E^{δ} and every A with A^{δ}).

Lemma 4.7 *For any δ and $\phi \in \mathcal{L}_{CTL}(\delta)$, $\vdash_{\mathcal{S}_{CTL}(\delta)} \phi$ implies that $\vdash_{\mathcal{S}_{MCTL}} \phi$.*

Proof. Straightforward induction on the length of the proof. □

When t is a state of a model $M \in \mathcal{M}_{CTL}(\delta)$ and $\delta' \neq \delta$, let

$$at(\delta', t, M) = \{\psi : \psi \in At^{\delta'}, \psi \text{ is of XU form}, \psi \in L(t)\} \cup \\ \{\neg\psi : \psi \in At^{\delta'}, \psi \text{ is of XU form}, \psi \notin L(t)\}$$

Proposition 4.8 *Let $\delta \in \Delta$ and $\phi \in \mathcal{L}_{CTL}(\delta)$. If ϕ is $\mathcal{S}_{MCTL}$-consistent, then there is a model $M' \in \mathcal{M}_{CTL}(\delta)$ with a state s' such that*

(i) $M', s' \models_{CTL} \phi$

(ii) *For all states t reachable from s' in M' and for all $\delta' \neq \delta$, $\bigwedge at(\delta', t, M')$ is $\mathcal{S}_{MCTL}$-consistent*

(iii) *There is no state t in M' such that $(t, s') \in R'$ (s' has no ingoing transitions)*

(iv) *M' is finite*

Proof. Let $XU^{\delta'}$ be the set of all formulae in $At^{\delta'}$ of XU form, and let $XU^{\delta'+}$ be $XU^{\delta'}$ closed under single negation, i.e., $XU^{\delta'+} = \{\alpha, \neg\alpha : \alpha \in At^{\delta'}, \alpha \text{ of XU form}\}$.

Let $Y^{\delta'}$ be the set of all $XU^{\delta'+}$-maximal $\mathcal{S}_{MCTL}$-inconsistent subsets of $XU^{\delta'+}$, i.e., all sets $y \subseteq XU^{\delta'+}$ such that either $\alpha \in y$ or $\neg\alpha \in y$ for any $\alpha \in XU^{\delta'}$ and $\vdash_{\mathcal{S}_{MCTL}} \bigwedge y \rightarrow \bot$. $Y^{\delta'}$ is finite because $XU^{\delta'+}$ is finite. Let

$$f(\delta') = \bigwedge y_1 \vee \cdots \vee \bigwedge y_k$$

where $Y^{\delta'} = \{y_1, \ldots, y_k\}$.

We show that

$$\gamma = \phi \wedge A^{\delta} \Box \bigwedge_{\delta' \neq \delta} \neg f(\delta')$$

is $\mathcal{S}_{MCTL}$-consistent. Assume the opposite: $\vdash_{\mathcal{S}_{MCTL}} \gamma \rightarrow \bot$. It follows that $\vdash_{\mathcal{S}_{MCTL}} A^{\delta} \Box \bigwedge_{\delta' \neq \delta} \neg f(\delta') \rightarrow \neg\phi$. However, for any $\delta' \neq \delta$ and $y \in Y^{\delta'}$ we have that $\vdash_{\mathcal{S}_{MCTL}} \neg \bigwedge y$, and thus that $\vdash_{\mathcal{S}_{MCTL}} \neg f(\delta')$ for any δ'. It follows that $\vdash_{\mathcal{S}_{MCTL}} \bigwedge_{\delta' \neq \delta} \neg f(\delta')$. By (Gen), we have that $\vdash_{\mathcal{S}_{MCTL}} A^{\delta} \Box \bigwedge_{\delta' \neq \delta} \neg f(\delta')$. But then we also have that $\vdash_{\mathcal{S}_{MCTL}} \neg\phi$, which contradicts the fact that ϕ is $\mathcal{S}_{MCTL}$-consistent. Thus, γ is $\mathcal{S}_{MCTL}$-consistent.

γ is $\mathcal{S}_{CTL}(\delta)$-consistent – otherwise it would not have been $\mathcal{S}_{MCTL}$-consistent by Lemma 4.7. By completeness of $\mathcal{S}_{CTL}(\delta)$ (Theorem 2.1), there is a finite model $M = (S, R, L) \in \mathcal{M}_{CTL}(\delta)$ such that $M, s \models_{CTL} \gamma$ for some s. Let t be reachable from s in M. Assume that $\bigwedge at(\delta', t, M)$ is not $\mathcal{S}_{MCTL}$-consistent for some $\delta' \neq \delta$. Then $at(\delta', t, M) = y_j$ for some j, so $M, t \models_{CTL} f(\delta')$. It follows that $M, s \models_{CTL} E^{\delta} \Diamond f(\delta')$, but this contradicts the fact that $M, s \models_{CTL} \gamma$. Thus, $\bigwedge at(\delta', t, M)$ is $\mathcal{S}_{MCTL}$-consistent. Also, $M, s \models_{CTL} \phi$.

To get a satisfying state with no ingoing transitions, let $M' = (S', R', L')$ where $S' = S \cup \{s'\}$ for some new state s'; $R' = R \cup \{(s', t) : (s, t) \in R\}$; $L'(s') = L(s)$ and $L'(t) = L(t)$ for $t \neq s'$. It is easy to see that $M, s \models_{CTL} \psi$ iff $M', s' \models_{CTL} \psi$ for any ψ. In particular $M', s' \models_{CTL} \phi$. $\Box$

Definition 4.9 [General Models] A *general model* over Θ and Δ is a tuple $M = (S, T, U, \tau, \{R_{\delta} : \delta \in \Delta\}, L, K)$ where T and U partition S, $\tau(u) \in \Delta \cup \{\epsilon\}$ for each $u \in U$, $K(u) \subseteq \bigcup_{\delta' \neq \tau(u)} At^{\delta'}$ for each $u \in U$, and the other elements are as in a model. A general model is finite if S finite.

Satisfaction of a formula $\phi \subseteq subf(\phi_0)$ in a pointed general model is defined as follows. Let $s \in S$.

$$M, s \models \top$$

$$M, s \models p \quad \Leftrightarrow p \in L(s) \quad (p \in \Theta)$$

$$M, s \models \neg\phi \quad \Leftrightarrow M, s \not\models \phi$$

$$M, s \models \phi \vee \psi \quad \Leftrightarrow M, s \models \phi \text{ or } M, s \models \psi$$

$$M, s \models A^{\delta} X \phi \Leftrightarrow \begin{cases} \forall (x \in paths(R_{\delta}, s)) M, x[1] \models \phi & s \in T \text{ or } (s \in U \text{ and } \delta = \tau(s)) \\ A^{\delta} X \phi \in K(s) & s \in U \text{ and } \delta \neq \tau(s) \end{cases}$$

and similarly for the other temporal atoms.

We now define a sequence $M_0, M_1, \ldots$ of finite general models $M_j = (S^j, T^j, U^j, \tau^j, \{R^j_{\delta} : \delta \in \Delta\}, L^j, K^j)$ such that $\hat{s} \in S^j$ for all j for some state

$\hat{s}$, having the three following properties for any j:

(i) $M_j, \hat{s} \models \phi_0$

(ii) For every $t \in U^j$ and $\delta \neq \tau^j(t)$, $\bigwedge at(\delta, t, M_j)$ is $\mathcal{S}_{MCTL}$-consistent

(iii) For every $t \in U^j$, each $\alpha \in K^j(t)$ is of XU form

where

$$at(\delta, s, M_j) = \{\psi : \psi \in At^\delta, \psi \text{ is of XU form}, \psi \in K^j(s)\} \cup \{\neg\psi : \psi \in At^\delta, \psi \text{ is of XU form}, \psi \notin K^j(s)\}$$

It might be instructive to refer to the example in the previous section as an illustration of the construction.

M_0 has a single state $\hat{s}$, such that $\hat{s} \in U^0$ and $\tau(\hat{s}) = \epsilon$. If we view *every* temporal atom in ϕ_0 which is not in the scope of another temporal operator as a primitive proposition, ϕ_0 is a purely propositional formula. Because $\mathcal{S}_{MCTL}$ contains propositional logic and ϕ_0 is of XU form, ϕ_0 is equivalent to a formula on disjunctive normal form $(A_1^1 \wedge \cdots \wedge A_m^1) \vee \cdots \vee (A_1^k \wedge \cdots A_m^k)$, where for each $1 \leq j \leq k$ and $1 \leq i \leq m$, either $A_i^j = B_i$ or $A_i^j = \neg B_i$, where $\{B_1, \ldots, B_m\} = \Theta \cup \{\alpha \in At : \alpha \text{ of XU form}\}$. Since ϕ_0 is $\mathcal{S}_{MCTL}$-consistent, some $\xi = (A_1^j \wedge \cdots \wedge A_m^j)$ is $\mathcal{S}_{MCTL}$-consistent. Let X be the set of positive atoms A_i^j in ξ, and let $Y = \{B_1, \ldots, B_m\} \backslash X$ be the negative atoms. I.e, $\xi = \bigwedge(X \cup \{\neg y : y \in Y\})$ is $\mathcal{S}_{MCTL}$-consistent. Set $L^0(\hat{s}) = X \cap \Theta$ and $K^0(\hat{s}) = X \setminus L^0(\hat{s})$. (i) clearly holds, because M^0 interprets ϕ_0 simply as a propositional formula using the valuations $L^0(\hat{s})$ and $K^0(\hat{s})$ and thus we see immediately that $M_0, \hat{s} \models \xi$. (ii) holds, because $at(\delta, \hat{s}, M_0) \subseteq X \cup \{\neg y : y \in Y\}$. (iii) holds immediately, because every atom in X is of XU form.

M_{j+1} is obtained from M_j as follows. Informally, the idea is to take, for every δ and every state s in U^j, the set $at(\delta, s, M_j)$, and replace it with a model $M \in \mathcal{M}_{CTL}(\delta)$ for $at(\delta, s, M_j)$ rooted in s. Formally, we define M_{j+1} as follows. For every $u \in U^j$ and every $\delta \neq \tau^j(u)$, we have that $\bigwedge at(\delta, u, M_j)$ is $\mathcal{S}_{MCTL}$-consistent by (ii), so take $\phi = \bigwedge at(\delta, u, M_j)$ and let $M' = (S', R', L') \in \mathcal{M}_{CTL}(\delta)$ and s' be as in Proposition 4.8. W.l.o.g. we assume that every state in M' is reachable from s'. Let $M_{u,\delta} = (S_{u,\delta}, R_{u,\delta}, L_{u,\delta}) \in \mathcal{M}_{CTL}(\delta)$ be equal to M' except that $L_{u,\delta}(s') = \emptyset$ and $L_{u,\delta}(t) = \{\alpha \in L'(t) : \alpha \text{ of XU form}\}$ for any $t \in S' \setminus \{s'\}$, and let $t_{u,\delta} = s'$. We have that each $\alpha \in at(\delta, u, M_j)$ starts with a (possibly negated) $E^{\delta'} \bigcirc$ or $A^{\delta'} \bigcirc$ operator (for some δ'). Together with the fact that s' does not have any ingoing transitions, this implies that changing $L'(s')$ does not affect the truth of $\bigwedge at(\delta, u, M_j)$ in s'. Furthermore, removing atoms not of XU form from $L'(t)$ does not affect the truth of $\bigwedge at(\delta, u, M_j)$ in s' either, because all atoms in $at(\delta, u, M_j)$ are of XU form. Thus, all the four points in Proposition 4.8 still hold for $M_{u,\delta}$ and $t_{u,\delta}$; in particular we have that $M_{u,\delta}, t_{u,\delta} \models_{CTL} \bigwedge at(\delta, u, M_j)$.

Let

$$
\begin{aligned}
&S^{j+1} = S^j \cup \bigcup_{u \in U^j, \delta \neq \tau^j(u)} (S_{u,\delta} \setminus \{t_{u,\delta}\})\\
&T^{j+1} = T^j \cup U^j\\
&U^{j+1} = \bigcup_{u \in U^j, \delta \neq \tau^j(u)} (S_{u,\delta} \setminus \{t_{u,\delta}\})\\
&\tau^{j+1}(v) = \delta \text{ iff } v \in S_{u,\delta} \text{ (for some } u)\\
&R^{j+1}_\delta = R^j_\delta \cup \bigcup_{u \in U^j, \delta \neq \tau^j(u)} \{(x', y) : (x, y) \in R_{u,\delta}\}\\
&\text{where } x' = u \text{ if } x = t_{u,\delta} \text{ and } x' = x \text{ otherwise}\\
&L^{j+1}(t) = \begin{cases} L^j(t) & t \in S^j \\ L_{u,\delta}(t) \cap \Theta & t \in S_{u,\delta} \end{cases}\\
&K^{j+1}(t) = L_{u,\delta}(t) \setminus \Theta \text{ when } t \in S_{u,\delta}
\end{aligned}
$$

Since S^j is finite and each $S_{u,\delta}$ is finite (guaranteed by Proposition 4.8) and both U^j and Δ are finite, S^{j+1} is finite.

We argue that (ii) holds for M_{j+1}. Let $t \in U^{j+1}$ and $\delta' \neq \tau^{j+1}(t)$. $t \in S_{u,\delta}$ for some $u \in U^j$ and some $\delta \neq \tau^j(u)$, which implies that $\tau^{j+1}(t) = \delta$ and thus that $\delta \neq \delta'$. We have that $\bigwedge at(\delta', t, M_{j+1}) = \bigwedge at(\delta', t, M_{u,\delta})$ is $\mathcal{S}_{MCTL}$-consistent by Proposition 4.8.

We argue that (iii) holds for M_{j+1}. Let $t \in U^{j+1}$. We have that $K^{j+1}(t) = L_{u,\delta}(t) \setminus \Theta$, for some u, δ. (iii) holds immediately, because every formula in $L_{u,\delta}$ is of XU form by construction.

We now argue that also (i) holds for M_{j+1}. First we show that for any $v \in U^j$ and any $\beta \in Subf(\phi_0)$ of XU form

$$M_j, v \models \beta \Leftrightarrow M_{j+1}, v \models \beta \tag{1}$$

by induction on the structure of β.

- $\beta = p$: $L^j(v) = L^{j+1}(v)$.
- $\beta \in \{E^{\delta'} \bigcirc \gamma, E^{\delta'} \bigcirc E^{\delta'}(\gamma_1 \mathcal{U} \gamma_2), A^{\delta'} \bigcirc \gamma, A^{\delta'} \bigcirc E^{\delta'}(\gamma_1 \mathcal{U} \gamma_2) : \gamma, \gamma_1, \gamma_2$ of XU form$\}$: First assume that $\delta' \neq \tau^j(v)$. $M_j, v \models \beta$ iff $\beta \in K^j(v)$ iff $\beta \in at(\delta', v, M_j)$ iff $M_{v,\delta'}, t_{v,\delta'} \models \beta$ (the fact that $at(\delta', v, M_j)$ is closed under single negation gives us both directions). From the construction of $R^{\delta'}_{j+1}$, the only δ'-transitions from v are transitions from $t_{v,\delta'}$ in $R_{v,\delta}$: we have that

  $$(t_{v,\delta'}, t) \in R_{v,\delta'} \Leftrightarrow (v, t) \in R^{\delta'}_{j+1}$$

  for any t. Furthermore, we have that

  $$M_{v,\delta'}, t \models \alpha \Leftrightarrow M_{j+1}, t \models \alpha$$

  for any t such that $(v, t) \in R^{\delta'}_{j+1}$ and for any α: the submodel of M_{j+1} generated by t is equivalent to the submodel of $M_{v,\delta'}$ generated by t – this holds because $t_{v,\delta}$ does not have any ingoing transitions and thus v does not have any ingoing δ'-transitions – and these two submodels interpret formulae in exactly the same way. It follows that $M_{v,\delta'}, t_{v,\delta'} \models \beta$ iff $M_{j+1}, v \models \beta$.

  Second, assume that $\delta' = \tau(v)$. Observe that $paths(R^j_{\delta'}, v) = paths(R^{j+1}_{\delta'}, v)$, and for any state w along a path we have that $w \in U^j$ by construction since

$v \in U^j$, so $M_j, w \models \alpha$ iff $M_{j+1}, w \models \alpha$ for $\alpha \in \{\gamma, \gamma_1, \gamma_2\}$ by the induction hypothesis. It follows that $M_j, v \models \beta$ iff $M_{j+1}, v \models \beta$.

- Propositional connectives: Straightforward.

We now argue that for any $v \in S_j$ and any $\psi \in Subf(\phi_0)$ of XU form,

$$M_j, v \models \psi \Leftrightarrow M_{j+1}, v \models \psi$$

That (i) holds for M_{j+1} follows immediately. We argue by structural induction on ψ. For $\psi \in \Theta$, we have that $M_j, v \models \psi \Leftrightarrow M_{j+1}, v \models \psi$ because $L^j(v) = L^{j+1}(v)$. Assume that ψ is a temporal atom in $\{E^\delta \bigcirc \gamma, E^\delta \bigcirc E^\delta(\gamma_1 \mathcal{U} \gamma_2), A^\delta \bigcirc \gamma, A^\delta \bigcirc A^\delta(\gamma_1 \mathcal{U} \gamma_2) : \gamma, \gamma_1, \gamma_2 \text{ of XU form}\}$, and consider first the case that $v \in T_j$. For any δ, the δ-paths in M_j starting in v are exactly the same as the δ-paths in M_{j+1} starting in v. By the induction hypothesis, we have that $M_j, w \models \gamma$ iff $M_{j+1}, w \models \gamma$ for any w along any of these paths and any $\gamma \in Subf(\psi)$ of XU form, which shows that $M_j, v \models \psi$ iff $M_{j+1}, v \models \psi$. Consider, second, that $v \in U_j$, in which case we immediately have the required result by (1). The cases for the propositional connectives are straightforward. This concludes the argument that (i) holds for M_{j+1}.

Let the degree of a formula α, denoted $deg(\alpha)$, be the maximum number of nested temporal operators *of alternating type* in the formula. For example, $deg(p) = 0$, $deg(E^\delta \bigcirc p) = 1$, $deg(E^\delta \bigcirc A^{\delta'}(p \mathcal{U} q)) = 2$ (two dimensions of alternating type), $deg(E^\delta \bigcirc A^\delta(p \mathcal{U} q)) = 1$ (two temporal operators but not of alternating type), $deg((E^\delta \bigcirc p) \wedge (E^{\delta'} \bigcirc q)) = 1$ (two dimensions but not nested), $deg(E^\delta \bigcirc E^{\delta'}(p \mathcal{U} A^{\delta'} \bigcirc E^\delta \bigcirc q)) = 4$, etc. Formally we can define $deg(\alpha)$ as follows. Let $dim(\alpha)$ be the set of dimensions of all occurrences of temporal operators in the formula α which are not in the scope of any other temporal operator: $dim(p) = \emptyset$; $dim(\neg\phi) = dim(\phi)$; $dim(\phi_1 \vee \phi_2) = dim(\phi_1) \cup dim(\phi_2)$; $dim(E^\delta \bigcirc \phi) = dim(A^\delta \bigcirc \phi) = dim(E^\delta(\phi_1 \mathcal{U} \phi_2)) = dim(A^\delta(\phi_2 \mathcal{U} \phi_2)) = \{\delta\}$. Finally, $deg(p) = 0$; $deg(\neg\phi) = deg(\phi)$; $deg(\phi_1 \vee \phi_2) = max(deg(\phi_1), deg(\phi_2))$;

$$deg(E^\delta \bigcirc \phi) = deg(A^\delta \bigcirc \phi) = \begin{cases} deg(\phi) & dim(\phi) \setminus \{\delta\} = \emptyset \\ deg(\phi) + 1 & \text{otherwise} \end{cases}$$

(increase the degree whenever there is a dimension different from δ in ϕ);

$$deg(E^\delta(\phi_1 \mathcal{U} \phi_2)) = deg(A^\delta(\phi_2 \mathcal{U} \phi_2)) = \begin{cases} deg(\phi) & dim(\phi) \setminus \{\delta\} = \emptyset \\ deg(\phi) + 1 & \text{otherwise} \end{cases}$$

where $\phi = \phi_1$ if $deg(\phi_1) > deg(\phi_2)$ and $\phi = \phi_2$ otherwise.

A general model $M = (S, T, U, \tau, \{R_\delta : \delta \in \Delta\}, L, K)$ is a generalisation of a proper model $M' = (S, \{R_\delta : \delta \in \Delta\}, L)$. We say that the satisfaction relationship between a formula ϕ and a pointed general model (M, s), i.e., the question of whether $M, s \models \phi$ or not, is *classical* if the definition (as given recursively above) does not involve any state from U (and thus not τ or K either). If the satisfaction relationship is classical, then satisfaction only depends on the underlying (proper) model.

Lemma 4.10 *For any $\alpha \in Subf(\phi_0)$, the satisfaction relationship between α and (M_{m+j+1}, v) is classical when $deg(\alpha) = j$ and $v \in U^m$.*

Proof. Directly from the definition of satisfaction, we have that when $v \in U^m$ then for every $k > m$ if $x \in paths(R^k_\delta, v)$ for some δ then

(i) if $\tau^m(v) = \delta$, then for any i, $x[i] \in U^m$

(ii) if $\tau^m(v) \neq \delta$, then for any i, $x[i] \in U^m \cup U^{m+1}$

This means that when we evaluate a formula in a state $v \in U^m$, only a "switch" in dimension can involve states from U^{m+1}. For example, in the evaluation the formula $E^\delta(A^\delta \bigcirc p \mathcal{U} E^\delta \bigcirc q)$ of degree 1 in a state $v \in U^m$ in a model M_k where $k > m$, only states $u \in U^m \cup U^{m+1}$ are involved. In the evaluation of the degree 2 formula $E^\delta(A^\delta \bigcirc p \mathcal{U} E^{\delta'} \bigcirc q)$, only states in $U^m \cup U^{m+1} \cup U^{m+2}$ are involved. If there are j "switches" between dimensions, only states in $U^m \cup \cdots \cup U^{m+j}$ are involved. Thus, if the degree of α is j, the evaluation of α in $v \in U^m$ may involve states in U^{m+1}, U^{m+2}, .., U^{m+j}, but not states from U^{m+j+1}. This means that the satisfaction relationship between α and (M_{m+j+1}, v) is classical – it does not depend on any state from U^{m+j+1}. □

Finally, we define a $\mathcal{M}_{MCTL}$ model for ϕ_0. Let $j = \deg(\phi_0)$. We have that $M_{j+1}, \hat{s} \models \phi_0$ holds, and since $\hat{s} \in U^0$, the satisfaction relationship between ϕ_0 and $(M_{j+1}, \hat{s})$ is classical – the fact that $M_{j+1}, \hat{s} \models \phi_0$ does not depend on U^{j+1} (or τ^{j+1} or K^{j+1}). Take $M = (S, \{R_\delta : \delta \in \Delta\}, L)$ such that $S = S^{j+1}$, $R_\delta = R^{j+1}_\delta$, and $L(s) = L^{j+1}(s)$. Because $M_{j+1}, \hat{s} \models \phi_0$ does not depend on U^{j+1}, we also have that $M, \hat{s} \models \phi_0$. Since M_{j+1} is finite, M is finite.

4.3 Complexity

Now, we know that the satisfiability problem for CTL is EXPTIME-complete, and this gives us an EXPTIME lower bound for MCTL satisfiability (since CTL is — very obviously – a fragment of MCTL). But the construction described above also gives us an EXPTIME upper bound, thus giving the following.

Theorem 4.11 *The satisfiability problem for* MCTL *is* EXPTIME*-complete.*

Consider the decision procedure outlined in section 4.1. The idea behind this procedure is to use a constructive CTL decision procedure (such as the tableau method described in [4]) as a sub-routine for constructing components of a model for the input formula, each component corresponding to a different dimension. The use of the sub-routine is analytic, in that, each time we call the CTL satisfiability checking sub-routine, we are working with strict sub-formulae of the input formula. Thus, the overall running time of the procedure described in section 4.1 for a formula ϕ over Θ is $O(2^{l \cdot m \cdot n})$ where $l = |dim(\phi)|$ is the number of dimensions in ϕ, $m = |\Theta|$ is the number of atomic propositions in ϕ, and $n = deg(\phi)$ is the degree of ϕ.

5 Discussion

Model construction techniques similar to the one we have used are found in several works on transfer of properties to fusions. As discussed by Fajardo and Finger [6], many proofs of meta-logical properties of fusions in the literature [8,9,10,11] employ the same strategy of (i) studying the modalisation/temporalisation of a generic logic;

(ii) studying the (finite) *iterated modalisations* of two modal logics and (iii) viewing the fusion as a union of iterated modalisations. While the proof strategy used in this paper do not employ that strategy directly, there certainly are similarities.

References

[1] Ågotnes, T., W. van der Hoek, C. S. Juan A. Rodriguez-Aguilar and M. Wooldridge, *On the logic of normative systems*, in: M. M. Veloso, editor, *Proceedings of the Twentieth International Joint Conference on Artificial Intelligence (IJCAI 2007)* (2007), pp. 1175–1180.

[2] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic," Cambridge Tracts in Theoretical Computer Science **53**, Cambridge University Press, Cambridge University Press, 2001.

[3] Clarke, E. M., O. Grumberg and D. A. Peled, "Model Checking," MIT Press, 2000.

[4] Emerson, E. A., *Temporal and modal logic*, in: J. van Leeuwen, editor, *Handbook of Theoretical Computer Science Volume B: Formal Models and Semantics*, Elsevier Science Publishers B.V.: Amsterdam, The Netherlands, 1990 pp. 996–1072.

[5] Emerson, E. A. and J. Y. Halpern, *Decision procedures and expressiveness in the temporal logic of branching time*, Journal of Computer and System Sciences **30** (1985).

[6] Fajardo, R. and M. Finger, *How not to combine modal logics.*, in: B. Prasad, editor, *IICAI* (2005), pp. 1629–1647.

[7] Fine, K. and G. Schurz, *Transfer theorems for multimodal logics*, in: J. Copeland, editor, *Logic and Reality: Essays on the Legacy of Arthur Prior*, Oxford University Press, Oxford, 1996 pp. 169–213.

[8] Finger, M. and D. M. Gabbay, *Adding a temporal dimension to a logic system*, Journal of Logic, Language, and Information **1** (1992), pp. 203–233.

[9] Finger, M. and D. M. Gabbay, *Combining temporal logic systems*, Notre Dame Journal of Formal Logic **37** (1996), pp. 204–232.

[10] Finger, M. and M. A. Weiss, *The unrestricted addition of a temporal dimension to a logic system*, in: *3rd International Conference on Temporal Logic (ICTL2000)*, Leipzig, Germany, 2000.

[11] Finger, M. and M. A. Weiss, *The unrestricted combination of temporal logic systems*, Logic Journal of the IGPL **10** (2002), pp. 165–189.

[12] Gabbay, D. M., A. Kurucz, F. Wolter and M. Zakharyaschev, "Many-Dimensional Modal Logics: Theory and Applications," Elsevier, Amsterdam, 2003.

[13] Goranko, V. and S. Passy, *Using the universal modality: Gains and questions*, Journal of Logic and Computation **2** (1992), pp. 5–30.

[14] Gottlob, G. and C. Koch, *Monadic queries over tree-structured data*, in: *Proceedings of the 17th IEEE Symposium on Logic in Computer Science (LICS-02))* (2002), pp. 189–202.

[15] Kracht, M. and F. Wolter, *Simulation and transfer results in modal logic - a survey*, Studia Logica **59** (1997), pp. 149–177.

[16] Wolter, F., *Fusions of modal logics revisited*, in: M. Kracht, M. de Rijke, H. Wansing and M. Zakharyaschev, editors, *Advances in Modal Logic*, CSLI, Stanford, CA, 1998 .

[17] Wooldridge, M., "An Introduction to Multiagent Systems," John Wiley & Sons, 2002.