LECTURE 7: PROPOSITIONAL LOGIC (1)

Software Engineering

Mike Wooldridge

1 What is a Logic?

- When most people say 'logic', they mean either *propositional logic* or *first-order predicate logic*.
- However, the precise definition is quite broad, and literally hundreds of logics have been studied by philosophers, computer scientists and mathematicians.
- Any 'formal system' can be considered a logic if it has:
  - a well-defined *syntax*;
  - a well-defined *semantics*; and
  - a well-defined *proof-theory*.

- The *syntax* of a logic defines the syntactically acceptable objects of the language, which are properly called *well-formed formulae* (wff). (We shall just call them formulae.)
- The *semantics* of a logic associate each formula with a *meaning*.
- The *proof theory* is concerned with manipulating formulae according to certain rules.

2 Propositional Logic

- The simplest, and most abstract logic we can study is called *propositional logic*.
- **Definition:** A *proposition* is a statement that can be either *true* or *false*; it must be one or the other, and it cannot be both.
- EXAMPLES. The following are propositions:
  - the reactor is on;
  - the wing-flaps are up;
  - John Major is prime minister.

  whereas the following are not:
  - are you going out somewhere?
  - 2+3

- It is possible to determine whether any given statement is a proposition by prefixing it with:

  It is true that . . .

  and seeing whether the result makes grammatical sense.
- We now define *atomic* propositions. Intuitively, these are the set of smallest propositions.
- **Definition:** An *atomic proposition* is one whose truth or falsity does not depend on the truth or falsity of any other proposition.
- So all the above propositions are atomic.

- Now, rather than write out propositions in full, we will abbreviate them by using *propositional variables*.
- It is standard practice to use the lower-case roman letters

  $$p, q, r, \ldots$$

  to stand for propositions.
- If we do this, we must define what we mean by writing something like:

  Let p be *John Major is prime Minister*.
- Another alternative is to write something like *reactor_is_on*, so that the interpretation of the propositional variable becomes obvious.

2.1 The Connectives

- Now, the study of atomic propositions is pretty boring. We therefore now introduce a number of *connectives* which will allow us to build up *complex propositions*.
- The connectives we introduce are:

$$\begin{array}{ll} \wedge & \text{and } (\& \text{ or } .) \\ \vee & \text{or } (| \text{ or } +) \\ \neg & \text{not } (\sim) \\ \Rightarrow & \text{implies } (\supset \text{ or } \rightarrow) \\ \Leftrightarrow & \text{iff} \end{array}$$

- (Some books use other notations; these are given in parentheses.)

And

- Any two propositions can be combined to form a third proposition called the *conjunction* of the original propositions.
- **Definition:** If p and q are arbitrary propositions, then the *conjunction* of p and q is written

  $$p \wedge q$$

  and will be true iff both p and q are true.

- We can summarise the operation of $\wedge$ in a *truth table*. The idea of a truth table for some formula is that it describes the behaviour of a formula under all possible interpretations of the primitive propositions the are included in the formula.
- If there are n different atomic propositions in some formula, then there are 2^n different lines in the truth table for that formula. (This is because each proposition can take one 1 of 2 values — *true* or *false*.)
- Let us write T for truth, and F for falsity. Then the truth table for $p \wedge q$ is:

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Or

- Any two propositions can be combined by the word 'or' to form a third proposition called the *disjunction* of the originals.
- **Definition:** If p and q are arbitrary propositions, then the *disjunction* of p and q is written

  $$p \vee q$$

  and will be true iff either p is true, or q is true, or both p and q are true.

- The operation of $\vee$ is summarised in the following truth table:

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

If… Then…

- Many statements, particularly in mathematics, are of the form:

  if p *is true then* q *is true.*

  Another way of saying the same thing is to write:

  p *implies* q.

- In propositional logic, we have a connective that combines two propositions into a new proposition called the *conditional*, or *implication* of the originals, that attempts to capture the sense of such a statement.

- **Definition:** If p and q are arbitrary propositions, then the *conditional* of p and q is written

  $$p \Rightarrow q$$

  and will be true iff either p is false or q is true.

- The truth table for $\Rightarrow$ is:

p	q	$p \Rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

- The $\Rightarrow$ operator is the hardest to understand of the operators we have considered so far, and yet it is extremely important.
- If you find it difficult to understand, just remember that the $p \Rightarrow q$ means 'if p is true, then q is true'.

  If p is false, then we don't care about q, and by default, make $p \Rightarrow q$ evaluate to T in this case.
- Terminology: if ϕ is the formula $p \Rightarrow q$, then p is the *antecedent* of ϕ and q is the *consequent*.

Iff

- Another common form of statement in maths is:

  p *is true if, and only if,* q *is true*.

- The sense of such statements is captured using the *biconditional* operator.
- **Definition:** If p and q are arbitrary propositions, then the *biconditional* of p and q is written:

  $$p \Leftrightarrow q$$

  and will be true iff either:

  1. p and q are both true; or
  2. p and q are both false.

- The truth table for $\Leftrightarrow$ is:

p	q	$p \Leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

- If $p \Leftrightarrow q$ is true, then p and q are said to be *logically equivalent*. They will be true under exactly the same circumstances.

Not

- All of the connectives we have considered so far have been *binary*: they have taken *two* arguments.
- The final connective we consider here is *unary*. It only takes *one* argument.
- Any proposition can be prefixed by the word 'not' to form a second proposition called the *negation* of the original.
- **Definition:** If p is an arbitrary proposition then the *negation* of p is written

  $$\neg p$$

  and will be true iff p is false.
- Truth table for $\neg$:

p	$\neg p$
F	T
T	F

Comments

- We can *nest* complex formulae as deeply as we want.
- We can use *parentheses* i.e.,),(, to *disambiguate* formulae.
- EXAMPLES. If p, q, r, s and t are atomic propositions, then all of the following are formulae:

  $$p \wedge q \Rightarrow r$$
  $$p \wedge (q \Rightarrow r)$$
  $$(p \wedge (q \Rightarrow r)) \vee s$$
  $$((p \wedge (q \Rightarrow r)) \vee s) \wedge t$$

  whereas none of the following is:

  $$p \wedge$$
  $$p \wedge q)$$
  $$p\neg$$

3 Tautologies & Consistency

- Given a particular formula, can you tell if it is true or not?
- No — you usually need to know the truth values of the component atomic propositions in order to be able to tell whether a formula is true.
- **Definition:** A *valuation* is a function which assigns a truth value to each primitive proposition.
- In Modula-2, we might write:

```
PROCEDURE Val(p : AtomicProp):
    BOOLEAN;
```

- Given a valuation, we can say for any formula whether it is true or false.

- EXAMPLE. Suppose we have a valuation v, such that:

$$\begin{aligned} v(p) &= F \\ v(q) &= T \\ v(r) &= F \end{aligned}$$

Then we truth value of $(p \vee q) \Rightarrow r$ is evaluated by:

$$\begin{aligned} & (v(p) \vee v(q)) \Rightarrow v(r) && (1) \\ &= (F \vee T) \Rightarrow F && (2) \\ &= T \Rightarrow F && (3) \\ &= F && (4) \end{aligned}$$

Line (3) is justified since we know that $F \vee T = T$.

Line (4) is justified since $T \Rightarrow F = F$.

If you can't see this, look at the truth tables for $\vee$ and $\Rightarrow$.

- When we consider formulae in terms of interpretations, it turns out that some have interesting properties.
- **Definition:**
  1. A formula is a *tautology* iff it is true under *every* valuation;
  2. A formula is *consistent* iff it is true under *at least one* valuation;
  3. A formula is *inconsistent* iff it is not made true under *any* valuation.
- Now, each line in the truth table of a formula correponds to a valuation.
- So, we can use truth tables to determine whether or not formulae are tautologies.