LECTURE 8: SETS

Software Engineering

Mike Wooldridge

1 What is a Set?

- The concept of a set is used throughout mathematics; its formal definition matches closely our intuitive understanding of the word.
- **Definition:** A *set* is an *unordered* collection of *distinct* objects.
- We can build sets containing any objects that we like, but usually we consider sets whose elements have *some property in common*.
- The objects comprising a set are called its *elements* or *members*.
- Note that:
  - sets do not contain *duplicates*;
  - the order of elements in a set is *not significant*.

2 Defining Sets — Enumeration

- By convention, a set can be defined by *enumerating* its components in curly brackets.
- EXAMPLE.

  $$Vowels == \{a, e, i, o, u\}$$

  EXAMPLE.

  $$Weekend == \{Sat, Sun\}$$

- The double-equals symbol (==) is read 'is defined to be'; it gives us a method for naming sets, so that we can subsequently use them.
- (We can't use sets unless they have been previously defined in some way.)

- In Z, another equivalent method for defining sets is allowed …
- EXAMPLE.

$$\begin{array}{rcl} \mathit{Vowels} & ::= & a \\ & | & e \\ & | & i \\ & | & o \\ & | & u \end{array}$$

- Another standard convention is to allow number ranges …
- EXAMPLE.

$$\mathit{Days} == 1 \mathrel{..} 365$$

- Some sets occur so often that they have been pre-defined and given standard names.

  $\mathbb{N}$ the *natural numbers*:

  $$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

  $\mathbb{N}_1$ the natural numbers greater than 0:

  $$\mathbb{N}_1 = \{1, 2, 3, \ldots\}$$

  Z the *integers*:

  $$Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

  $\mathbb{B}$ the *truth values*:

  $$\mathbb{B} = \{\mathit{true}, \mathit{false}\}$$

3 Relational Operators for Sets

- The set theory in Z provides us with a variety of *relational operators* for expressing the properties of sets.

3.1 Set Membership

- To express the fact that an object x is a *member* of set S, we write

  $$x \in S$$

  Note that an expression with this form is either *true* or *false*; the thing is either a member or not.

- EXAMPLES.

  $$a \in \{a, e, i, o, u\}$$
  $$1 \in \mathbb{N}$$
  $$23 \in Days$$

- Note that the r.h.s. of the expression $x \in S$ must be an expression which returns a set of values of the same type as the expression on the l.h.s returns.
- EXAMPLE. This isn't allowed:

  $$\{1, 2, 3\} \in \{4, 5, 6\}$$

  But this is:

  $$\{1, 2, 3\} \in \{\{6, 7, 8\}, \{1, 2, 3\}\}$$

  and so is this:

  $$\{1, 2, 3\} \in \{\{6, 2, 1\}\}$$

  The first legal expression above evaluates to *true*; the second evaluates to *false*.
- Note that Z differs in this respect from what might be called *naive set theory*: Z is said to be a *typed* set theory, (because we must check the types on the l.h.s and r.h.s. of our expressions.)

3.2 Set Equality

- **Definition:** Let S and T be arbitrary sets of the same type. Then

  $$S = T$$

  is *true* iff S and T contain precisely the same members.
- Note that the *order* in which values occur in the set is not significant.
- EXAMPLES. The following is a true expression:

  $$\{1, 2, 3\} = \{2, 3, 1\}$$

  as is this

  $$Days = 1 \mathrel{..} 365$$

  however, the following is not:

  $$Days = \mathbb{N}.$$

3.3 Subsets

- **Definition:** Let S and T be arbitrary sets of the same type. Then

  $$S \subseteq T$$

  is true iff *every member of S is also a member of y.*

- EXAMPLES. The following are true expressions:

  $$\{1, 2, 3\} \subseteq \mathbb{N};$$
  $$\{true\} \subseteq \mathbb{B};$$
  $$\mathbb{N} \subseteq \mathbb{Z};$$
  $$\mathbb{N}_1 \subseteq \mathbb{N};$$
  $$\{1, 2, 3\} \subseteq \{1, 2, 3\}.$$

  whereas the following are not:

  $$\{1, 2, 3, 4\} \subseteq \{1, 2, 3\};$$
  $$\mathbb{Z} \subseteq \mathbb{N}.$$

- Note that equal sets are subsets of each other; we can express this fact in the following theorem about sets:

  $$(S = T) \Rightarrow (S \subseteq T)$$

3.4 Proper Subsets

- As we just saw, if two sets are equal, then they are subsets of each other. This may seem strange! We introduce another relational operator that more accurately captures our intuitive understanding of what 'subset' means.
- **Definition:** Let S and T be arbitrary sets of the same type. Then:

  $$S \subset T$$

  is true iff both $S \subseteq T$ and $S \neq T$.
- If $S \subset T$ then then S is said to be a *proper subset* of T.
- Note that we can formally define the $\subset$ relation using concepts introduced earlier:

  $$S \subset T \Leftrightarrow (S \subseteq T) \wedge (S \neq T)$$

- *Note the use of $\Leftrightarrow$ to introduce definitions.*

4 The Empty Set

- There is a special set that has the property of having no members; this set is called the *empty set*, and is denoted either

  $$\{\}$$

  or (more usually)

  $$\emptyset$$

- Note that if x is *any* value, then the expression

  $$x \in \emptyset$$

  must be false. (Nothing is a member of the empty set.)

- Similarly, if S is an arbitrary set, then

  $$\emptyset \subseteq S$$

  must be true; this includes of course

  $$\emptyset \subseteq \emptyset$$

  (The emptyset is a subset of every set.)

- But note that this is *not* a theorem:

  $$S \subset \emptyset$$

  Why?

5 The Powerset Operator

- **Definition:** Let S be an arbitrary set. Then the set of all subsets of S is given by

$$\mathbb{P}\, S.$$

- EXAMPLES.
  - If
  $$S = \{1\}$$
  then
  $$\mathbb{P}\, S = \{\emptyset, \{1\}\}$$
  - If
  $$S = \{1, 2\}$$
  then
  $$\mathbb{P}\, S = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$
  - If
  $$S = \{a, b, c\}$$
  then
  $$\mathbb{P}\, S = ?$$

- We can again use $\Leftrightarrow$ to introduce $\mathbb{P}$ as a *derived* operator:

$$T \in \mathbb{P}\, S \Leftrightarrow (T \subseteq S)$$

- Note that if S has n members, then $\mathbb{P}\, S$ has 2^n members.
- For this reason, the powerset of a set S is sometimes denoted by 2^S. (But not by us.)

6 Manipulating Sets

6.1 Set Union

- The *union* of two sets is a third set that contains the members of both.
- The symbol for union is $\cup$.
- **Definition:** Let S and T be arbitrary sets. Then

  $$x \in (S \cup T)$$

  is true iff either $x \in S$ or $x \in T$.
- Note that we can introduce this as a derived operator:

  $$x \in S \cup T \Leftrightarrow x \in S \vee x \in T.$$

- EXAMPLES.

  $$\{a, e, i\} \cup \{o, u\} = \{a, e, i, o, u\};$$
  $$\{a, e\} \cup \{e, i\} = \{a, e, i\};$$
  $$\emptyset \cup \{a, e\} = \{a, e\}.$$

6.2 Set Intersection

- The *intersection* of two sets is a third set that contains only elements common to both.
- **Definition:** Let S and T be arbitrary sets. Then

  $$x \in (S \cap T)$$

  iff both $x \in S$ and $x \in T$.
- As a derived operator:

  $$x \in (S \cap T) \Leftrightarrow (x \in S) \wedge (x \in T).$$
- EXAMPLES.

  $$\{a, e, i\} \cap \{o, u\} = \emptyset;$$
  $$a \neg \in \{a, e, i\} \cap \{e, i\};$$
  $$\mathbb{N} = \mathbb{N} \cap \mathbb{Z}.$$

6.3 Set Difference

- The notation $S \setminus T$ denotes the set obtained from S by removing from it all the elements of T tha occur in it; in other words, you take T away from S.
- **Definition:** Let S and T be arbitrary sets. Then

  $$x \in (S \setminus T)$$

  is true iff $x \in S$ and $x \notin T$.
- As a derived operator:

  $$\ldots$$

- EXAMPLES.

  $$\{a, e, i, o, u\} \setminus \{o\} = \{a, e, i, u\}$$
  $$\emptyset \setminus \{1, 2, 3\} = \emptyset.$$

6.4 Theorems about Sets

Let S, T, and U be arbitrary sets of the same type. Then:

$$
\begin{aligned}
S \cup S &= S \\
S \cap S &= S \\
(S \cup T) \cup U &= S \cup (T \cup U) \\
(S \cap T) \cap U &= S \cap (T \cap U) \\
S \cup T &= T \cup S \\
S \cap T &= T \cap S \\
S \cup (T \cap U) &= (S \cup T) \cap (S \cup U) \\
S \cap (T \cup U) &= (S \cap T) \cup (S \cap U) \\
S \cup \emptyset &= S \\
S \cap \emptyset &= \emptyset
\end{aligned}
$$

6.5 Cardinality

- The *cardinality* of a set is the number of elements in it.
- **Definition:** If S is an arbitrary set, then the *cardinality* of S is denoted $\#S$.
- Note that for an arbitrary set S:

$$\#S \in \mathbb{N}.$$

- EXAMPLES.

$$\#\{1, 2, 3\} = 3;$$
$$\#\emptyset = 0;$$
$$\#(\{1, 2, 3\} \cap \{2\}) = 1.$$

7 Defining Sets: Comprehension

- We have seen how sets may be defined by *numeration* — listing their contents.
- This technique is impractical for large sets!
- *Comprehension* is a way of defining sets:
  - in terms of their properties;
  - in terms of other sets.
- Comprehension is best explained via examples:

  $\{n : \mathbb{N} \mid (n \geq 0) \wedge (n \leq 3)\} = \{0, 1, 2, 3\}$

  $\{n : \mathbb{Z} \mid (n > 1) \wedge (n < 1)\} = \emptyset$

  $\{n : \mathbb{N} \mid (n \geq 0) \wedge (n \leq 3) \bullet n + 1\} = \{1, 2, 3, 4\}$

- In general, a set comprehension expression will have three parts:
  - *a signature* — specifies the base sets from which values are extracted;
  - *a predicate* — the defining property of the set;
  - *a term* — specifies how the actual values in the set are computed.
- Generally, we omit the term part — we just have a signature and a predicate, as in examples (1) and (2), above.