# LECTURE 15: RELATIONS

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# 1 Introduction

• We saw in earlier lectures that a function is just a set of maplets, for example:

 $tel == \{mikew \mapsto 1531, eric \mapsto 1489\}.$ 

A maplet is just an ordered pair, so another way of writing this is

 $tel == \{(mikew, 1531), (eric, 1489)\}$ 

Formally, if  $f : T_1 \rightarrow T_1$ , then f is a subset of the cartesian product of  $T_1 \times T_1$ :

 $f \subseteq T_1 \times T_2.$ 

But functions can't be *defined* in this way, as they must have the uniqueness property — the following is not a function:

 $\{(mike, 1531), (eric, 1531)\}.$ 

- A more general way of capturing a relationship between two sets is to use a *relation*.
- Relations are similar to functions, but do not have the uniquness property.

- Just as we have a variety of function construction arrows, so we have the relation constructor arrow, '↔'.
- **Definition:** If *T*<sub>1</sub> and *T*<sub>2</sub> are arbitrary types, then

 $T_1 \leftrightarrow T_2$ 

is an expression giving the set of all relations between  $T_1$  and  $T_2$ . It may be defined:

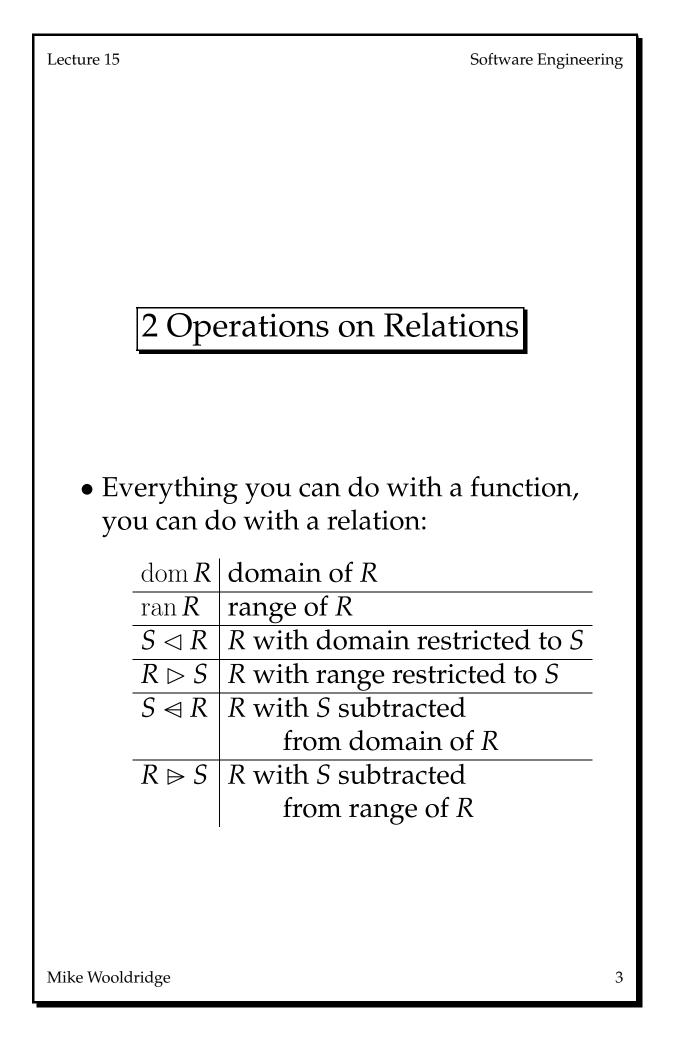
$$T_1 \leftrightarrow T_2 == \mathbb{P}(T_1 \times T_2).$$

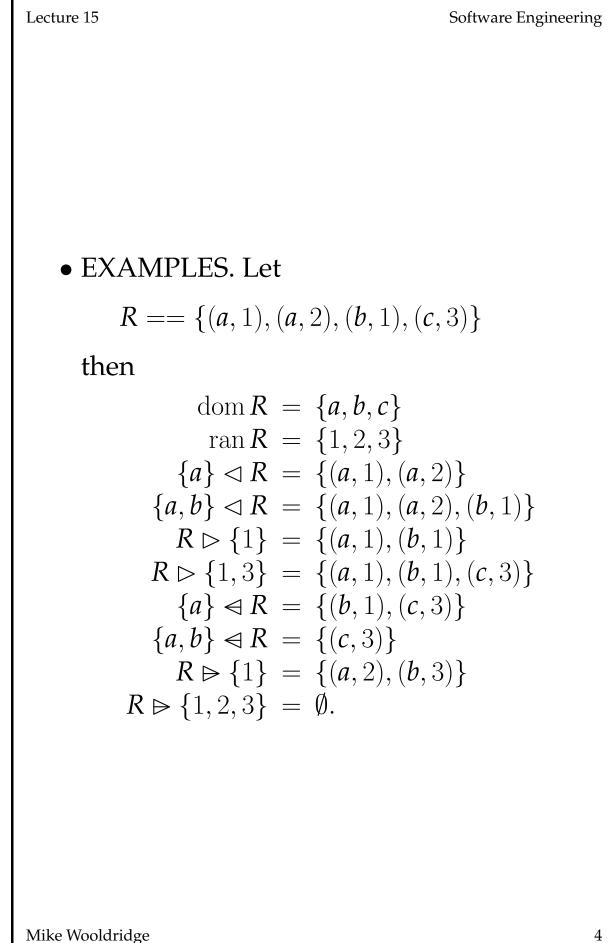
• EXAMPLE. Let

$$A == \{a, b, c\}$$
  
 $B == \{1, 2\}$ 

then

$$\{(a,1),(a,2)\}\in A\leftrightarrow B.$$





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# 2.1 The Inverse of a Relation

- Additionally, we can take the *inverse* of a relation.
- **Definition:** If

$$R:T_1\leftrightarrow T_2$$

then

 $R^{\sim}:T_1:T_2$ 

such that  $(y, x) \in R^{\sim}$  iff  $(x, y) \in R$ . Formally:

$$R^{\sim} == \{x: T_1; y: T_1 \mid (x, y) \in R \bullet (y, x)\}$$

• EXAMPLE. If

$$R == \{(a, 1), (a, 2), (b, 1), (c, 3)\}$$

then

$$R^{\sim} = \{(1, a), (1, 2), (1, b), (3, c)\}.$$

• Note that you can always take the inverse of a function, since functions are just special kinds of relation, but you do not always get a function as a result.

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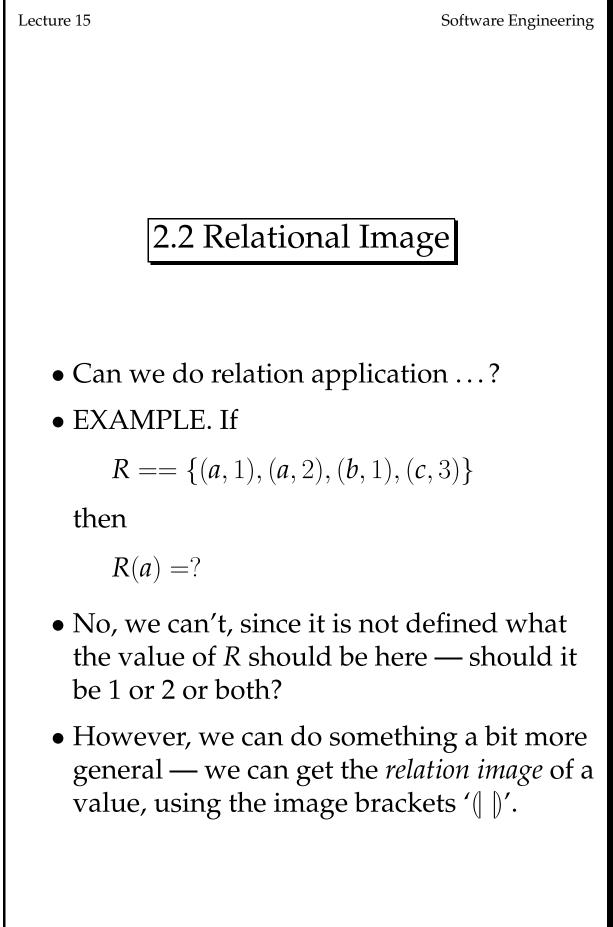
#### • EXAMPLE. Suppose we had

 $f == \{a \mapsto 1, b \mapsto 1\}$ 

Now *f* is certainly a function (though it is not one-to-one.) But take the inverse of *f*:

 $f^{\sim} = \{(1, a), (1, b)\}.$ 

Although  $f^{\sim}$  is a relation, it is *not* a function.



• **Definition:** If

 $\begin{array}{l} S: {\rm I\!P}\, T_1 \\ R: T_1 \leftrightarrow T_2 \end{array}$ 

then

R(|S|)

is an expression of type  $\mathbb{P} T_2$ , such that

 $y \in R(|S|) \Leftrightarrow \exists x : T_1 \bullet x \in S \land (x, y) \in R$ 

(i.e., *y* is in R(|S|) iff there is some *x* in *S* such that *R* relates *x* to *y*).

• EXAMPLE. If

$$R == \{(a, 1), (a, 2), (b, 1), (c, 3)\}$$

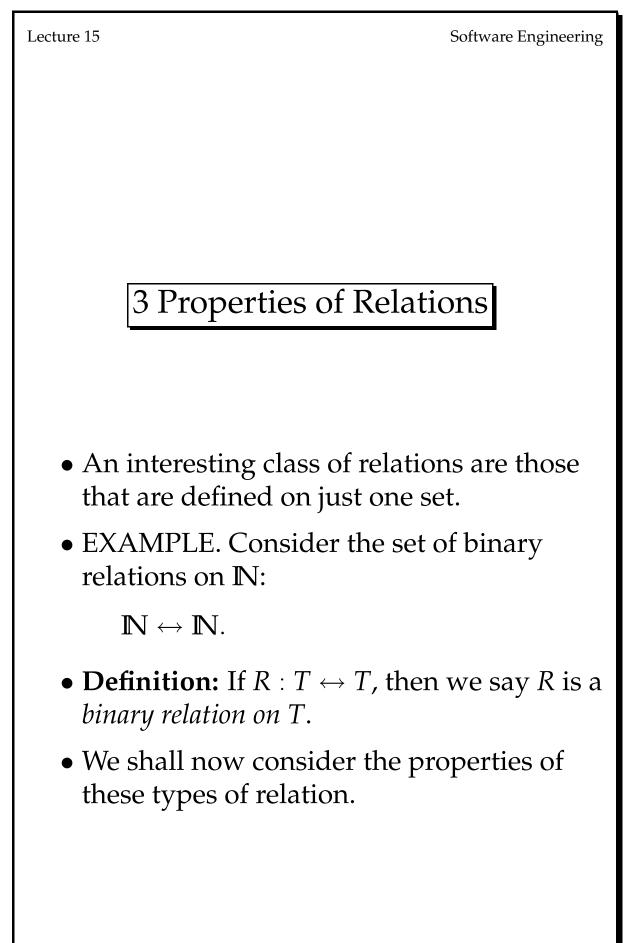
then

$$R(|\{a\}|) = \{1,2\}$$
  

$$R(|\{a,b\}|) = \{1,2\}$$
  

$$R(|\{a,c\}|) = \{1,2,3\}$$
  

$$R(|\{b\}|) = \{1\}$$



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# 3.1 Reflexivity

- Suppose we have a relation *R* on *T* such that for any element *x* ∈ (dom *R* ∪ ran *R*), we have (*x*, *x*) ∈ *R*. Then *R* is said to be *reflexive*.
- EXAMPLES.
  - **1.** {(1, 1), (1, 2), (2, 2), (1, 3), (3, 3)} ... *is* a reflexive relation on **N**.
  - 2. {(1,1), (1,2), (2,2), (1,3)}
    … *is not* a reflexive relation on **N** (because (3,3) is not a member).
  - 3. The subset relation,  $\subseteq$ , is reflexive, because  $S \subseteq S$ , for any set S.
  - 4. The 'less that' relation, <, is not reflexive, because it is not true that *n* < *n*, for any value *n*.
  - 5. The 'is the father of' relation (defined on the set of people) is *not* reflexive, because it is not true that *p* is the father of *p*, for any person *p*.

# 3.2 Symmetry

- A relation *R* is said to be *symmetric* if whenever  $(x, y) \in R$ , we have  $(y, x) \in R$ .
- EXAMPLES.
  - 1.  $\{(2,1), (1,2), (2,3), (3,2)\}$ 
    - $\dots$  *is* a symmetric relation on  $\mathbb{N}$ .
  - 2. {(2,1), (1,2), (2,3)}
    ... *is not* a symmetric relation on **N** (because (2,3) is a member, but (3,2) is not).
  - 3. The equality relation, '=', is a symmetric relation, since a = b implies b = a.
  - 4. The subset relation, ' $\subseteq$ ', is not a symmetric relation, since it is not generally the case that  $S \subseteq T$  implies  $T \subseteq S$ .
  - 5. The 'is father of relation' is not symmetric.



# 3.3 Transitivity

- A relation *R* is said to be *transitive* iff whenever we have  $(x, y) \in R$  and  $(y, z) \in R$ , we also have  $(x, z) \in R$ .
- EXAMPLES.
  - 1.  $\{(2,1), (1,2), (2,3), (3,2)\}$ 
    - $\dots$  *is not* a transitive relation, as it does not contain (1, 3).
  - 2. The less than relation, <, *is* transitive, since if a < b and b < c then a < c.
  - 3. The 'is an ancestor of' relation is transitive.
  - 4. Equality is transitive.

# 3.4 Equivalence Relations

- **Definition:** If a relation is reflexive, symmetric, and transitive, then it is called an *equivalence* relation.
- The general idea behind equivalence relations is that they classify objects which are 'alike' in some respect.
- EXAMPLES.
  - 1. Equality is an equivalence relation.
  - 2. The relation 'is the same species as', defined on the set of all animals, is an equivalence relation.
  - 3. The relation 'owns the same make car as', defined on the set of people, is an equivalence relation.
  - 4. Neither < nor  $\subseteq$  are equivalence relations.
- Equivalence relation are often written  $\equiv$  or  $\sim$ .

# **3.5 Reflexive and Transitive Closures**

• **Definition:** If *R* is a relation on some set *T*, then the *reflexive closure* of *R* is the smallest reflexive relation containing *R*, and is given by the expression

 $R^+$ .

• **Definition:** If *R* is a relationon some set *T*, then the *transitive closure* of *R* is the smallest transitive relation containing *R*, and is given by the expression

 $R^*$ .