Many-Valued Multiple-Expert modal models

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Abstract

In AI and other branches of Computer Science expert models are often studied and used. Here we examine experts which evaluate formulas of modal logic. An expert model consists of a set of experts and a domination relation which dictates how each expert's opinion is dependent upon the opinions expressed by other experts. Such multiple-expert modal models were introduced and investigated by Fitting in 1991-92. Fitting also introduced Heyting-style semantics for modal logics, called many-valued modal models, and showed their equivalence to multiple-expert modal models.
In this paper, multiple-expert modal models are extended to a system where experts reason in a many-valued fashion: they assign truth-values taken from finite Heyting algebras. These many-valued multiple-expert modal models are shown to be as well embeddable – and thus equivalent – to many-valued modal models.

Keywords: Modal logic, many-valued logic, expert models.
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1 Introduction

In AI and other branches of Computer Science **expert models** are often studied and used. In rough terms, an expert is an entity expressing an opinion upon the value of a given function throughout its domain. In our setting, experts evaluate formulas of modal logic. An expert model consists of a set of experts and a domination relation which classifies experts as **dominant** or **dominated**, and thus dictates how each expert's opinion must be in accordance to opinions expressed by other experts.
Such expert models were introduced and investigated by Fitting in [Fit92a] and [Fit92b]. There the experts would assign truth-values to formulas in two-valued (true/false) manner. Fitting called these models *multiple-expert modal models*. Moreover, he introduced Heyting-style semantics for modal logics – called many-valued modal models – and showed their equivalence to multiple-expert modal models.
In this paper, we examine multiple-expert modal models in which experts reason in a many-valued fashion: they assign truth-values taken from finite Heyting algebras (of truth-values). These expert models, which generalize those in [Fit92b], are called *many-valued multiple-expert modal models (*`MVMEm`*-models)* and are shown to be as well embeddable – and thus equivalent – to many-valued modal models. This implies that `MVMEm`-models are equivalent to 'ordinary' multiple-expert modal models.

1.1 Dominant and dominated experts

In order to clarify the behavior of the domination relation between experts, we examine a simple example from [Fit92b] in which we assume that assignments are two-valued ($\top$: true, $\bot$: false) and that formulas do not contain modal operators.
We examine the case of two non-independent experts: let e, f be experts that assign truth-values from the set $\{\top, \bot\}$ to formulas of propositional logic. Moreover, suppose that e **dominates** f; this means that

for every formula ϕ, if e assigns $\top$ to ϕ, then f also assigns $\top$ to ϕ (1)
(**domination condition**).

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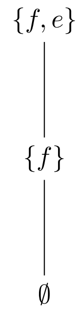
The above condition is identical to the behavior of Kripke intuitionistic models, that is Kripke semantics for intuitionistic propositional logic (see for example [SU98]). To add more to this Kripke-style semantical consideration, suppose we have the formulas ϕ, ψ that are assigned $\perp$ by e, whereas f assigns $\top$ to ϕ and $\perp$ to ψ. Recall that e dominates f; up to this point the domination condition is valid. However, if we let the experts assign truth-values to $\phi \supset \psi$ independently (each taking into consideration only his own assignments to ϕ and ψ), then e would assign $\top$ to it whereas f would assign $\perp$ – breaking the domination condition. Hence, in the case of implication, experts must form their opinion taking into consideration the opinion of the other (in fact, of their dominated) experts.

In a different approach, suppose we take as 'total' truth-value of a formula ϕ to be the set of experts that assign $\top$ to ϕ. This leads to a consideration of a many-valued interpretation, and the possible truth-values are three: $\emptyset$, $\{f\}$, $\{e, f\}$. The truth-value $\{e\}$ is not assignable to any formula, due to the domination condition. The underlying algebra for the set of truth-values is a *Heyting algebra*, having the diagram on the side. Hence, what we are essentially considering is a semantical interpretation following Heyting models for intuitionistic propositional logic.



Both the above above semantics were extended by Fitting to models for propositional modal logic. Before we examine those models, let us briefly mention some definitions and properties for Heyting algebras.

1.2 Finite Heyting algebras and finite distributive lattices

We will be considering sets of truth values that form Heyting algebras. The latter are defined as follows.

Definition 1.1
Let $\langle \mathcal{T}, \vee, \wedge \rangle$ be a lattice (usually denoted $\mathcal{T}$) and let $\alpha, \beta \in \mathcal{T}$. The pseudo-complement of α relative to β, denoted $\alpha \Rightarrow \beta$, is the greatest element of S, if such an element exists, where

$$S = \{\gamma \in \mathcal{T} \mid \alpha \wedge \gamma \leq \beta\}$$

$\mathcal{T}$ is relatively pseudo-complemented if, for all $\alpha, \beta \in \mathcal{T}$, $\alpha \Rightarrow \beta$ exists.
A relatively pseudo-complemented bounded lattice is called a Heyting algebra. $\dashv$

An important property of lattices we are going to use is distributivity.

Definition 1.2
A lattice $\mathcal{T}$ is distributive if one of the following equivalent statements hold.

$$\forall \alpha, \beta, \gamma \in \mathcal{T}.\ \alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$
$$\forall \alpha, \beta, \gamma \in \mathcal{T}.\ \alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$$

$\dashv$

In the case of finite lattices, distributivity and relative pseudo-complementation are equivalent, and hence finite Heyting algebras are exactly finite distributive lattices.

Theorem 1.3 ([Bir67]) *Let $\mathcal{T}$ be a finite lattice. Then, $\mathcal{T}$ is relatively pseudo-complemented iff $\mathcal{T}$ is distributive.*

The latter theorem is very convenient for checking whether certain finite algebras are Heyting algebras, since distributivity can be checked easily:

Theorem 1.4 (Sholander, see [Bir67]) *An algebra $\langle A, \vee, \wedge \rangle$ is a distributive lattice iff for all $\alpha, \beta, \gamma \in A$,*

$$\alpha \wedge (\alpha \vee \beta) = \alpha$$
$$\alpha \wedge (\beta \vee \gamma) = (\gamma \wedge \alpha) \vee (\beta \wedge \alpha)$$

Finally, in several points below we are going to use some standard properties of finite distributive lattices, which can be found in [Bir67] or [Ras74].

2 Defining two different semantics

In this section we give formal definitions for the observations of section 1.1. We define two different styles of semantics for modal logics, namely Heyting and Kripke semantics, both extending semantics for intuitionistic propositional logic.
For clarity, we first give a formal definition of the language of modal logic we are going to use, $\mathcal{L}^{mod}$.

Definition 2.1
The language $\mathcal{L}^{mod}$ is defined by:

- The set of logical symbols, which consists of:
  1. Symbols for propositional variables: p, q, etc.
     The set of propositional variables is denoted Var and is countable.
  2. Symbols for logic operators: $\wedge, \vee, \supset$.
  3. Symbols for modal operators: $\Box, \Diamond$.
- The set of non-logical symbols, which consists of the countably infinite set $Cons$ of propositional constants, in which we distinguish the constants $\top, \bot$.

$\dashv$

The set of atomic formulas will be denoted AF ($= Var \cup Cons$), and the set of all formulas For. The set $Cons$ of constants is going to serve us in interpreting truth-values in the language, once we utilize many-valued semantics for modal logic.
Now, many-valued modal models where introduced by Fitting in [Fit92b] as an extension of Heyting semantics for intuitionistic propositional logic to modal logic. Each such model is based on a finite Heyting algebra $\mathcal{T}$ (ie. a finite distributive lattice), which intuitively corresponds to a set of truth-values. We reproduce the definition in two-step fashion.

Definition 2.2
A Many-Valued modal frame on a finite Heyting algebra $\mathcal{T}$ is a structure $\mathcal{F}_{\mathcal{T}} = \langle \mathcal{W}, \mathcal{R}, n \rangle$, where:

- $\mathcal{W}$ is a non-empty set, the set of states.
- $\mathcal{R} : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{T}$ is a many-valued accessibility relation.
- $n : Cons \rightarrow \mathcal{T}$ is a surjective naming function, giving to any constant of the language $\mathcal{L}^{mod}$ an interpretation in the lattice of truth-values; in particular, $n(\top) = \top$ and $n(\bot) = \bot$.

$\mathcal{F}_{\mathcal{T}}$ is called a $\mathcal{T}$-modal frame. $\dashv$

Since n is onto, it gives to any truth-value $\alpha \in \mathcal{T}$ constant representatives in the language. In fact, we will rarely ever denote $c \in Cons$ by its original name "c"; instead, it will be denoted as $\widehat{n(c)}$ – at the risk, of course, of identifying distinct constants. Thus, we will use letters $\alpha, \beta, \gamma, ...$ for members of $\mathcal{T}$, and $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, ...$ for members of $Cons$.
From $\mathcal{T}$-modal frames we move on to $\mathcal{T}$-modal models.

Definition 2.3 (Many-Valued modal models, or $\mathcal{T}$-modal models. [Fit92b])
A Many-Valued modal model on a finite Heyting algebra $\mathcal{T}$ is a structure $\mathcal{M}_{\mathcal{T}} = \langle \mathcal{F}_{\mathcal{T}}, v \rangle$, where:

- $\mathcal{F}_{\mathcal{T}}$ is a $\mathcal{T}$-modal frame.
- $v : \mathcal{W} \times AF \rightarrow \mathcal{T}$ is a valuation function, with $\mathcal{W}$ being the set $\mathcal{W}$ of $\mathcal{M}_{\mathcal{E},\mathcal{T}}$.
  For each $\alpha \in \mathcal{T}$ and all $A \in \mathcal{W}$, we impose $v(A, \hat{\alpha}) = \alpha$.
  Moreover, v is extended to $\bar{v} : \mathcal{W} \times For \rightarrow \mathcal{T}$, as follows. Let ϕ, ψ be formulas, ϕ' an atomic formula, and $A \in \mathcal{W}$; then,
  - $\bar{v}(A, \phi') = v(A, \phi')$

– $\bar{v}(A, \phi \wedge \psi) = \bar{v}(A, \phi) \wedge \bar{v}(A, \psi)$

– $\bar{v}(A, \phi \vee \psi) = \bar{v}(A, \phi) \vee \bar{v}(A, \psi)$

– $\bar{v}(A, \phi \supset \psi) = \bar{v}(A, \phi) \Rightarrow \bar{v}(A, \psi)$

– $\bar{v}(A, \Box\phi) = \bigwedge_{B \in \mathcal{W}} \big(\mathcal{R}(A, B) \Rightarrow \bar{v}(B, \phi)\big)$

– $\bar{v}(A, \Diamond\phi) = \bigvee_{B \in \mathcal{W}} \big(\mathcal{R}(A, B) \wedge \bar{v}(B, \phi)\big)$

$\bar{v}$ will be simply denoted v.

$\mathcal{M}$ is called a $\mathcal{T}$-modal model. $\dashv$

Extensions of Kripke intuitionistic semantics to modal logic, called multiple-expert modal models, were introduced in [Fit92b]. We extend that definition by defining many-valued multiple-expert modal models. The idea is to consider models where experts assign truth-values from a finite Heyting algebra $\mathcal{T}$ to modal formulas, and have in mind a many-valued accessibility function into $\mathcal{T}$ to interpret modal transition. The domination condition (1) seen above is extended to the condition:

$$\begin{aligned}&\text{whenever an expert } e \text{ dominating expert } f \text{ assigns a truth-value } \alpha \in \mathcal{T}\\ &\text{to a formula } \phi,\ f \text{ assigns to } \phi \text{ a truth-value } \beta \in \mathcal{T} \text{ such that } \alpha \leq \beta\end{aligned} \tag{2}$$

Clearly, if $\mathcal{T} = \{\top, \bot\}$ then (2) reduces to (1). A similar condition is imposed to modal transitions: if $\mathcal{W}$ is the set of modal states and $\mathcal{R}_e$ and $\mathcal{R}_f$ stand for transition functions of experts e, f then

if expert e dominates f, then for all $A, B \in \mathcal{W}$, $\mathcal{R}_e(A, B) \leq \mathcal{R}_f(A, B)$.

Let us first define frames for these models. For this, it will be necessary to define the notion of $\mathcal{D}$-compatible functions, where $\mathcal{D}$ is the domination relation between experts.

Definition 2.4
A Many-Valued Multiple-Expert modal frame (`MVMEm`-frame) is a structure $\mathcal{F}_{\mathcal{E},\mathcal{T}} = \langle \mathcal{E}, \mathcal{D}, \mathcal{W}, \mathcal{R}, n \rangle$, where $\mathcal{T}$ is a finite Heyting algebra and:

- $\mathcal{E}$ is a finite non-empty set, the set of experts.
- $\mathcal{D} : \mathcal{E} \times \mathcal{E} \to \{\top, \bot\} \subseteq \mathcal{T}$ is a partial order in $\mathcal{E}$ and corresponds to the domination relation[1].
- $\mathcal{W}$ is a non-empty set, the set of states.
- $\mathcal{R} : \mathcal{E} \times \mathcal{W} \times \mathcal{W} \to \mathcal{T}$ is the accessibility function and is monotone in its first argument.
  For brevity we write $\mathcal{R}_e(A, B)$ instead of $\mathcal{R}(e, A, B)$. Monotonicity amounts to the following domination condition. For every $f, e \in \mathcal{E}$ and $A, B \in \mathcal{W}$, if $e\mathcal{D}f$ then $\mathcal{R}_e(A, B) \leq \mathcal{R}_f(A, B)$.
- $n : Cons \to \mathcal{S}$ is a surjective naming function, where $\mathcal{S}$ is the set of $\mathcal{D}$-compatible functions from $\mathcal{E}$ to $\mathcal{T}$.

For $\mathcal{E}, \mathcal{D}, \mathcal{T}$ as above, a function $\mathfrak{a} : \mathcal{E} \to \mathcal{T}$ is $\mathcal{D}$-compatible if it is monotone between partial orders, that is if, for every $e, f \in \mathcal{E}$, if $e\mathcal{D}f$ then $\mathfrak{a}(e) \leq \mathfrak{a}(f)$. $\dashv$

The first argument of $\mathcal{R}$ signifies that accessibility between states is different for each expert. Note that the domination condition on $\mathcal{R}$ is equivalent to the statement $\mathcal{R}_f(A, B) = \bigvee\{\mathcal{D}(e, f) \wedge \mathcal{R}_e(A, B) \mid e \in \mathcal{E}\}$. Now, elements of $\mathcal{S}$ are what we called in the introductive example *total truth values*, modified for the many-valued setting. So `MVMEm`-frames assign to constants not only truth values (i.e. elements $\alpha \in \mathcal{T}$, now encoded $\{(e, \alpha) \mid e \in \mathcal{E}\}$), but also 'total' truth values (i.e. $\mathcal{D}$-compatible functions $\mathcal{E} \to \mathcal{T}$).
We can move on to many-valued multiple-expert modal models.

[1] Clearly, we can view $\mathcal{D}$ as a partial order defined by: $e\mathcal{D}f \iff \mathcal{D}(e, f) = \top$; read as "$e$ dominates f". The function view is very useful, since the domain of $\mathcal{D}$ is then inside $\mathcal{T}$. The partial order view, on the other hand, secures transitivity, reflexivity and the absence of non-trivial cycles in domination.

Definition 2.5 (Many-Valued Multiple-Expert modal models)
A Many-Valued Multiple-Expert modal model (`MVMEm`-model) is a structure $\mathcal{M}_{\mathcal{E},\mathcal{T}} = \langle \mathcal{F}_{\mathcal{E},\mathcal{T}}, w \rangle$, where:

- $\mathcal{F}_{\mathcal{E},\mathcal{T}} = \langle \mathcal{E}, \mathcal{D}, \mathcal{W}, \mathcal{R}, n \rangle$ is a `MVMEm`-frame.
- $w \ : \ \mathcal{E} \times \mathcal{W} \times AF \to \mathcal{T}$ is a valuation and is monotone in its first argument.
  For brevity we write $w_e(A, \phi)$ instead of $w(e, A, \phi)$. Monotonicity amounts to the following domination condition. For every $e, f \in \mathcal{E}$, $A \in \mathcal{W}$ and $\phi \in AF$, if $e\mathcal{D}f$ then $w_e(A, \phi) \leq w_f(A, \phi)$.
  For every $e \in \mathcal{E}$, $A \in \mathcal{W}$ and propositional constant $\hat{\mathfrak{a}}$, we impose $w_e(A, \hat{\mathfrak{a}}) = \mathfrak{a}(e)$.

$\dashv$

As with accessibility functions, we note that each expert has his own evaluation function w_e. Valuation w is extended to the set of (all) formulas in a way that preserves the domination condition.

Definition 2.6
Let $\mathcal{M}_{\mathcal{E},\mathcal{T}} = \langle \mathcal{E}, \mathcal{D}, \mathcal{W}, \mathcal{R}, n, w \rangle$ be a `MVMEm`-model. We extend w to $\bar{w} \ : \ \mathcal{E} \times \mathcal{W} \times For \to \mathcal{T}$ follows.
Let ϕ, ψ be formulas, ϕ' an atomic formula, $e \in \mathcal{E}$ and $A \in \mathcal{W}$, then,

- $\bar{w}_e(A, \phi') \ = \ w_e(A, \phi')$
- $\bar{w}_e(A, \phi \wedge \psi) \ = \ \bar{w}_e(A, \phi) \wedge \bar{w}_e(A, \psi)$
- $\bar{w}_e(A, \phi \vee \psi) \ = \ \bar{w}_e(A, \phi) \vee \bar{w}_e(A, \psi)$
- $\bar{w}_e(A, \phi \supset \psi) \ = \ \bigwedge_{f \in \mathcal{E}} \Big(\mathcal{D}(e, f) \Rightarrow (\bar{w}_f(A, \phi) \Rightarrow \bar{w}_f(A, \psi)) \Big)$
- $\bar{w}_e(A, \Box\phi) \ = \ \bigwedge_{f \in \mathcal{E}} \Big(\mathcal{D}(e, f) \Rightarrow \bigwedge_{B \in \mathcal{W}} (\mathcal{R}_f(A, B) \Rightarrow \bar{w}_f(B, \phi)) \Big)$
- $\bar{w}_e(A, \Diamond\phi) \ = \ \bigvee_{B \in \mathcal{W}} \Big(\mathcal{R}_e(A, B) \wedge \bar{w}_e(B, \phi) \Big)$

For brevity $\bar{w}$ will be simply denoted w. $\dashv$

The clauses of implication and modal operators above may seem strange at first sight, yet they are the many-valued analogs of the respective clauses in definition 2.2.
In the case of implication, recall that $\mathcal{D}(e, f) \in \{\top, \bot\}$. Now, for any $\alpha \in \mathcal{T}$, $\top \Rightarrow \alpha = \alpha$ and $\bot \Rightarrow \alpha = \top$, so

$$\bigwedge_{f \in \mathcal{E}} \Big(\mathcal{D}(e, f) \Rightarrow (w_f(A, \phi) \Rightarrow w_f(A, \psi)) \Big) = \bigwedge_{(f \in \mathcal{E}) \& \ e\mathcal{D}f} \Big(w_f(A, \phi) \Rightarrow w_f(A, \psi) \Big)$$

Thus, expert e assigns to $\phi \supset \psi$ the least element of the set of truth-values that would have been assigned by the experts dominated by e (in these experts e is included, since $\mathcal{D}$ is a partial order), had they been in position to assign truth-values independently.
The cases of modal operators follow exactly the above line of reasoning, with the only addition that here one has to take into account the many-valued accessibility function.

Note 2.7 It is not difficult to see that the multiple-expert modal models of [Fit92b] are `MVMEm`-models over $\mathcal{T} = \{\top, \bot\}$.

The next proposition shows that the valuation we defined for `MVMEm`-models preserves the domination condition (2).

Proposition 2.8 *Let $\mathcal{M}_{\mathcal{E},\mathcal{T}} = \langle \mathcal{E}, \mathcal{D}, \mathcal{W}, \mathcal{R}, w \rangle$ be a* `MVMEm`*-model. Then, for every $A \in \mathcal{W}$, every formula ϕ, and every $e, f \in \mathcal{E}$, if $\mathcal{D}(e, f) = \top$ then $w_e(A, \phi) \leq w_f(A, \phi)$.*

Proof: The proof is done by induction on ϕ. The case of atomic formulas is proven by definition of w; whereas the cases of $\wedge, \vee$ are relatively simple: for example, for every $A \in \mathcal{W}$, $e, f \in \mathcal{E}$ and $\phi, \psi \in For$, we have that, if $\mathcal{D}(e,f) = \top$, then,

$$w_e(A, \phi \wedge \psi) \;=\; w_e(A,\phi) \wedge w_e(A,\psi) \;\overset{IH}{\leq}\; w_f(A,\phi) \wedge w_f(A,\psi) \;=\; w_f(A, \phi \wedge \psi)$$

and similarly for the case of $\phi \vee \psi$. For the case of $\phi \supset \psi$, note that, if $\mathcal{D}(e,f) = \top$, then, since $\mathcal{D}$ is transitive, for every $g \in \mathcal{E}$ with $\mathcal{D}(f,g) = \top$, we have $\mathcal{D}(e,g) = \top$; hence, $\mathcal{D}(f,g) \leq \mathcal{D}(e,g)$. Moreover, for every $\alpha, \beta, \gamma \in \mathcal{T}$, if $\alpha \leq \beta$, then $\beta \Rightarrow \gamma \leq \alpha \Rightarrow \gamma$. Thus, for every $e, f, g \in \mathcal{E}$ with $\mathcal{D}(e,f) = \top$, $A \in \mathcal{W}$ and $\phi, \psi \in For$, we have $\mathcal{D}(f,g) \;\leq\; \mathcal{D}(e,g)$,

$$\begin{array}{rrcl}
\therefore & \mathcal{D}(e,g) \Rightarrow (w_g(A,\phi) \Rightarrow w_g(A,\psi)) & \leq & \mathcal{D}(f,g) \Rightarrow (w_g(A,\phi) \Rightarrow w_g(A,\psi)) \\
\therefore & \bigwedge_{g \in \mathcal{E}} \Big(\mathcal{D}(e,g) \Rightarrow (w_g(A,\phi) \Rightarrow w_g(A,\psi))\Big) & \leq & \bigwedge_{g \in \mathcal{E}} \Big(\mathcal{D}(f,g) \Rightarrow (w_g(A,\phi) \Rightarrow w_g(A,\psi))\Big) \\
\therefore & w_e(A, \phi \supset \psi) & \leq & w_f(A, \phi \supset \psi)
\end{array}$$

Similarly, for every $e, f, g \in \mathcal{E}$, $A \in \mathcal{W}$, and $\phi \in For$ we have $\mathcal{D}(f,g) \;\leq\; \mathcal{D}(e,g)$,

$$\begin{array}{rrcl}
\therefore & \mathcal{D}(e,g) \Rightarrow \bigwedge_{B \in \mathcal{W}} (\mathcal{R}_g(A,B) \Rightarrow w_g(B,\phi)) & \leq & \mathcal{D}(f,g) \Rightarrow \bigwedge_{B \in \mathcal{W}} (\mathcal{R}_g(A,B) \Rightarrow w_g(B,\phi)) \\
\therefore & \bigwedge_{g \in \mathcal{E}} \Big(\mathcal{D}(e,g) \Rightarrow \bigwedge_{B \in \mathcal{W}} (\mathcal{R}_g(A,B) \Rightarrow w_g(B,\phi))\Big) & \leq & \bigwedge_{g \in \mathcal{E}} \Big(\mathcal{D}(f,g) \Rightarrow \bigwedge_{B \in \mathcal{W}} (\mathcal{R}_g(A,B) \Rightarrow w_g(B,\phi))\Big) \\
\therefore & w_e(A, \Box\phi) & \leq & w_f(A, \Box\phi)
\end{array}$$

Finally, for every $e, f \in \mathcal{E}$, $A, B \in \mathcal{W}$, and $\phi \in For$, if $\mathcal{D}(e,f) = \top$, then, using the IH,

$$\begin{array}{rrcl}
 & w_e(B,\phi) \;\leq\; w_f(B,\phi) \quad , & & \mathcal{R}_e(A,B) \;\leq\; \mathcal{R}_f(A,B) \\
\therefore & \mathcal{R}_e(A,B) \wedge w_e(B,\phi) & \leq & \mathcal{R}_f(A,B) \wedge w_f(B,\phi) \\
\therefore & \bigvee_{B \in \mathcal{W}} (\mathcal{R}_e(A,B) \wedge w_e(B,\phi)) & \leq & \bigvee_{B \in \mathcal{W}} (\mathcal{R}_f(A,B) \wedge w_f(B,\phi)) \\
\therefore & w_e(A, \Diamond\phi) & \leq & w_f(A, \Diamond\phi)
\end{array}$$

■

Note 2.9 According to the previous proposition, for every formula ϕ and every state $A \in \mathcal{W}$, the function $x \longmapsto w(x, A, \phi)$ mapping $\mathcal{E}$ to $\mathcal{T}$ is a $\mathcal{D}$-compatible function.

3 Equivalence between the two semantics

In [Fit92b] it was shown that many-valued modal models are equivalent to multiple-expert modal models, ie. `MVMEm`-models over $\{\top, \bot\}$. Formally, the result reads as follows.

Theorem 3.1 ([Fit92b]) *There is an embedding from Multiple-Expert modal models to Many-Valued modal models.*
Moreover, there is an embedding from Many-Valued modal models to Multiple-Expert modal models.

In this section we generalize the above result by showing that `MVMEm`-models are equivalent to Many-Valued modal models, and thus to ('ordinary') multiple-expert modal models. In order to prove this, it suffices to show that `MVMEm`-models are embeddable to many-valued modal models.
First we show that the $\mathcal{D}$-compatible functions of a `MVMEm`-frame $\mathcal{F}_{\mathcal{E},\mathcal{T}}$ form a finite distributive lattice.

Proposition 3.2 *Let $\mathcal{F}_{\mathcal{E},\mathcal{T}} = \langle \mathcal{E}, \mathcal{D}, \mathcal{W}, \mathcal{R}, n \rangle$ be a Many-Valued Multiple-Expert modal frame. Then the algebra with domain*

$$\mathcal{S} \;:=\; \{\mathfrak{a} : \mathcal{E} \to \mathcal{T} \mid \mathfrak{a} \textit{ is } \mathcal{D}\textit{-compatible}\}$$

and with meet and join operations $\wedge, \vee$ defined pointwise, is a finite distributive lattice.
Moreover, for every $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}$ and $e \in \mathcal{E}$,

$$(\mathfrak{a} \Rightarrow \mathfrak{b})(e) \;\leq\; \mathfrak{a}(e) \Rightarrow \mathfrak{b}(e)$$

Proof: Note that, by definition, for every $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}$ and $e \in \mathcal{E}$,

$$(\mathfrak{a} \wedge \mathfrak{b})(e) \;=\; \mathfrak{a}(e) \wedge \mathfrak{b}(e) \;\;,\;\; (\mathfrak{a} \vee \mathfrak{b})(e) \;=\; \mathfrak{a}(e) \vee \mathfrak{b}(e) \;\;,\;\; \mathfrak{a} \leq \mathfrak{b} \Longleftrightarrow \forall e \in \mathcal{E}.\, \mathfrak{a}(e) \leq \mathfrak{b}(e)$$

We first need to show that $\mathcal{S}$ is closed under the operations $\vee, \wedge$; that is, for every $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}$, to show that $\mathfrak{a} \wedge \mathfrak{b}$ and $\mathfrak{a} \vee \mathfrak{b}$ are $\mathcal{D}$-compatible functions.
For every $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}$ and $e, f \in \mathcal{E}$ with $\mathcal{D}(e, f) = \top$, since $\mathfrak{a}, \mathfrak{b}$ are $\mathcal{D}$-compatible, we have

$$\begin{aligned}(\mathfrak{a} \wedge \mathfrak{b})(e) &= \mathfrak{a}(e) \wedge \mathfrak{b}(e) \;\leq\; \mathfrak{a}(f) \wedge \mathfrak{b}(f) \;=\; (\mathfrak{a} \wedge \mathfrak{b})(f)\\ (\mathfrak{a} \vee \mathfrak{b})(e) &= \mathfrak{a}(e) \vee \mathfrak{b}(e) \;\leq\; \mathfrak{a}(f) \vee \mathfrak{b}(f) \;=\; (\mathfrak{a} \vee \mathfrak{b})(f)\end{aligned}$$

Hence, $\mathcal{S}$ is closed under $\wedge$ and $\vee$. Moreover, since $\mathcal{E}$ and $\mathcal{T}$ are finite, $\mathcal{S}$ is finite. Therefore, by Sholander's theorem (theorem 1.4), we only need to show that, for every $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{S}$ and $e \in \mathcal{E}$,

$$(\mathfrak{a} \wedge (\mathfrak{a} \vee \mathfrak{b}))(e) = \mathfrak{a}(e)$$

$$(\mathfrak{a} \wedge (\mathfrak{b} \vee \mathfrak{c}))(e) = ((\mathfrak{c} \wedge \mathfrak{a}) \vee (\mathfrak{b} \wedge \mathfrak{a}))(e)$$

which trivially hold, since $\mathfrak{a}(e), \mathfrak{b}(e)$ and $\mathfrak{c}(e)$ are elements of the distributive lattice $\mathcal{T}$.
Finally, for each $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}$,

$$\mathfrak{a} \wedge (\mathfrak{a} \Rightarrow \mathfrak{b}) \leq \mathfrak{b} \;\;,\therefore \forall e \in \mathcal{E}.\, \mathfrak{a}(e) \wedge (\mathfrak{a} \Rightarrow \mathfrak{b})(e) \leq \mathfrak{b}(e) \;\;,\therefore \forall e \in \mathcal{E}.\, (\mathfrak{a} \Rightarrow \mathfrak{b})(e) \leq \mathfrak{a}(e) \Rightarrow \mathfrak{b}(e)$$

■

We can now define many-valued modal models derived from `MVMEm`-models.

Definition 3.3
Let $\mathcal{M}_{\mathcal{E},\mathcal{T}} = \langle \mathcal{E}, \mathcal{D}, \mathcal{W}, \mathcal{R}, n, w \rangle$ be a `MVMEm`-model. We define the $\mathcal{S}$-modal model $\mathcal{M}_{\mathcal{S}} = \langle \mathcal{W}, \mathcal{P}, n, v \rangle$ as follows.

- $\mathcal{S} := \{\mathfrak{a} \in \mathcal{T}^{\mathcal{E}} \mid \mathfrak{a} \text{ is } \mathcal{D}\text{-compatible}\}$, with meet and join defined pointwise.
- $\mathcal{W}$ is the set $\mathcal{W}$ of $\mathcal{M}_{\mathcal{E},\mathcal{T}}$.
- For every $A, B \in \mathcal{W}$, $\mathcal{P}(A, B) = \{(e, \mathcal{R}_e(A, B)) \mid e \in \mathcal{E}\}$. By definition, $\mathcal{P} \;:\; \mathcal{W} \times \mathcal{W} \to \mathcal{S}$.
- n is as in $\mathcal{M}_{\mathcal{E},\mathcal{T}}$.
- For every $A \in \mathcal{W}$ and $\phi \in Var$, $v(A, \phi) = \{(e, w_e(A, \phi)) \mid e \in \mathcal{E}\}$.

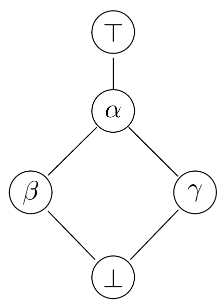
$\mathcal{M}_{\mathcal{S}}$ is called the derived $\mathcal{S}$-modal model of $\mathcal{M}_{\mathcal{E},\mathcal{T}}$. ⊣

Note that, by definitions 2.3 and 2.5, for every $A \in \mathcal{W}$ and $c \in Cons$ we have

$$v(A, c) = n(c) = \{(e, w_e(A, c)) \mid e \in \mathcal{E}\}$$

Hence, by note 2.9, n and v_A return elements of $\mathcal{S}$.
In order to demonstrate the construction, let's consider the following example.

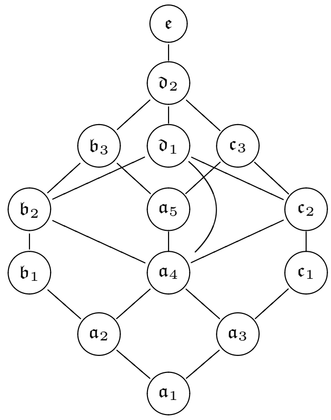
Example 3.4 Let us suppose that we have the `MVMEm`-model of the initial two experts e, f, with e dominating f. This time the experts are assigning truth-values to modal formulas from the lattice $\mathcal{T}$ shown on the side. The set of states is taken, for sake of simplicity, to be $\mathcal{W} = \{A\}$.
Now, since e dominates f, the truth-value that f assigns to each formula is greater or equal to the one that e assigns to the same formula. With this simple rule we can derive the set of total truth-values, the elements of which are shown in the first line of the table below.



	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_3$	$\mathfrak{a}_4$	$\mathfrak{a}_5$	$\mathfrak{b}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_3$	$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{c}_3$	$\mathfrak{d}_1$	$\mathfrak{d}_2$	$\mathfrak{e}$
e	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	β	β	β	γ	γ	γ	α	α	$\top$
f	$\bot$	β	γ	α	$\top$	β	α	$\top$	γ	α	$\top$	α	$\top$	$\top$

We compute the meet of each pair of elements in $\mathcal{S}$ as in the following table, from which we deduce the diagram for $\mathcal{S}$.

$\bigwedge$	$\mathfrak{a}_2$	$\mathfrak{a}_3$	$\mathfrak{a}_4$	$\mathfrak{a}_5$	$\mathfrak{b}_1$	$\mathfrak{b}_2$	$\mathfrak{b}_3$	$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{c}_3$	$\mathfrak{d}_1$	$\mathfrak{d}_2$	$\mathfrak{e}$
$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_1$
$\mathfrak{a}_2$		$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathfrak{a}_2$
$\mathfrak{a}_3$			$\mathfrak{a}_3$	$\mathfrak{a}_3$	$\mathfrak{a}_1$	$\mathfrak{a}_3$	$\mathfrak{a}_3$	$\mathfrak{a}_3$	$\mathfrak{a}_3$	$\mathfrak{a}_3$	$\mathfrak{a}_3$	$\mathfrak{a}_3$	$\mathfrak{a}_3$
$\mathfrak{a}_4$				$\mathfrak{a}_4$	$\mathfrak{a}_2$	$\mathfrak{a}_4$	$\mathfrak{a}_4$	$\mathfrak{a}_3$	$\mathfrak{a}_4$	$\mathfrak{a}_4$	$\mathfrak{a}_4$	$\mathfrak{a}_4$	$\mathfrak{a}_4$
$\mathfrak{a}_5$					$\mathfrak{a}_2$	$\mathfrak{a}_4$	$\mathfrak{a}_5$	$\mathfrak{a}_3$	$\mathfrak{a}_4$	$\mathfrak{a}_5$	$\mathfrak{a}_4$	$\mathfrak{a}_5$	$\mathfrak{a}_5$
$\mathfrak{b}_1$						$\mathfrak{b}_1$	$\mathfrak{b}_1$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathfrak{b}_1$	$\mathfrak{b}_1$	$\mathfrak{b}_1$
$\mathfrak{b}_2$							$\mathfrak{b}_2$	$\mathfrak{a}_3$	$\mathfrak{a}_4$	$\mathfrak{a}_4$	$\mathfrak{b}_2$	$\mathfrak{b}_2$	$\mathfrak{b}_2$
$\mathfrak{b}_3$								$\mathfrak{a}_3$	$\mathfrak{a}_4$	$\mathfrak{a}_5$	$\mathfrak{b}_2$	$\mathfrak{b}_3$	$\mathfrak{b}_3$
$\mathfrak{c}_1$									$\mathfrak{c}_1$	$\mathfrak{c}_1$	$\mathfrak{c}_1$	$\mathfrak{c}_1$	$\mathfrak{c}_1$
$\mathfrak{c}_2$										$\mathfrak{c}_2$	$\mathfrak{c}_2$	$\mathfrak{c}_2$	$\mathfrak{c}_2$
$\mathfrak{c}_3$											$\mathfrak{c}_2$	$\mathfrak{c}_3$	$\mathfrak{c}_3$
$\mathfrak{d}_1$												$\mathfrak{d}_1$	$\mathfrak{d}_1$
$\mathfrak{d}_2$													$\mathfrak{d}_2$



The following theorem shows the equivalence between a `MVMEm`-model and its derived $\mathcal{S}$-modal model.

Theorem 3.5 *Let $\mathcal{M}_{\mathcal{E},\mathcal{T}} = \langle \mathcal{E}, \mathcal{D}, \mathcal{W}, \mathcal{R}, n, w \rangle$ be a* `MVMEm`*-model and $\mathcal{M} = \langle \mathcal{W}, \mathcal{P}, n, v \rangle$ its derived $\mathcal{S}$-modal model. Then, for every $A \in \mathcal{W}$ and $\phi \in For$,*

$$v(A, \phi) = \{(e, w_e(A, \phi)) \mid e \in \mathcal{E}\} \quad \text{i.e., for every } e \in \mathcal{E}, \quad w_e(A, \phi) = v(A, \phi)(e)$$

Proof: The proof is done by induction on ϕ. The case of atomic formulas is proven by the previous definition.
Now, suppose we have formulas ϕ, ψ. Then, for every $A \in \mathcal{W}$ and $e \in \mathcal{E}$, we have

$$\begin{aligned} v(A, \phi \wedge \psi)(e) \ &= \ (v(A, \phi) \wedge v(A, \psi))(e) = \ v(A, \phi)(e) \wedge v(A, \psi)(e) \\ &\overset{IH}{=} \ w_e(A, \phi) \wedge w_e(A, \psi) \ = \ w_e(A, \phi \wedge \psi) \end{aligned}$$

and similarly for the case of $\phi \vee \psi$. For the case of $\phi \supset \psi$, let $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}$, such that

$$\begin{aligned} \mathfrak{a} \ &= \ v(A, \phi \supset \psi) \ = \ v(A, \phi) \Rightarrow v(A, \psi) \\ \mathfrak{b} \ &= \ \{(e, w_e(A, \phi \supset \psi)) \mid e \in \mathcal{E}\} \end{aligned}$$

We want to show that $\mathfrak{a} = \mathfrak{b}$. First, we have

$$\begin{aligned} \mathfrak{b} \wedge v(A, \phi) \ &\overset{IH}{=} \ \{(e, w_e(A, \phi \supset \psi)) \mid e \in \mathcal{E}\} \wedge \{(e, w_e(A, \phi)) \mid e \in \mathcal{E}\} \\ &= \ \{(e, w_e(A, \phi \supset \psi) \wedge w_e(A, \phi)) \mid e \in \mathcal{E}\} \\ &\overset{(*)}{\leq} \ \{(e, w_e(A, \psi)) \mid e \in \mathcal{E}\} \overset{IH}{=} \ v(A, \psi) \ , \text{ hence } \quad \mathfrak{b} \leq \mathfrak{a} \end{aligned}$$

where $(*)$ holds because, for every $e \in \mathcal{E}$,

$$\begin{aligned} w_e(A, \phi \supset \psi) \wedge w_e(A, \phi) \ &= \ \bigwedge_{f \in \mathcal{E}} \Big(\mathcal{D}(e, f) \Rightarrow (w_f(A, \phi) \Rightarrow w_f(A, \psi)) \Big) \wedge w_e(A, \phi) \\ &\leq \ (\mathcal{D}(e, e) \Rightarrow (w_e(A, \phi) \Rightarrow w_e(A, \psi))) \wedge w_e(A, \phi) \\ &\overset{\mathcal{D}(e,e) = \top}{=} \ (w_e(A, \phi) \Rightarrow w_e(A, \psi)) \wedge w_e(A, \phi) \\ &= \ w_e(A, \phi) \wedge w_e(A, \psi) \ \leq \ w_e(A, \psi) \end{aligned}$$

Moreover, suppose that $\mathfrak{a} \neq \mathfrak{b}$, that is there exists $e \in \mathcal{E}$ such that $\mathfrak{a}(e) > \mathfrak{b}(e)$; then

$$\begin{array}{rcl}(v(A,\phi) \Rightarrow v(A,\psi))(e) & > & w_e(A, \phi \supset \psi) \\ & = & \bigwedge_{g\in\mathcal{E}} \Big(\mathcal{D}(e,g) \Rightarrow (w_g(A,\phi) \Rightarrow w_g(A,\psi))\Big) \\ & \stackrel{IH}{=} & \bigwedge_{g\in\mathcal{E}} \Big(\mathcal{D}(e,g) \Rightarrow (v(A,\phi)(g) \Rightarrow v(A,\psi)(g))\Big) \\ & = & \bigwedge_{g\in\mathcal{E},\mathcal{D}(e,g)=\top} \Big(v(A,\phi)(g) \Rightarrow v(A,\psi)(g)\Big)\end{array}$$

If, for every $g \in \mathcal{E}$ with $\mathcal{D}(e,g)$, we have $(v(A,\phi) \Rightarrow v(A,\psi))(e) \ \leq\ v(A,\phi)(g) \Rightarrow v(A,\psi)(g)$, then we also have $(v(A,\phi) \Rightarrow v(A,\psi))(e) \ \leq\ \bigwedge_{g\in\mathcal{E},\mathcal{D}(e,g)=\top} \Big(v(A,\phi)(g) \Rightarrow v(A,\psi)(g)\Big)$.
Thus, there must exist $f \in \mathcal{E}$ with $\mathcal{D}(e,f) = \top$, such that

$$(v(A,\phi) \Rightarrow v(A,\psi))(e) \ \not\leq\ v(A,\phi)(f) \Rightarrow v(A,\psi)(f)$$

However, since $v(A,\phi) \Rightarrow v(A,\psi)$ is $\mathcal{D}$-compatible ($\mathfrak{a} \in \mathcal{S}$), we have

$$(v(A,\phi) \Rightarrow v(A,\psi))(e) \ \leq\ (v(A,\phi) \Rightarrow v(A,\psi))(f) \quad \text{hence,}$$

$$(v(A,\phi) \Rightarrow v(A,\psi))(f) \ \not\leq\ v(A,\phi)(f) \Rightarrow v(A,\psi)(f)$$

which is a contradiction, by proposition 3.2. Therefore, $w_e(A, \phi \supset \psi) = v(A,\phi)(e)$.

For the case of $\Box\phi$ we argue in a similar way. Let $\mathfrak{c}, \mathfrak{d} \in \mathcal{S}$, such that

$$\begin{array}{rcl}\mathfrak{c} & = & v(A,\Box\phi) \ = \ \bigwedge_{B\in\mathcal{W}} \Big(\mathcal{P}(A,B) \Rightarrow v(B,\phi)\Big) \\ \mathfrak{d} & = & \{(e, w_e(A,\Box\phi)) \mid e \in \mathcal{E}\}\end{array}$$

We are going to show that $\mathfrak{c} = \mathfrak{d}$. First, for every $B \in \mathcal{W}$,

$$\begin{array}{rcl}\mathfrak{d} \wedge \mathcal{P}(A,B) & = & \{(e, w_e(A,\Box\phi)) \mid e \in \mathcal{E}\} \wedge \{(e, \mathcal{R}_e(A,B)) \mid e \in \mathcal{E}\} \\ & = & \{(e, \mathcal{R}_e(A,B) \wedge w_e(A,\Box\phi)) \mid e \in \mathcal{E}\} \\ & \stackrel{(**)}{\leq} & \{(e, w_e(B,\phi)) \mid e \in \mathcal{E}\} \ \stackrel{IH}{=} \ v(B,\phi) \\ \therefore \mathfrak{d} & \leq & \mathcal{P}(A,B) \Rightarrow v(B,\phi) \quad \text{hence} \quad \mathfrak{d} \leq \mathfrak{c}\end{array}$$

where $(**)$ holds because, for every $e \in \mathcal{E}$,

$$\begin{array}{rcl}\mathcal{R}_e(A,B) \wedge w_e(A,\Box\phi) & = & \mathcal{R}_e(A,B) \wedge \bigwedge_{g\in\mathcal{E}} \Big(\mathcal{D}(e,g) \Rightarrow \bigwedge_{\Gamma\in\mathcal{W}} (\mathcal{R}_g(A,\Gamma) \Rightarrow w_g(\Gamma,\phi))\Big) \\ & \leq & \mathcal{R}_e(A,B) \wedge (\mathcal{D}(e,e) \Rightarrow \bigwedge_{\Gamma\in\mathcal{W}} (\mathcal{R}_e(A,\Gamma) \Rightarrow w_e(\Gamma,\phi))) \\ & \stackrel{\mathcal{D}(e,e)=\top}{=} & \mathcal{R}_e(A,B) \wedge \bigwedge_{\Gamma\in\mathcal{W}} (\mathcal{R}_e(A,\Gamma) \Rightarrow w_e(\Gamma,\phi)) \\ & \leq & \mathcal{R}_e(A,B) \wedge (\mathcal{R}_e(A,B) \Rightarrow w_e(B,\phi)) \\ & = & \mathcal{R}_e(A,B) \wedge w_e(B,\phi) \ \leq \ w_e(B,\phi)\end{array}$$

Further, suppose that $\mathfrak{c} \neq \mathfrak{d}$, that is there exists $e \in \mathcal{E}$ such that $\mathfrak{c}(e) > \mathfrak{d}(e)$; then

$$v(A,\Box\phi)(e) \ > \ w_e(A,\Box\phi) \ = \ \bigwedge_{g\in\mathcal{E}} \Big(\mathcal{D}(e,g) \Rightarrow \bigwedge_{B\in\mathcal{W}} (\mathcal{R}_g(A,B) \Rightarrow w_g(B,\phi))\Big)$$

then, arguing as in the case of $\phi \supset \psi$, there exists $f \in \mathcal{E}$ with $\mathcal{D}(e,f)$ such that

$$v(A,\Box\phi)(e) \ \not\leq\ \bigwedge_{B\in\mathcal{W}} (\mathcal{R}_f(A,B) \Rightarrow w_f(B,\phi))$$

However, since $v(A, \Box\phi)$ is $\mathcal{D}$-compatible, we have $v(A, \Box\phi)(e) \leq v(A, \Box\phi)(f)$; thus,

$$v(A, \Box\phi)(f) \not\leq \bigwedge_{B \in \mathcal{W}} \Big(\mathcal{R}_f(A, B) \Rightarrow w_f(B, \phi)\Big) \iff$$
$$\Big(\bigwedge_{B \in \mathcal{W}} \Big(\mathcal{P}(A, B) \Rightarrow v(B, \phi)\Big)\Big)(f) \not\leq \bigwedge_{B \in \mathcal{W}} \Big(\mathcal{R}_f(A, B) \Rightarrow w_f(B, \phi)\Big) \iff$$
$$\bigwedge_{B \in \mathcal{W}} \Big((\mathcal{P}(A, B) \Rightarrow v(B, \phi))(f)\Big) \not\leq \bigwedge_{B \in \mathcal{W}} \Big(\mathcal{P}(A, B)(f) \Rightarrow v(B, \phi)(f)\Big)$$

Contradiction, since, by proposition 3.2, for every $B \in \mathcal{W}$ we have

$$(\mathcal{P}(A, B) \Rightarrow v(B, \phi))(f) \leq \mathcal{P}(A, B)(f) \Rightarrow v(B, \phi)(f)$$
$$\therefore \bigwedge_{B \in \mathcal{E}} \Big((\mathcal{P}(A, B) \Rightarrow v(B, \phi))(f)\Big) \leq \bigwedge_{B \in \mathcal{W}} \Big(\mathcal{P}(A, B)(f) \Rightarrow v(B, \phi)(f)\Big)$$

The case of $\Diamond\phi$ is relatively simple: we have that

$$w_e(A, \Diamond\phi) = \bigvee_{B \in \mathcal{W}} \Big(\mathcal{R}_e(A, B) \wedge w_e(B, \phi)\Big) \stackrel{IH}{=} \bigvee_{B \in \mathcal{W}} \Big(\mathcal{P}(A, B)(e) \wedge v(B, \phi)(e)\Big)$$
$$= \bigvee_{B \in \mathcal{W}} \Big((\mathcal{P}(A, B) \wedge v(B, \phi))(e)\Big) = \Big(\bigvee_{B \in \mathcal{W}} \Big(\mathcal{P}(A, B) \wedge v(B, \phi)\Big)\Big)(e) = v(A, \Diamond\phi)(e)$$

■

Corollary 3.6 `MVMEm`*-models are equivalent to Many-Valued modal models. Therefore, any* `MVMEm`*-model over a finite Heyting algebra $\mathcal{T}$ is embeddable to a* `MVMEm`*-model over* $\{\top, \bot\}$.

Proof: The first claim follows from theorems 3.1 and 3.5, by observing that any Multiple-Expert modal model is a `MVMEm`-model over $\{\top, \bot\}$. From this, the second claim follows straightforwardly.
■

4 Conclusion

In this paper, multiple-expert modal models are extended to a system where the experts reason in a many-valued fashion, resulting to many-valued multiple-expert modal models. The interesting result is that these latter models are embeddable to many-valued modal models. Combined with the results in [Fit92b], we have that many-valued multiple-expert modal models are embeddable also to 'ordinary' multiple-expert modal models. Hence, many-valuedness does not give a strictly larger class of models.

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