AJM-games revisited

Nikos Tzevelekos

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AJM vs. HO

The Mother of All Toy Languages

PCF:

- Introduced by Plotkin (1977), embodying Scott's LCF (1969).
- Syntax:

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{}{\Gamma \vdash n : \mathsf{nat}}$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A . M : A \to B}$$

$$\frac{\Gamma \vdash M : \mathsf{nat} \quad \Gamma \vdash M : \mathsf{nat}}{\Gamma \vdash M \pm N : \mathsf{nat}} \qquad \frac{\Gamma \vdash M : \mathsf{nat} \quad \Gamma \vdash N, N' : A}{\Gamma \vdash \mathsf{if0}\; M\; N\; N' : A}$$

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$$(\lambda x.M)N \longrightarrow M\{N/x\}$$

$$\mathsf{if0}\, n\, N\, N' \longrightarrow \begin{cases} N & \text{if } n = 0 \\ N' & \text{if } n > 0 \end{cases}$$

$$\mathbf{Y}M \longrightarrow M(\mathbf{Y}M)$$

$$\ldots$$

Full abstraction for PCF

- Solved by use of *game semantics* around 1994:
  - AJM: Abramsky, Jagadeesan & Malacaria;
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  - Two different formalisms, solving the same problem.

Features of two game models: HO vs. AJM

$A = \langle M_A, \lambda_A, \vdash_A \rangle$ is an *arena*:

- M_A a set of *moves*.
- λ_A is a *labelling* function.
- $\vdash_A$ is a *justification* relation.

$$\lambda_A : M_A \to \{PQ, PA, OQ, OA\}$$
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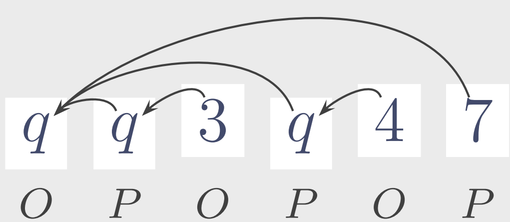

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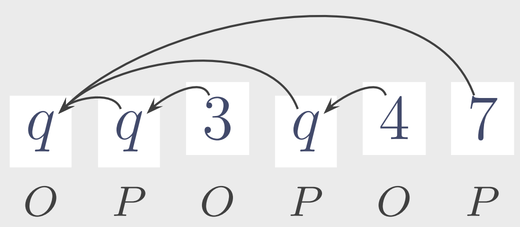

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- P cannot see the whole history of a play, but only a part of it, what we call the *P-view*.

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AJM-strategies are *history-free*:

- P cannot see the whole history of a play, but only the last move.

A paradox

At PCF types:

- Innocent HO-strategies precisely correspond to PCF terms.
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- Some details are missing here...
- AJM play-equivalence ($\approx_A$) is less transparent than it seems.

Play-equivalence under scrutiny

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  - A skeleton is a set of plays,
  - and σ is history-free if it has a history-free skeleton.

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Proof:

- Work in AJM-games; recover pointers via *pointifixion*.
- History-free $\implies$ innocent [DHR].
- Every PCF-game is *storeful*.
- Innocent $\implies$ history-free.

New-AJM games

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A recent re-formulation of AJM games by AJ:

- Justification is made explicit in games: $A = \langle M_A, \lambda_A, \mathrm{j}_A, P_A, \approx_A \rangle$.
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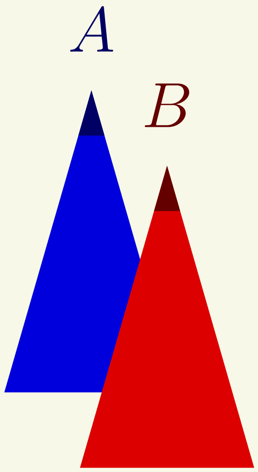
Other structure:

- From AJM: $\multimap$, $\otimes$, $!$
- product: $\&$ *of AJM*

Connectives pictorially



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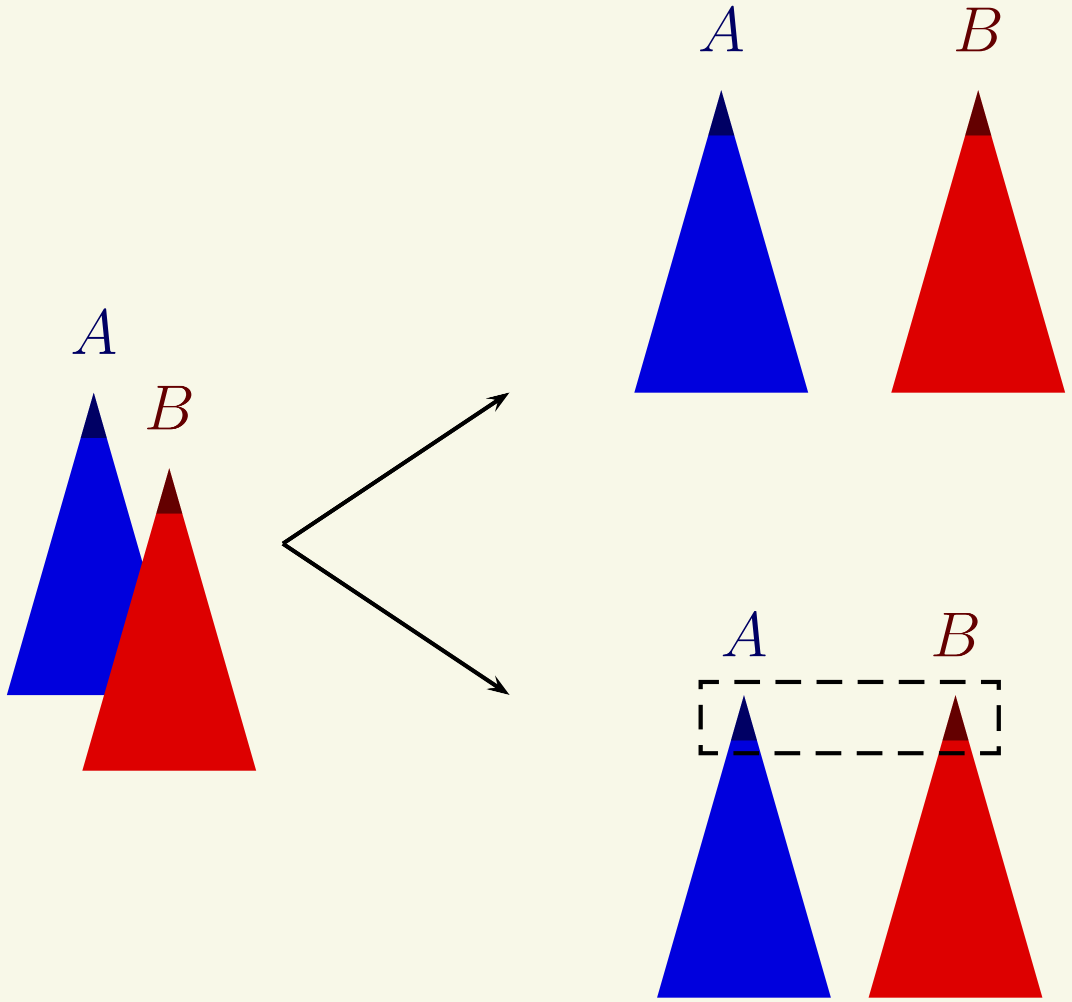

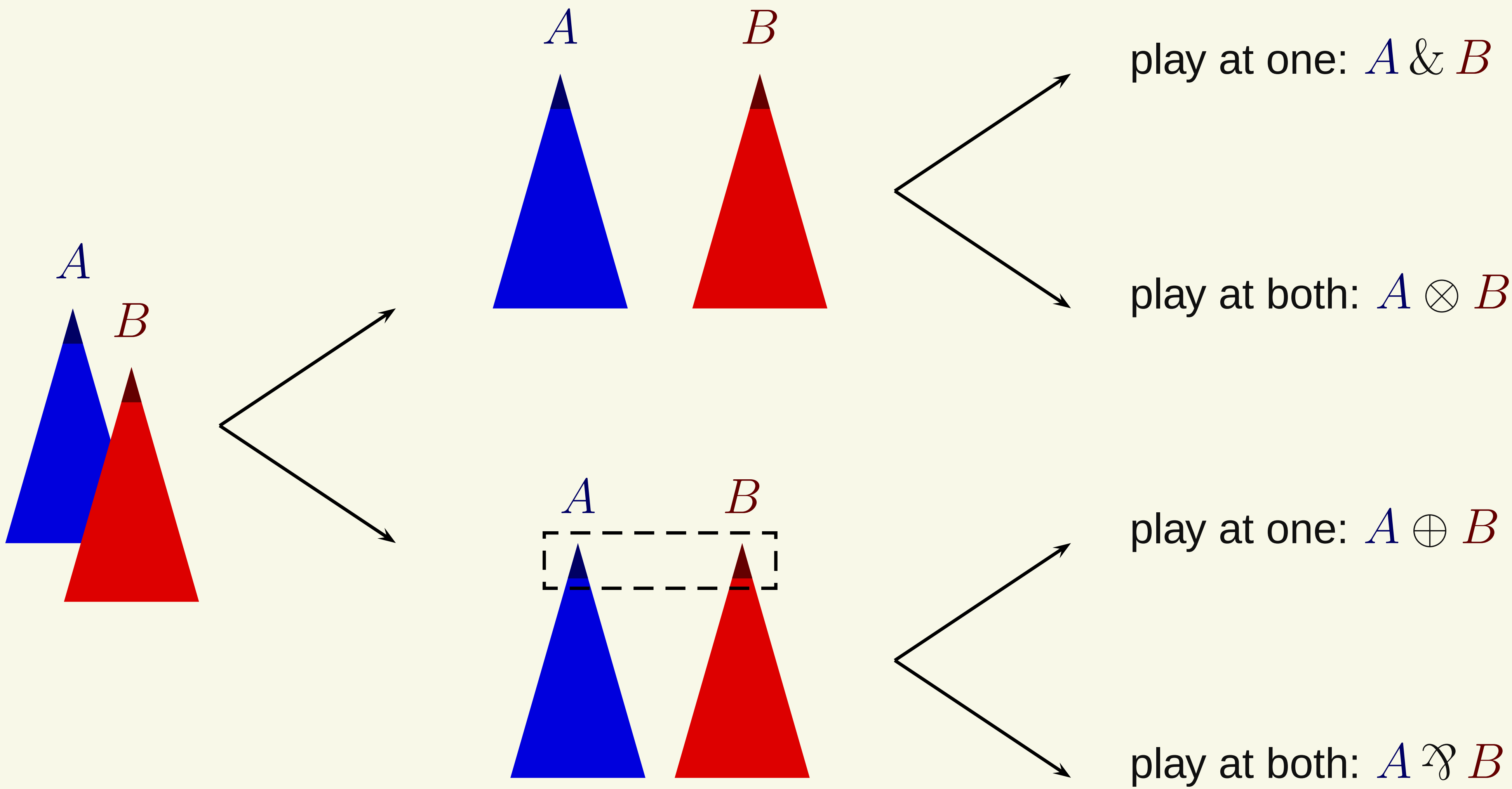
A
B
A
B
A
B
play at one: $A \,\&\, B$
play at both: $A \otimes B$
play at one: $A \oplus B$
play at both: $A \parr B$

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- ...these yield a model of linear logic (i.e. linear $\lambda\mu$-calculus).

Access Control

A calculus of access control (DCC)

- There is a lattice $\mathcal{L}$ of *access levels*.
- Types (all $l \in \mathcal{L}$):

$$A, B \ ::= \ \texttt{unit} \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid T_l A$$

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  - $\texttt{unit} \supseteq l$, and if $A \supseteq l$ then $T_{l'} A \supseteq l$
- Terms: λ-calculus plus:

$$\frac{\Gamma \vdash s : A}{\Gamma \vdash \eta_l s : T_l A} \qquad \frac{\Gamma \vdash s : T_l A \quad \Gamma, x : A \vdash t : B}{\Gamma \vdash \texttt{bind}\ x = s\ \texttt{in}\ t : B}\ B \supseteq l$$

A computation of type A cannot use an input of level l unless $A \supseteq l$

In game semantics [AJ]:

- Moves have *levels*: $A = \langle M_A, \lambda_A, \mathtt{j}_A, \mathtt{lev}_A, P_A, \approx_A \rangle$
- E.g: $\mathtt{lev}_{T_l A}(m) = \mathtt{lev}_A(m) \sqcup l$.
- *A move can only be played if its level is below that of its justifier.*

Games for access control

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Classical versions?

- how to allow $T_l A \longrightarrow T_l A \ \vee \ B$

  and disallow $T_l A \longrightarrow A \ \vee \ T_l B$