Full Abstraction for Reduced ML

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What this talk is about

Denotational semantics of Reduced ML (Reduced ML).

- Reduced ML (Stark '94) is a functional language with *nominal* integer references.
- Its denotational modelling had not been addressed before in a satisfactory way.
- We provide such a model in *Nominal Games*.

Nominal references

Names in programming languages are *atomic* constructs which can be:

- created fresh locally,
- compared for equality,
- passed around via function application.

…*Locality, Distinguishability, Mobility.*

These are the basic specifications, describing a very 'simple' nominal language, the ν-calculus (Pitts & Stark 93).

Other uses for names: π-calculus, Java, ML, etc.

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Nominal references can also be:

- dereferenced and updated.

Reduced ML: syntax

Simply-typed λ-calculus with integers and references.

Assume a countably infinite set of names, $\mathbb{A}$, with elements $a, b, \ldots$.

- Types:

$$\theta ::= \ \mathsf{unit} \mid \mathsf{int} \mid \mathsf{int\,ref} \mid \theta \rightarrow \theta$$

- Terms:

$$\begin{aligned} M ::= \ & x \mid \lambda x.M \mid M\,M \mid () \mid \Omega \\ & \mid n \mid M \odot M \mid \mathsf{if}\ M\ \mathsf{then}\ M\ \mathsf{else}\ M \\ & \mid a \mid \mathsf{ref}\,M \mid\ !\,M \mid M := M \end{aligned}$$

$$V ::= \ n \mid () \mid a \mid x \mid \lambda x.M$$

- Typing judgments have the shape

$$u, \Gamma \vdash M : \theta$$

where u is a finite subset of $\mathbb{A}$.

$$\frac{}{u, \Gamma \vdash () : \mathsf{unit}} \qquad \frac{}{u, \Gamma \vdash \Omega : \mathsf{unit}} \qquad \frac{i \in \mathbb{Z}}{u, \Gamma \vdash i : \mathsf{int}}$$

$$\frac{(x : \theta) \in \Gamma}{u, \Gamma \vdash x : \theta} \qquad \frac{u, \Gamma \vdash M : \theta \to \theta' \quad u, \Gamma \vdash N : \theta}{u, \Gamma \vdash MN : \theta'} \qquad \frac{u, \Gamma \oplus \{x : \theta\} \vdash M : \theta'}{u, \Gamma \vdash \lambda x^{\theta}.M : \theta \to \theta'}$$

$$\frac{u, \Gamma \vdash M_1 : \mathsf{int} \qquad u, \Gamma \vdash M_2 : \mathsf{int}}{u, \Gamma \vdash M_1 \odot M_2 : \mathsf{int}} \qquad \frac{u, \Gamma \vdash M : \mathsf{int} \qquad u, \Gamma \vdash N_0, N_1 : \theta}{u, \Gamma \vdash \mathsf{if}\ M\ \mathsf{then}\ N_1\ \mathsf{else}\ N_0 : \theta}$$

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where u is a finite subset of $\mathbb{A}$.

$$\frac{a \in u}{u, \Gamma \vdash a : \mathsf{int\,ref}} \qquad \frac{u, \Gamma \vdash M : \mathsf{int}}{u, \Gamma \vdash \mathsf{ref}\ M : \mathsf{int\,ref}}$$

$$\frac{u, \Gamma \vdash M : \mathsf{int\,ref}}{u, \Gamma \vdash\ !M : \mathsf{int}}$$

$$\frac{u, \Gamma \vdash M : \mathsf{int\,ref} \quad u, \Gamma \vdash N : \mathsf{int}}{u, \Gamma \vdash M := N : \mathsf{unit}}$$

Reduced ML: reduction rules

Reduction rules have the shape

$$s, M \longrightarrow s', M'$$

where $s, s' : \mathbb{A} \to_{\mathrm{fin}} \mathbb{Z}$.

- $s, (\lambda x.M)V \longrightarrow M[V/x]$
- $s, \mathsf{if}\ 0\ \mathsf{then}\ N_1\ \mathsf{else}\ N_0 \longrightarrow s, N_0$
- $s, \mathsf{if}\ 1\ \mathsf{then}\ N_1\ \mathsf{else}\ N_0 \longrightarrow s, N_1$
- $\dots$
- $s, \mathsf{ref}\ i \longrightarrow s(a \mapsto i), a$ with $a \notin \mathsf{dom}(s)$
- $s, a := i \longrightarrow s(a \mapsto i), ()$
- $s, !a \longrightarrow s, i$ with $s(a) = i$.

Notation:

- $M; N \ \equiv\ (\lambda x.N)M$, some x not free in N.
- new (x, M) in $N \ \equiv\ (\lambda x^{\mathsf{int\,ref}}.N)(\mathsf{ref}\ M)$

$$s, \mathsf{new}\,(x, M)\,\mathsf{in}\,N \ \longrightarrow\!\!\!\!\rightarrow\ s', \mathsf{new}\,(x, i)\,\mathsf{in}\,N \ \longrightarrow\!\!\!\!\rightarrow\ s'(a \mapsto i), N[a/x]$$

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Examples

- Scope extrusion. E.g: $\lambda x.\,\mathsf{ref}\,x \;:\; \mathsf{int} \rightarrow \mathsf{int\,ref}$.

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- Functions with local state. E.g. define $\mathsf{Counter} : \mathsf{unit} \to \mathsf{int}$ as:

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- Name equality. Define $\mathsf{Test} : \mathsf{int\,ref} \to \mathsf{int\,ref} \to \mathsf{int}$ as:

$$\begin{aligned}\lambda x^{\mathsf{int\,ref}}.\lambda y^{\mathsf{int\,ref}}.\mathsf{new}\,(z, 0), (x', !\,x), (y', !\,y)\,\mathsf{in}&\\ x := 0\,; y := 1\,;(\mathsf{if}\;!\,x\,\mathsf{then}\,z := 1\,\mathsf{else}\,())\,;&\\ x := \,!\,x'\,; y := \,!\,y'\,;!\,z&\end{aligned}$$

Given $\vdash M : \text{unit}$ we write $M \Downarrow$ if, for some s,

$$\emptyset, M \longrightarrow\!\!\!\!\rightarrow s, ()$$

Definition

Terms-in-context $\emptyset, \Gamma \vdash M_1 : \theta$ and $\emptyset, \Gamma \vdash M_2 : \theta$ are **equivalent** (written $\Gamma \vdash M_1 \cong M_2$) if

$$C[M_1] \Downarrow \quad \Longleftrightarrow \quad C[M_2] \Downarrow$$

for any context $C[-]$ such that $\vdash C[M_1], C[M_2] : \text{unit}$.

Examples

$$\vdash \ \mathsf{new}\,(x,0),(y,0)\,\mathsf{in}\,M \ \cong \ \mathsf{new}\,(y,0),(x,0)\,\mathsf{in}\,M$$

$$\vdash \ \mathsf{new}\,(y,0)\,\mathsf{in}\,(\lambda x^{\mathsf{int\,ref}}.\,\mathsf{Test}(x,y)) \ \cong \ \lambda x^{\mathsf{int\,ref}}.\,0$$

$$f : \mathsf{int\,ref} \to \mathsf{unit} \ \vdash \ \mathsf{new}\,(x,0)\,\mathsf{in}\,(fx\,;x:=0\,;fx)$$
$$\cong \ \mathsf{new}\,(x,0),(y,0)\,\mathsf{in}\,(fx\,;fy)$$

A denotational model for Reduced ML is **fully abstract** if

$$\Gamma \vdash M_1 \cong M_2 : \theta \iff [\![\Gamma \vdash M_1 : \theta]\!] = [\![\Gamma \vdash M_2 : \theta]\!]$$

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This had not been solved satisfactorily.

- Traditional denotational models:
  global/fixed scope (no locality/mobility), soundness only.
- Game semantics (Abramsky & McCusker '97):
  "bad" variables.
- Nominal game semantics (Laird '04, Laird '08):
  references to names, not integers.
- Nominal game semantics (AGMOS '04, Tz. '07):
  FA via semantical quotienting.

Nominal Games

- Two participants: O (environment) and P (program).
- Games specified by plays, i.e *justified* sequences of *moves*. Moves can contain names, i.e. elements of $\mathbb{A}$.
- Moves selected from *arenas*. Examples:

$$[\![\mathsf{unit}]\!] = \{*_P\} \qquad [\![\mathsf{int}]\!] = \{i_P \mid i \in \mathbb{Z}\}$$

$$[\![\mathsf{int\,ref}]\!] = \{a_P \mid a \in \mathbb{A}\}$$

$$[\![\mathsf{int\,ref} \to \mathsf{unit}]\!] = \quad *_P \quad a_O \quad *_P$$

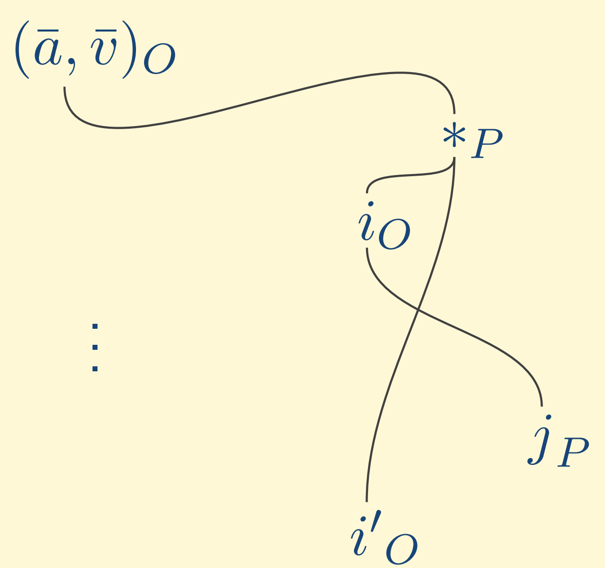
- Founded on *(linear) nominal sets* (Gabbay & Pitts '99).

Strategies

- Programs are interpreted as *strategies* for P.
  - These are *deterministic* up to choice of fresh names,
  - saturated wrt name-permutations.

Example: how does $[\![u, \Gamma \vdash \lambda x.M : \mathsf{int} \to \mathsf{int}]\!]$ look like?

$$[\![u,\Gamma]\!] \longrightarrow [\![\mathsf{int} \to \mathsf{int}]\!]$$

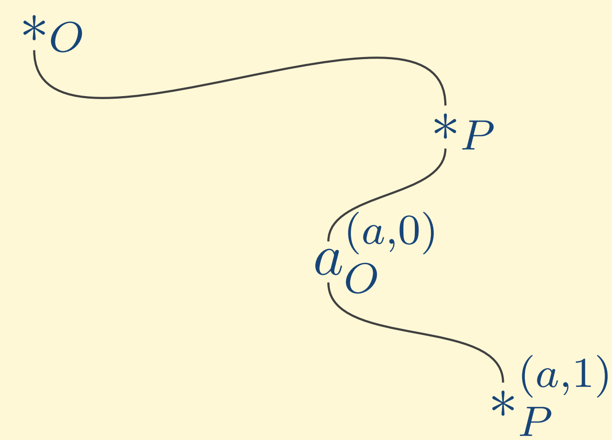


etc.

Local state

- We use moves attached with store:

$$[\![\emptyset,\emptyset]\!] \longrightarrow [\![\mathsf{int\,ref} \to \mathsf{unit}]\!]$$



Notation: $\overset{\frown}{\underset{O}{*}\ \underset{P}{*}}\ \underset{O}{a^{(a,0)}}\ \underset{P}{*^{(a,1)}}$

- Stores should not introduce fresh names (*frugality*).

Considerations:

- Which names are available to P?
- Which names should appear in the store?

Examples

- Consider a function of type int $\rightarrow$ int.

$$u, \Gamma \vdash \lambda x^{\mathsf{int}}.\, M$$

  Can it ‘remember’ x’s from different function calls?

- Now consider a function of type int ref $\rightarrow$ int.

$$u, \Gamma \vdash \lambda x^{\mathsf{int\,ref}}.\, N$$

  Can it ‘remember’ x’s from different function calls?

- Therefore, P cannot have full information of the history of a play.
- In particular, he can only see his own names and those of the current *closure*.
  Put differently, P is *purely functional* namewise.
- But he can see everything else..

Name innocence

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  Put differently, P is *purely functional* namewise.
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In game semantics:

current closure $\longmapsto$ P-*view*

pure behaviour $\longmapsto$ *Innocence*

General innocence:

- Introduced by (Hyland & Ong '94).
- Captures PCF.

Restrictions on plays

The P-view of a play s (written $\ulcorner s \urcorner$) is the subplay corresponding to the computation that has been performed under (and up to) the current closure.

We define the set of a play's P-**available** names:

$$\mathsf{Av}_P(s) = P(s) \cup \nu(\ulcorner s \urcorner)$$

P-**availability** If P plays an O-name then it must have been P-available.

$$s' p^S \text{ a play and } a \in (\nu(p) \cap O(s')) \implies a \in \mathsf{Av}_P(s')$$

P-**storage** Only P-available names appear in the store.

$$(s' m^S \text{ a play}) \implies \mathsf{dom}(S) = \mathsf{Av}_P(s' m^S)$$

Restrictions on strategies

Conditions on plays:

- P cannot play unavailable names.
- P cannot change/use the values of unavailable names.

But he can still *see* them..

E.g. can a strategy σ_{bad} contain both the following plays?

$$\begin{array}{cccccc} * & * & a^{(a,0)} & *^{(a,1)} & a^{(a,0)} & *^{(a,1)} \\ O & P & O & P & O & P \end{array}$$

$$\begin{array}{cccccc} * & * & a^{(a,0)} & *^{(a,1)} & b^{(b,0)} & *^{(b,0)} \\ O & P & O & P & O & P \end{array}$$

- We can define a notion of **renamings** invisible to P. (Note these are not just name-permutations..)
- This induces an equivalence relation $\overset{\mathrm{r}}{\sim}$ on plays.

Definition

A strategy σ is **blind** if $s \in \sigma$ implies $s' \in \sigma$ for all $s' \overset{\mathrm{r}}{\sim} s$.

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breaks determinacy!

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Proposition
There is a category $\mathcal{G}$ having arenas as objects and blind strategies as arrows.

Proposition
$\mathcal{G}$ is a sound model of Reduced ML.

Theorem
Finitary blind strategies are definable.

Theorem
Two Reduced ML terms are equivalent iff they generate the same sets of *complete symmetric* plays.

Conclusion

Notes:

- The fully abstract model of Reduced ML is effectively presentable.
- Nevertheless, Reduced ML is undecidable (Murawski '03).

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What has been achieved?

- We have a handle on program equivalence in Reduced ML.
- We have new notions and intuitions in nominal games (name-availability, blindness).

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THANKS!