A universal characterisation of locally determined $\omega$-colimits

Ohad Kammar
<ohad.kammar@cl.cam.ac.uk>
Programming, Logic, and Semantics Group
University of Cambridge Computer Laboratory

Abstract—Characterising colimiting $\omega$-cocoones of projection pairs in terms of least upper bounds of their embeddings and projections is important to the solution of recursive domain equations. We present a universal characterisation of this local property as $\omega$-cocontinuity of locally continuous functors. We present a straightforward proof using the enriched Yoneda embedding. The proof can be generalised to Cattani and Fiore’s notion of locality for adjoint pairs.

15 MINUTE TALK OUTLINE

In the category theoretic solution of recursive domain equations [SP82], several technical results hinge upon the fact that the universality of $\omega$-cocoones of projection pairs can be characterised locally in terms of least upper bounds (lubs) of their embeddings and projections. To fix terminology and notation, consider an $O$-category $K$. Let $K_{\text{PR}}$ be the $O$-category consisting of projection pairs $f : A \to B$ given by $f = \langle f^L : A \to B, f^R : B \to A \rangle$ where $f^R \circ f^L = \text{id}_A$ and $f^L \circ f^R \leq \text{id}_B$.

Definition ([SP82, Definition 8]). We say that a cocone $(C, c)$ for an $\omega$-chain of projection pairs is locally determined when

$$\text{\bigcup}_{n \in \mathbb{N}} c^L_n \circ c^R_n = \text{id}_C.$$ 

When all colimiting $\omega$-cocoones of projection pairs are locally determined, we say that the $O$-category has locally determined $\omega$-colimits of projection pairs.

For example, the category $\omega\text{-CPO}$ of (not necessarily pointed) $\omega$-cpos and continuous functions has locally determined $\omega$-colimits.

The importance of these cocoones lies in the fact that every locally determined cocone is colimiting. As any locally continuous functor $F : K \to L$ gives a continuous functor $F_{\text{PR}} : K_{\text{PR}} \to L_{\text{PR}}$, given by $F_{\text{PR}} f := \langle F f^L, F f^R \rangle$, and locally determined $\omega$-cocoones are preserved by these functors. Our contribution is to show the converse:

Theorem. An $\omega$-colimiting cocone of projection pairs is locally determined if and only if it is preserved by every locally continuous functor.

Let $\hat{K}$ be the $O$-category of $O$-presheaves, namely locally continuous functors and natural transformations from $K^{\text{op}}$ to $\omega\text{-CPO}$. Let $y : K \to \hat{K}$ be the enriched Yoneda embedding $y x := \omega\text{-CPO}(\_, x)$. Then, following from general principles [Kel82, Section 2.4], $y$ is locally continuous and fully faithful.

As is well-known, lubs and colimits in $O$-functor categories are given pointwise. The same argument shows that $\omega$-colimits of projection pairs are also given componentwise in $O$-functor categories. Therefore:

Proposition. If $K$, $L$ are $O$-categories and $L$ has locally determined $\omega$-colimits of projection pairs, then so does the $O$-functor category $L^K$. In particular, every $O$-presheaf category $\hat{K}$ has locally determined $\omega$-colimits.

We complete the proof of our theorem. Let $(C, c)$ be any colimiting cocone that is preserved (in particular) by the locally continuous Yoneda embedding. As $\hat{K}$ has locally determined $\omega$-colimits:

$$y \left( \text{\bigcup}_{n \in \mathbb{N}} c^L_n \circ c^R_n \right) = \text{\bigcup}_{n \in \mathbb{N}} y(c^L_n) \circ y(c^R_n) = y(\text{id})$$

By the faithfulness of the Yoneda embedding we deduce that $(C, c)$ is locally determined.

Corollary. An $O$-category has locally determined $\omega$-colimits of projection pairs if and only if every locally continuous functor from it yields an $\omega$-cocontinuous functor on projection pairs.

Much of the theory of recursive domain equations generalises to adjoint pairs $\langle f^L, f^R \rangle$ where $f^L \circ f^R \leq \text{id}$ and $\text{id} \leq f^R \circ f^L$. Cattani et al. [CFW98], [CF07] generalised locally determined cocoones as follows:

Definition (cf. [CF07, Theorem 1.5]). We say that a cocone $(C, c)$ for an $\omega$-chain $\Delta$ of adjoint pairs is locally determined when

$$\text{\bigcup}_{n \in \mathbb{N}} c^L_n \circ c^R_n = \text{id}_C$$ and, for all $n \in \mathbb{N}$:

$$\text{\bigcup}_{m \geq n} \Delta^R_{m \leq n} \circ \Delta^L_{m \geq n} = c^R_n \circ c^L_n$$

When all colimiting $\omega$-cocoones of adjoint pairs are locally determined, we say that the $O$-category has locally determined $\omega$-colimits of projection pairs.

As $\omega\text{-CPO}$ has locally determined $\omega$-colimits of adjoint pairs, almost identical proofs show the following:

Theorem. An $\omega$-colimiting cocone of adjoint pairs is locally determined if and only if it is preserved by every locally continuous functor.

Corollary. An $O$-category has locally determined $\omega$-colimits of adjoint pairs if and only if every locally continuous functor from it yields an $\omega$-cocontinuous functor on adjoint pairs.

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REFERENCES


