Computational Complexity; slides 6, HT 2023 Further topics: Space Hierarchy Theorem, Gap Theorem; NP-intermediate problems: Ladner's theorem, Search problems and total search problems

Paul W. Goldberg (Dept. of CS, Oxford)

HT 2023

Overview of next 2 lectures

Main complexity classes covered so far:

$$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} = \mathsf{NPSPACE} \subseteq \mathsf{EXP} \subseteq \mathsf{NEXP} \subseteq \mathsf{EXPSPACE} = \mathsf{NEXPSPACE} \subseteq \ldots$$

Next: space hierarchy theorem, and strict containments it gives us; Gap theorem

Then: "NP-intermediate" problems:

- Ladner's theorem
- search problems where solutions are guaranteed to exist

recall: Time Hierarchy theorem

proper complexity function f: roughly, an increasing function that can be computed by a TM in time $f(n) + n$

For $f(n) \geq n$ a proper complexity function, we have

$$\text{TIME}(f(n)) \text{ is a proper subset of } \text{TIME}((f(2n+1))^3).$$

It follows that P is a proper subset of EXP.

Proof used "time-bounded halting language" H_f and a "diagonalising machine"

$$H_f := \{\langle M, w \rangle \ : \ M \text{ accepts } w \text{ after } \leq f(|w|) \text{ steps}\}$$

Space Hierarchy Theorem

Let $S, s : \mathbb{N} \rightarrow \mathbb{N}$ be functions such that

1. S is "space constructible", and
2. $S(n) \geq n$,
3. $s = o(S)$.

Then DSPACE(s) $\subsetneq$ DSPACE(S).

Space-constructible functions

Definition.

$f : \mathbb{N} \to \mathbb{N}$ is space constructible if $f(n) \geq \log n$ and $f(n)$ can be computed from input $1^n := \underbrace{1 \ldots 1}_{n \text{ times}}$ in space $\mathcal{O}(f(n))$.

Most standard functions are space-constructible:

- All polynomial functions (e.g. $3n^3 - 5n^2 + 1$)
- All exponential functions (e.g. 2^n)

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For any space-constructible function f we can build a counter that goes off after $f(n)$ cells have been used on inputs of length n.

Consequence: As polynomials are space constructible, we can enforce that in an n^k-space bounded NTM M all computations halt after using $\mathcal{O}(n^k)$ space.

Definition.

$f : \mathbb{N} \to \mathbb{N}$ is time constructible if $f(n) \geq n \log n$ and $f(n)$ can be computed from input $1^n := \underbrace{1 \ldots 1}_{n \text{ times}}$ in time $\mathcal{O}(f(n))$.

Similar points apply for time constructible functions (as for space constructible ones, previous slide).

Proof of Space Hierarchy Theorem — Part I

Construct S-space bounded TM $\mathcal{D}$ as follows.

1. On input $\langle M, w \rangle$, let $n = |\langle M, w \rangle|$.
2. If the input is not of the form $\langle M, w \rangle$, then reject.
3. Compute $S(n)$ and mark off this much tape. If later stages ever exceed this allowance, then reject.
4. Simulate M on input $\langle M, w \rangle$ while counting number of steps used in simulation; if count ever exceeds $2^{S(n)}$, then reject.

   The simulation introduces only a constant factor c space overhead.
5. If M *accepts*, then reject; otherwise accept.

$\mathcal{L}(\mathcal{D}) = \{\langle M, w \rangle : \mathcal{D} \text{ accepts } \langle M, w \rangle\}$.

By construction, $\mathcal{L}(\mathcal{D}) \in \text{DSPACE}(S)$

Proof of Space Hierarchy Theorem — Part II

Claim. $\mathcal{L}(\mathcal{D}) \notin \text{DSPACE}(s)$

Towards a contradiction,

let $\mathcal{B}$ be a s space bounded TM with $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{D})$.

- As $s = o(S)$ there is $n_0 \in \mathbb{N}$ such that $S(n) \geq c \cdot s(n)$ for all $n \geq n_0$.
- Hence, for almost all inputs $\langle \mathcal{B}, w \rangle$ (length of $\langle \mathcal{B}, w \rangle \geq n_0$)

  $\mathcal{D}$ completely simulates the run of $\mathcal{B}$ on $\langle \mathcal{B}, w \rangle$
- Hence, for almost all $w \in \{0,1\}^*$

  $\langle \mathcal{B}, w \rangle \in \mathcal{L}(\mathcal{D}) \iff \mathcal{B}$ does not accept $\langle \mathcal{B}, w \rangle$ (Def of $\mathcal{D}$)

  $\langle \mathcal{B}, w \rangle \in \mathcal{L}(\mathcal{B}) \iff \mathcal{B}$ accepts $\langle \mathcal{B}, w \rangle$. (Def of "$\mathcal{L}(\mathcal{B})$")

A Hierarchy of Complexity Classes

Consequence of hierarchy theorems:

- LOGSPACE $\subsetneq$ PSPACE $\subsetneq$ EXPSPACE
- P $\subsetneq$ EXP

Relation between complexity classes covered so far:

$$
\begin{array}{ccccccccc}
\mathsf{L} & \subseteq & \mathsf{NL} & \subseteq & \mathsf{P} & \subseteq & \mathsf{NP} & \subseteq \\
\neq & & \neq & & \neq & & \neq & \\
\mathsf{PSPACE} & = & \mathsf{NPSPACE} & \subseteq & \mathsf{EXP} & \subseteq & \mathsf{NEXP} & \subseteq \\
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\end{array}
$$

The Gap Theorem

Question. Given more resources, can we always solve more problems?

How much more resources do we need to be able to solve more problems? (Can we solve strictly more problems in time $2^{2^{f(n)}}$ than in $f(n)$?)

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Theorem. (Gap theorem for time complexity)

For every total computable function $g : \mathbb{N} \to \mathbb{N}$ with $g(n) \geq n$ there is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\mathrm{DTIME}\big(f(n)\big) = \mathrm{DTIME}\big(g(f(n))\big)$$

Analogously for space complexity.
contrast with Time hierarchy theorem

For $f(n) \geq n$ a proper complexity function, we have $\mathrm{TIME}(f(n))$ is a proper subset of $\mathrm{TIME}((f(2n+1))^3)$.

The Gap Theorem

Special case (Papadimitriou's book, theorem 7.3): There is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that TIME($f(n)$)=TIME($2^{f(n)}$). Proof works by constructing f such that no TM, on input of length n, halts between $f(n)$ and $2^{f(n)}$ steps.

Corollaries of Gap theorem. There are computable functions f such that

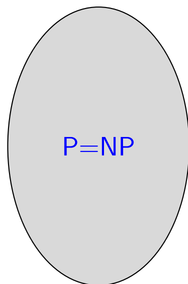
- $\mathrm{DTIME}(f) = \mathrm{DTIME}(2^f)$
- $\mathrm{DTIME}(f) = \mathrm{DTIME}(2^{2^f})$
- $\mathrm{DTIME}(f) = \mathrm{DTIME}\left(2^{2^{\cdot^{\cdot^{2}}}} \right\} f(n) \text{ times} \Big)$

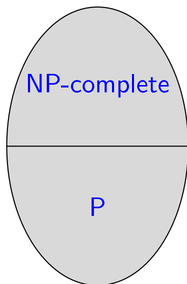
However, the functions f are not time (space) constructible.

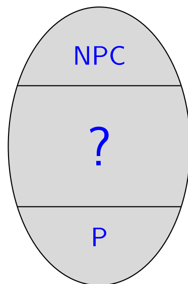
NP-Intermediate Problems

Question.

- Can we classify any problem in NP as polynomial or NP-complete?
- Which of the following diagrams corresponds to a true picture of NP?







Ladner's theorem

background

Cook/Levin (1971): SAT is NP-complete
Karp (1972): many other diverse NP problems of interest also NP-complete

Ladner's Theorem (1975)
If P $\neq$ NP then there is a language in NP that is neither in P not NP-complete.

Proof. padding, diagonalisation; sketch on next slide...
(details in Papadimitriou Chapter 14; Arora/Barak Ch.3).

Proof idea

Diagonalisation; let M_i be i-th Turing machine...

For $f : \mathbb{N} \longrightarrow \mathbb{N}$ let $\mathrm{SAT}_f = \{\varphi 1^{n^{f(n)}} \ : \ \varphi \in \mathrm{SAT} \text{ and } n = |\varphi|\}$

Q: How hard is SAT_f for f constant? $f(n) = n$?

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Q: How hard is SAT_f for f constant? $f(n) = n$?

Let $f(n)$ be smallest $i < \log\log n$ such that for every bit-string x with $|x| < \log n$, M_i on input x outputs $\text{SAT}_f(x)$ within $i|x|^i$ steps; if no such i, set $f(n) = \log\log n$.

$f(n)$ can be computed from n in $O(n^3)$ time

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Claim. $\mathrm{SAT}_f \in \mathrm{P}$ iff $f = O(1)$.

Then if $\mathrm{SAT}_f \in \mathrm{P}$, solved by some TM M_i — for $n > 2^{2^i}$, $f(n) \leq i$ — f never gets larger than a constant.

If SAT_f is NP-complete, consider reduction from SAT to SAT_f. Reduction must map instances of SAT to instances of SAT_f only polynomially larger...

NP-Intermediate Problems

Ladner's theorem gives an *artificial* problem between P and NP. Other candidates exist, however. Keep in mind, *unconditional* NP-intermediateness is too much to hope for...
We can base this property on stronger assumptions than P$\neq$NP.

Garey and Johnson 1979.

In their text book they highlight three problems whose complexity was undecided:

- Linear Programming
- Primes/Composite
- Graph Isomorphism

The first 2 of these now known to belong to P.
Total search problems (FACTORING, Nash equilibrium computation, and others) are NP-intermediate assuming they're not in P, and NP$\neq$co-NP.

Overview

We noted that NP problems have "search" counterparts that are of equal difficulty.

(recall FSAT: find a satisfying assigment)

For search problems having guaranteed solutions, we'll see that a novel classification is needed...

For NP-complete problems, e.g. SAT, suppose we want to compute a satisfying assignment, not just test for satisfiability. This is at least as challenging as SAT...

Search versus decision

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If we had a SAT-oracle, proceed as follows.
For φ over variables $x_1, \ldots, x_n$, check if φ is satisfiable, if so, try φ with $x_1 \mapsto 0$ alternatively $x_1 \mapsto 1$, then proceed to x_2 etc.

Conclude that in a sense, computing a s.a. is no harder than SAT.

Complexity class **FNP**: functions checkable in poly-time.

Search versus decision

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Complexity class **FNP**: functions checkable in poly-time.

- FSAT is FNP-complete (via Cook-Levin)
- So are function versions of other NP-complete problems

Some apparently-hard “total” search problems in **FNP**

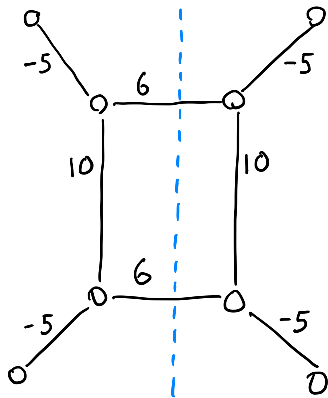
“total” — compute a total function, not a partial function

- PIGEONHOLE CIRCUIT:
  Input: a boolean circuit with n input gates and n output gates
  Output: either input vector x mapping to $\mathbf{0}$ or vectors x, x' mapping to the same output
- many problems of local optimisation, e.g. LOCAL-MAX-CUT of a weighted graph (next slide).
- FACTORING
- NASH: the problem of computing a Nash equilibrium of a game (comes in many versions depending on the structure of the game)
- many other problems associated with “non-constructive” existence results

LOCAL MAX-CUT example

LOCAL MAX-CUT on a weighted graph:

- is indeed a *total* search problem
- local (unlike global) optima are easy to check
- It "seems" hard! (though easy for unweighted graph)
- TFNP-hard? ...



Search problems as poly-time checkable relations

NP search problem is modelled as a relation $R(\cdot, \cdot)$ where

- $R(x, y)$ is checkable in time polynomial in $|x|$, $|y|$
- input x, find y with $R(x, y)$ (y as certificate)
- *total* search problem: $\forall x \exists y \quad (|y| = poly(|x|), R(x, y))$

SAT: x is boolean formula, y is satisfying bit vector.
Decision version of SAT is polynomial-time equivalent to search for y.

FACTORING: input (the "x" in $R(x, y)$) is number N, output (the "y") is prime factorisation of N. No decision problem!

contrast with "promise problems"

Reducibility among search problems

FP, FNP: search (or, function computation) problems where output of function is computable (resp., checkable) in poly time.

Any NP problem has FNP version "find a certificate".

Definition

Let R and S be search problems in FNP. We say that R (many-one) reduces to S, if there exist polynomial-time computable functions f, g such that

$$(f(x), y) \in S \implies (x, g(x, y)) \in R.$$

Key point: If S is polynomial-time solvable, then so is R. We say that two problems R and S are (polynomial-time) equivalent, if R reduces to S and S reduces to R.

Theorem: FSAT, the problem of finding a s.a. of a boolean formula, is FNP-complete.

Example

(To help motivate/understand that definition of reducibility)

Consider 2 versions of FACTORING: one using base-10 numbers, and the other version using base-2 numbers. Intuitively, these two problems have the same difficulty: there is a fast algorithm to factor in base 2, if and only if there is a fast algorithm to factor in base 10.

In trying to make that intuition mathematically precise, we get the definition of the previous slide.

TFNP

TFNP: "Total" function computation problems in NP

As we shall see, it looks like we really do need to introduce a new complexity class, in fact a collection of complexity classes...

Contrast with "promise problems", e.g. PROMISE SAT: SAT-instances where you've been promised there is a satisfying assignment. But such a promise isn't directly checkable.

Some total search problems seem hard. (F)NP-hard?

Theorem

There is an FNP*-complete problem in* TFNP *if and only if* NP=co-NP.

Proof: "if": if NP=co-NP, then any FNP-complete problem is in TFNP (which is F(NP∩co-NP)).

"only if": Suppose X∈TFNP is FNP-complete, and R is the binary relation for X.

Consider problem FSAT (given formula φ, find a satisfying assignment.) We have $\mathrm{FSAT} \leq_p$X.

Any unsatisfiable φ would get a certificate of unsatisfiability, namely the string y with $(f(\varphi), y) \in R$ and $g(y) =$"no" (or generally, anything other than a satisfying assignment).

N. Megiddo and C.H. Papadimitriou. On total functions, existence theorems and computational complexity. *Theoretical Computer Science*, **81**(2) pp. 317–324 (1991).

So what can we say about the hardness of FACTORING, and others?

FACTORING (for example) cannot be NP-hard unless NP $=$ co-NP. Unlikely! So FACTORING is in strong sense “NP-intermediate”.

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So what can we say about the hardness of Factoring, and others?

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$\rightsquigarrow$ task of classifying "hard" NP total search problems. Factoring and Pigeonhole Circuit are important in cryptography; other important problems include local optimisation

OK can we have, say, Factoring is TFNP-complete?
Good question! TFNP-completeness is as much as we can hope for, hardness-wise

TFNP doesn't (seem to) have complete problems (which needs syntactic description of "fully general" TFNP problem). (Similarly, RP, BPP, NP∩co-NP don't have complete problems)

Try to describe "generic" problem/language X in NP∩co-NP as pair of NTMs that accept X and $\overline{X}$: what goes wrong?

FACTORING, NASH, and the others, as "NP-intermediate"

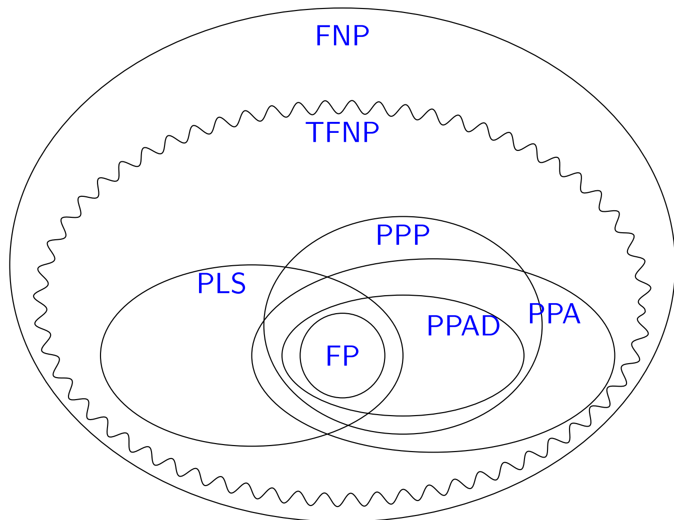
Advantage of (problems arising in) Ladner's theorem: you just have to believe P≠NP, to have NP-intermediate. For us, we have to believe that FACTORING (say) is not in FP, also that NP≠co-NP.

Disadvantage of Ladner's theorem: the NP-intermediate problems are unnatural (did not arise independently of Ladner's thm; problem definitions involve TMs/circuits)

Next: subclasses of TFNP that have complete problems

General idea: define classes in terms of "non-constructive" existence principles

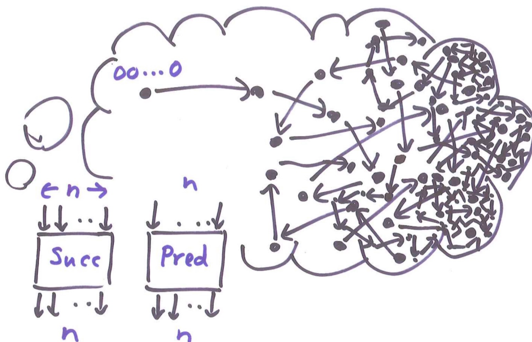
Some syntactic classes



Johnson, Papadimitriou, and Yannakakis. How easy is local search? *JCSS*, 1988.
C.H. Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *JCSS*, 1994.

PPAD

PPAD "given a source in a digraph having in/outdegree at most 1, there's another degree-1 vertex"



The End-of-Line problem

given Boolean circuits S, P with n input bits and n output bits and such that $P(0) = 0 \neq S(0)$, find x such that $P(S(x)) \neq x$ or $S(P(x)) \neq x \neq 0$.

PPAD

A problem X belongs to PPAD if $X \leq_P$ End-of-Line.

X is PPAD-complete if in addition, End-of-Line$\leq_P X$

Work by myself and others: Nash equilibrium computation is PPAD-complete

...which is taken to indicate it's a hard problem! Let's see why we believe "PPAD is hard"

PPA, PPP

PPA: Like PPAD, but the "implicit" graph is undirected.

The LEAF problem

given boolean circuit C with n inputs, $2n$ outputs. regard input as one of 2^n vertices, output as 2 neighbouring vertices.
If $\mathbf{0}$ has degree 1, find some other degree-1 vertex.

PPP ("polynomial pigeonhole principle"): defined in terms of:

The PIGEONHOLE CIRCUIT problem

given boolean circuit C with n inputs, n outputs.
Find *either* a bit-string that is mapped to $\mathbf{0}$, *or* two bitstrings that are mapped to the same bit-string

We have

- END-OF-LINE $\leq_p$ LEAF (hence PPAD$\subseteq$PPA)
- END-OF-LINE $\leq_p$ PIGEONHOLE CIRCUIT

PPAD is a subset of PPP

End-of-line reduces to Pigeonhole circuit:

Given S, P, circuits representing an End of line instance, build a circuit C_{PPP} that does the following:

C_{PPP} uses S, P to identify any neighbours of a vertex v in the End-of-line graph, then

- If v has no outgoing edge in the End-of-line graph, C_{PPP} maps v to itself.
  (so all isolated vertices are mapped to themselves)
- Otherwise, let (v, w) be a directed edge in the End-of-line graph.
  C_{PPP} maps v to w

Evidence of hardness

- Failure to find poly-time algorithms for most of these problems, indeed even sub-exponential algorithms.
- cryptographic hardness

- **Separation oracles**

Circuits viewed as proxies for unrestricted boolean functions: the search problems stay total even if the circuits in the defs are allowed to be any functions (not necessarily having small circuits)

Warm-up: in the context of END OF LINE/PPAD, if the circuits S and P were replaced with unrestricted boolean functions allowing "black-box access", the problem becomes impossible.
Call this "oracle PPAD"

Now define a "PPAD machine" to be a notional machine that, given black-box access to S and P, identifies a solution...
Such a machine can't used to solve oracle PPA!

Paul Beame, Stephen A. Cook, Jeff Edmonds, Russell Impagliazzo, Toniann Pitassi: The Relative Complexity of NP Search Problems. *JCSS* (1998)

Conclusion

This is ongoing work! The hardness of some of these complexity classes has been derived from various cryptographic assumptions (that are stronger than P$\neq$NP).

Further question include: do we "need" any other as-yet undefined classes of TFNP problems?
Can we base the hardness of (say) PPAD on weaker assumptions, ideally P$\neq$NP?

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Thanks!