MSc in Computer Science
MSc in Mathematics and the Foundations of Computer Science

Michaelmas Term 2017

Foundations of CS

Exercise class 5

1. Let T be a Turing Machine. A *tape configuration* for T is a string describing the tape contents, head position, and control state of T: we can take this to be a string of the form uqw, where the tape content is the string uw, the control state is q, and the head is on the first symbol of w.

   An *oscillating computation* of T is a sequence $c_1\#(c_2)^R\#c_3\ldots\ldots c_n$, where $\#$ is a new symbol and c_1 is the initial tape configuration of T followed by the reverse of the second configuration, and so forth up until the n^{th} configuration, which will be reversed if n is even and unreversed if n is odd.

   If Σ is the input alphabet of T, then we let Σ_{config} be the alphabet for oscillating configurations of T.

   Consider the language $NOTCOMPUTE(T) = \{\omega \in \Sigma^*_{config} : \omega$ is *not* an oscillating computation of $T\}$.

   - Argue that for any TM T, $NOTCOMPUTE(T)$ is context-free. Your argument does not have to be completely formal.
   - A context-free language is *universal* if it contains all strings (i.e. its complement is empty). Show that the set $\{\langle C\rangle : C$ is a context-free language and C is universal $\}$ is undecidable. That is, there is no algorithm for deciding whether a context-free grammar generates all strings. You can assume the result of the problem above, if that helps you!
   - Show that the equivalence problem for context-free grammars is undecidable

2. For each of the following propositional logic formulas ϕ, tell: whether ϕ is a validity a contradiction, or neither a validity or a contradiction. If ϕ is in the third category, give a model for ϕ and a model for $\neg\phi$.

   (a) $q \vee \neg(p \vee \neg(p \wedge q))$

   (b) $p \vee \neg(p \wedge q)$

   (c) $(p \rightarrow q) \vee \neg(q \rightarrow p)$

   (d) $p \wedge \neg(p \vee q)$

3. Can every formula be written as a 3-CNF? If so, prove this, if not give an example (and a proof that the example cannot be so expressed).

4. (this is Sipser 7.21 page 300).

   Let DOUBLE-SAT be $\{\langle\phi\rangle | \phi$ mentions symbols $p_1 \ldots p_n$ and has at least two worlds over $p_1 \ldots p_n$ that satisfy it $\}$.

   Show that DOUBLE-SAT is NP-complete.

5. The exclusive-or operator $\otimes$ works as follows. For propositional formula $\phi = \phi' \otimes \phi''$, ϕ evaluates to TRUE if and only if one but not both of ϕ' and ϕ'' evaluate to TRUE.

   The function $\phi_n = x_1 \otimes x_2 \otimes \ldots x_n$ can be expressed using the operators $\wedge$, $\vee$, $\neg$. Prove that it's possible to write down ϕ_n using these operators, so that the length of ϕ_n is polynomial in n.