In lecture 2, I present constructions that show closure properties of NFAs. The purpose of this note is to present the formal proof that the construction done for union "works". Officially we are showing the following result:

Theorem 0.1. *The class of regular languages is closed under union; that is, if L_1 is recognized by a NFA and L_2 is recognized by a NFA, then $L_1 \cup L_2$ is recognized by a NFA as well.*

Proof:

Suppose L_1 is recognized by NFA $A_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and L_2 is recognized by NFA $A_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.

Construct the NFA $A = (Q, \Sigma, \delta, q_0, F)$ where:

- $Q = Q_1 \cup Q_2 \cup \{q_0\}$ where q_0 is not in Q_1 or Q_2.
- $F = F_1 \cup F_2$
- δ is defined as:
  - $\delta(q_0, \epsilon) = \{q_1, q_2\}$,
  - $\delta(q_0, a) = \{\}$ for $a \neq \epsilon$
  - $\delta(q, a) = \delta_1(q, a)$ for a in $\Sigma \cup \{\epsilon\}$ and q in Q_1
  - $\delta(q, a) = \delta_2(q, a)$ for a in $\Sigma \cup \{\epsilon\}$ and q in Q_2

Note: The construction above is nothing but the formal definition of the construction given in the slides for closure of NFAs under union!

We claim that A accepts exactly $L_1 \cup L_2$. We need to show two directions: if a string is in $L_1 \cup L_2$ it is accepted by A, and if a string is accepted by A it is in $L_1 \cup L_2$.

Throughout this argument, we refer to the formal definition of what it means for an NFA to accept, as given in the slides and on page 54 of Sipser.

First, suppose a string ω is accepted by A. Let ω' be a rewriting of ω, with ϵ's possibly in between letters, and $r_0 \ldots r_n$ for some $n \geq 0$ a sequence of states such that ω' and $r_0 \ldots r_n$ witness acceptance according to the formal definition of acceptance of an NFA. Then $r_0 = q_0$, by the definition of acceptance. Since q_0 is not an accepting state of A, we must have $n > 0$, so there is an r_1 with $r_1 \in \delta(q_0, \omega'_1)$. But by the definition of A, ω'_1 must be ϵ and r_1 must be either q_1 or q_2, since there is only one transition out of q_0 that can lead to an accepting state.

Suppose r_1 is q_1. Note that every transition in A out of a state in Q_1 leads to another state in Q_1, and that for states in Q_1, the definition of A is the same as the definition of A_1. So $r_1 \ldots r_n$ must satisfy $r'_{i+1} \in \delta_1(r_i, \omega'_i)$ for $i \geq 1$ and $r_n \in F_1$. But then the sequence $r_1 \ldots r_n$ together with the string $\omega'_2 \ldots \omega'_n$ witness the acceptance of ω by A_1. Since ω was accepted by A_1, ω is in L_1.

Similarly if r_1 is q_2, then we argue symmetrically that ω is in L_2. In either case, we have ω is in $L_1 \cup L_2$ as required.

For the other direction, suppose a string ω is in $L_1 \cup L_2$. By definition of union, ω is in either L_1 or L_2. We first suppose ω is in L_1. Then since

A_1 recognizes L_1, we know by the definition of NFA acceptance that there is a sequence of states $r_0 \dots r_n$ and a padding of ω by epsilons $\omega'_1 \dots \omega'_n$ which witness acceptance in A_1. Since q_1 is the initial state of A_1, we must have $r_0 = q_1$. Now consider the sequence of states $q_0 r_1 \dots r_n$ and the string $\omega'' = \epsilon\omega'_1 \dots \omega'_n$. We claim that these together witness that ω is accepted by A. The sequence of states begins with the initial state of A, as required in the definition of acceptance. We have $r_1 \in \delta(q_0, \epsilon)$ by the definition of A, since $r_1 = q_1$. We also have $r_{i+1} \in \delta(r_i, \omega'_i)$ for $i \geq 0$ because $r_1 \dots r_n$ and ω' together witnessed the acceptance of ω by A_1 and A is defined identically to A for states in Q_1. So the second requirement in the definition of acceptance is fulfilled. Finally we have r_n is an accepting state of A, since r_n is an accepting state of A_1 and all accepting states of A_1 are accepting states of A.

So we have shown ω is accepted by A.

If ω is in L_2, we argue symmetrically to get that ω is accepted by A.