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Revenue maximization in a Bayesian double auction market

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ABSTRACT

We study double auction market design where the market maker wants to maximize its total revenue by buying low from the sellers and selling high to the buyers. We consider a Bayesian setting where buyers and sellers have independent probability distributions on the values of products on the market.

For the simplest setting, each seller has one kind of indivisible good with a bounded (integer) amount that can be sold to a buyer, who may demand a bounded number of copies. We develop a maximum mechanism for the market maker to maximize its own revenue.

For the more general case where each seller's product may be different, we consider a number of variants in terms of constraints on supplies and demands. For each of them, we develop a polynomial time computable truthful mechanism for the market maker to achieve a revenue at least a constant α times the revenue of any other truthful mechanism.

1. Introduction

We consider a double auction market maker who collects valuations from buyers and sellers about a certain product to decide on the prices each seller gets and each buyer pays. The buyers may want to buy many units and the sellers may have many units to part with. The buyers and sellers may have different valuations of the product, and the probability distributions of the valuations are public knowledge but each valuation, sampled from its distribution, is known only to its own buyer or seller. For simplicity, we assume that the probability distributions are independent. For the sellers and buyers, they know their own private values exactly. The market maker purchases the products from the sellers and sells them to the buyers. Our goal is to design a market mechanism that maximizes the revenue of the market maker. In other words, the market maker is to buy the same amount of products from the sellers as the amount sold to the buyers with the objective of maximizing the difference of its collected payment from the buyers and the total amount paid to the sellers. When in addition we assume public knowledge of distributions of buyers' private values from the previous sales, we call it a revenue maximization Bayesian double auction market maker.

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Table 1
Results.

	Dimension	Demand	Supply	Distribution	Results
Section 3	Single	Arbitrary	Arbitrary	Continuous/Discrete	Optimal
Section 4	Multi	Arbitrary	Arbitrary	Continuous	1/4-Approx
Section 4	Multi	Arbitrary	Arbitrary	Discrete	1/4-Approx
Section 5	Multi	Unlimited	Arbitrary	Discrete	Optimal
Section 5	Multi	Arbitrary	Unlimited	Discrete	Optimal

For the demand column, "Arbitrary" refers to the case where buyers can buy at most d_i items where d_i can be an arbitrary number and "Unlimited" means $d_i = +\infty$. The supply column is similar.

There have been many double auction institutions, each of which may be suitable for one type of market environment [9]. Ours is motivated by the growing use of discriminative pricing models over the Internet such as one that is studied in [7] for the prior-free market environment. A possible realistic setting for applications of our model could be Google's ad exchange where Google could play a market maker for advertisers and webpage owners [13]. One may also use it for a market model of Groupon. Our use of the Bayesian model is justified by the repeated uses of a commercial system by registered users. It allows the market maker to gain Bayesian information of the users' valuations of the products being sold. Therefore, the Bayesian model adequately describes the knowledge of the market maker, buyers and sellers for the optimal mechanism design.

Our results. We provide optimal or constant approximate mechanisms for various settings for double auction design. The parameters considered in our discussion are related to important market design issues. Those include one or multi dimensional problems (meaning, one product or multiple different types of products). The buyers can have demand constraints or not. The sellers can be supply constrained or not. Players' values may be drawn from a continuous or from a discrete distribution. The results are summarized in Table 1. In the Bayesian Mechanism Design problems, there are two computational processes involved. The first one is to design an optimal or approximate mechanism which can be viewed as a function mapping bidders' profiles to allocation and payment outcomes. Since the function maps potentially exponentially many profiles to outcomes, a succinct representation of the function is an important part in the Bayesian mechanism design. The second process is the implementation of the mechanism, i.e., given a bid profile, we run the mechanism to compute the outcome allocation and payment scheme. Our results imply that all mechanisms described in the table can be represented in polynomial size and can be found and implemented in polynomial time.

Related works. Auction design plays an important role in economics in general and especially in electronic commerce [12]. Of particular interest, is the problem of maximizing the auctioneer's revenue, referred as the optimal auction design problem. A number of research works have focused on this issue. Myerson, in his seminal paper [14], characterized the optimal auction for the single-item setting in the Bayesian model. Recently, efforts have been made on extending Myerson's results to border settings [8,16,18].

Unlike Myerson's optimal auction result, finding the optimal solution is not easy for multi-dimensional settings. Recent research interest has turned toward approximate mechanisms [1,5]. Cai et al. [4] presented a characterization of a rather general multi-dimensional setting and proposed an efficient mechanism for the special case where no bidders are demand constrained. Using similar ideas, Alaei et al. [2] present a general framework for reducing multi-agent service problems to single-agent ones.

The double auction design problem becomes more complicated since the market maker acts as the middle man to bring buyers and sellers together. A guide to the literature in micro-economics on this topic can be found in [9]. The profit maximization problem for the single buyer/single seller setting has been studied by Myerson and Satterthwaite [15]. Our optimal double auction is a direct extension of their work and, to our best knowledge, fills a clear gap in the economic theory of double auctions. Deshmukh et al. [7], studied the revenue maximization problem for double auctions where the auctioneer has no prior knowledge about bids. Their prior-free model is essentially different from ours. More auction mechanism design problems were studied by many researchers in recent years, but as far as we know, not in the context of optimal double auction design in the Bayesian setting. The most related one is by Jain and Wilkens [10], where they studied the market intermediation problem in a setting with a single unit-demand buyer and a group of sellers. They gave several constant approximate mechanisms with various buyer behavior assumptions. While our setting assumes the existence of a monopoly platform, Rochet and Tirole [17] and Armstrong [3] introduced several different models for the two-sided market and studied the platform competition problem.

2. Preliminaries

Throughout the paper we focus on Bayesian incentive compatible mechanisms only. Informally, a mechanism is Bayesian incentive compatible if it is optimal (in the expected utility) for each buyer and each seller to bid its true value of the items. We will formally define this concept later. As a consequence, we should consider their bids to be their true valuations and restrict our discussion to (direct revelation) mechanisms that result in less or equal utility if one deviates to report a false value.

Therefore, we will use the notation v_{ij} to represent the ith buyer's (true) bid for one of the jth seller's items and w_j for the jth seller's (true) bid. We will drop the "(true)" subsequently as deviations of bids from the true valuations will be clearly stated. The ith buyer's bid can be denoted by a vector v_i and bids of all buyers can be denoted by v or sometimes $(v_i; v_{-i})$ where v_{-i} is the joint profile of all other bidders. Similarly, we use w and $(w_j; w_{-j})$ for the sellers' bid.[3]

In our model, all players' bids are assumed to be distributed independently according to publicly known distributions, V for buyers, W for sellers. Note that we also assume that V and W should be bounded, i.e. $v_{ij} \in [\underline{v}_{ij}, \overline{v}_{ij}]$ and $w_j \in [\underline{w}_j, \overline{w}_j]$.

Before introducing the formal notations of double auction, we present some preliminaries for Myerson's classical auction [14], where there is one item to sell and there are n buyers, whose valuations are drawn independently from some distributions, each of whom are risk-neutral expected utility maximizers, bidding for the item. The auctioneer of the auction is the seller who would like to find a mechanism consisting of an allocation rule and payment rule given the bids of buyers such that the mechanism is Bayesian incentive compatible and individual rationality and the expected revenue of the seller is maximized. Myerson characterized the Bayesian incentive compatible mechanism as monotonicity of allocation rule and payment rule and based on this, he converted the revenue maximization problem into social welfare maximization problem and resolved the problem completely. This is also called one dimensional Bayesian mechanism design since each buyer's valuation is single dimensional.

The idea of double auction design is similar to Myerson's auction while maintaining some differences. First, the auctioneer is not the item holder but intermediary agent who buys items from sellers and sells the items to buyers while simultaneously maximizing his own revenue. We will still adopt Bayesian settings and assume all the participators are selfish, e.g. risk-neutral expected utility maximizers. Second, we need to clarify the utility of sellers and not only to make sure that buyers are Bayesian incentive compatible and individual rational but also to guarantee these two properties for sellers. Third, the mechanisms we consider include the single dimension case where sellers hold the same items, and similar to Myerson's result, the optimal mechanism for single dimension is resolved. We extended the analysis of single dimension to multi dimension by employing recent elegant techniques developed for one side multi dimensional mechanism design. More precise notations and definitions will be presented below.

The outcome of a mechanism M consists of four random variables (x, p, y, q) where x and p are the allocation function and payment functions for buyers, y and q for sellers. That is, buyer i receives item j with probability $x_{ij}(v, w)$ and pays $p_i(v, w)$; seller j sells her item with probability $y_j(v, w)$ and gets a payment $q_j(v, w)$. Thus, the expected revenue of the mechanism is $R(M) = \mathrm{E}_{v,w}[\sum_i p_i(v, w) - \sum_j q_j(v, w)]$ where $\mathrm{E}_{v,w}$ is short for $\mathrm{E}_{v\sim V, w\sim W}$.

In general, a buyer may buy more than one item from the mechanism. We assume buyers' valuation functions are additive, i.e. $v_i(S) = \sum_{j\in S} v_{ij}$. For each buyer i, let d_i denote the demand constraint for buyer i, i.e. buyer i cannot buy more than d_i items. Similarly, let k_j be the supply constraint for seller j, i.e. seller j cannot sell more than k_j items. By the Birkhoff-von Neumann theorem [11,8,6], it suffices to satisfy $\sum_j x_{ij} \le d_i$ and $y_j = \sum_i x_{ij} \le k_j$.

Let $U_i(v, w) = \sum_j x_{ij}(v, w)v_{ij} - p_i(v, w)$ be the expected utility of buyer i when the profile of all players is (v, w), which is identical to usual definition of utility for buyers [14]. Similarly, the expected utility of sellers is the expected selling price of his item minus the product of his true valuation for the item and the probability that the item is sold, and we use $T_j(v, w) = q_j(v, w) - y_j(v, w)w_j$ to be the expected utility of seller j when the profile of all players is (v, w). We proceed to formally define the concepts of Bayesian Incentive Compatibility of mechanisms and ex-interim Individual Rationality of the buyers and sellers:

Definition 1. A double auction mechanism M is said to be *Bayesian Incentive Compatible (BIC)* iff the following inequalities hold for all i, j, v, w.

$$\begin{aligned} &\mathrm{E}_{v_{-i},w}\big[U_i(v, w)\big] \ge \mathrm{E}_{v_{-i},w}\big[U_i\big(\big(v'_i; v_{-i}\big), w\big)\big] \\ &\mathrm{E}_{v,w_{-j}}\big[T_j(v, w)\big] \ge \mathrm{E}_{v,w_{-j}}\big[T_j\big(v, \big(w'_j; w_{-j}\big)\big)\big] \end{aligned} \tag{1}$$

We note that, if $U_i(v, w) \ge U_i((v'_i; v_{-i}), w)$ and $T_j(v, w) \ge T_j(v, (w'_j; w_{-j}))$ for all v, w, v'_i, w'_j, we say M is *Incentive Compatible*.

To illustrate incentive compatibility, two well-known auctions are sufficient, one is first price auction, that is, the bidder with highest bids wins and pays his bids, the other one is second price auction, that is, the bidder with highest bid wins and pay the second highest bids. The first price auction is not incentive compatible, for example, the second highest bid is \$8 and a bidder's true value is \$10, if he bids \$10, he wins but pays \$10 getting utility 0, however, if he lies by bidding \$9, he still wins and pay \$9, getting utility \$1 > 0, therefore, he has incentive to lie. The truthfulness of second price auction is well known [19].

[3] We use semi-colon to separate the profile of a special player with others and use comma to separate the buyers' profiles with sellers'.

Besides Bayesian incentive compatibility, another important concept is individual rationality, which requires each participant's (expected) utility be non negative, ensuring his participation into the game since no participation guarantees his utility is zero.

Definition 2. A double auction mechanism M is said to be *ex-interim Individual Rational (IR)* iff the following inequalities hold for all i, j, v, w.

$$\begin{aligned} &\mathrm{E}_{v_{-i},w}\big[U_i(v,w)\big] \geq 0 \\ &\mathrm{E}_{v,w_{-j}}\big[T_j(v,w)\big] \geq 0 \end{aligned} \tag{2}$$

Similarly, we note that, if $U_i(v,w) \geq 0$ and $T_j(v,w) \geq 0$ for all v, w, we say M is *ex-post Individual Rational.*

We say a mechanism is feasible if each buyer and seller are Bayesian incentive compatible and ex-interim individual rational, and simultaneously demand and supply constraints are satisfied.

Not all the mechanisms are individual rational, for example, consider a fixed price auction with one item to be sold and the price is set to be \$10. Some buyer may value the item \$8 and the item will be allocated to him and charge him \$10 if he participates, hence, he will not participate in this auction.

Finally, we present the formal definition of approximate mechanism.

Definition 3 *(α-Approximate Mechanism [18]).* Given a set $\mathbb{M}$ of any mechanisms, we say mechanism $M \in \mathbb{M}$ is an α-approximate mechanism in $\mathbb{M}$ iff for each mechanism $M' \in \mathbb{M}$, for any set of buyers and sellers $\alpha \cdot R(M') \leq R(M)$. A mechanism is *optimal* in $\mathbb{M}$ if it is a 1-approximate mechanism in $\mathbb{M}$.

3. Optimal single-dimensional double auction

In this section, we consider the single-dimensional double auction design problem where all sellers sell identical items, that is for all $j, j' \in [m]$, $v_{ij} = v_{ij'}$. Moreover, as shown in Table 1, in this section we assume the bidders' bids are drawn from continuous distributions.[4] Let f_i, F_i be the probability density function (PDF) and cumulative distribution function (CDF) for buyer i's value, g_j, G_j be the PDF and CDF for seller j's value.

Our mechanism can be viewed as a generalization of the classical Myerson's Optimal Auction [14]. It is well known that Myerson's approach is powerful and extensive in the single-dimensional setting. We strengthen this by showing that a similar optimal double auction can be found in this single-dimensional setting. In addition, in Section 4 this optimal mechanism will be used to construct a constant approximate mechanism for a multi-dimensional setting.

Recall that Myerson's virtual value function is defined as $c_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ for each buyer. In the double auction, we define the virtual value functions for buyers and sellers as $c_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ and $r_j(w_j) = w_j + \frac{G_j(w_j)}{g_j(w_j)}$. If $c_i(v_i)$ is not an increasing function of v_i or r_j is not increasing, by Myerson's ironing technique, we can use the ironed virtual value function $\bar{c}_i$ and $\bar{r}_j$. W.l.o.g., we assume the buyers are sorted in decreasing order with respect to $\bar{c}_i(v_i)$ and all sellers are in increasing order with respect to $\bar{r}_j(w_j)$. Let $D = \max_{i,j}\{\min\{\sum_{s=1}^{i} d_s, \sum_{t=1}^{j} k_j\} \mid \bar{c}_i(v_i) > \bar{r}_j(w_j)\}$. Thus, we can define the optimal auction in the spirit of maximizing virtual surplus.

$$x_i(v,w) = \begin{cases} d_i & \text{if } \sum_{s\leq i} d_s \leq D \\ D - \sum_{s<i} d_s & \text{if } \sum_{s<i} d_s < D < \sum_{s\leq i} d_s \\ 0 & \text{otherwise} \end{cases}$$

$$y_j(v,w) = \begin{cases} k_j & \text{if } \sum_{t\leq j} k_t \leq D \\ D - \sum_{s<j} k_s & \text{if } \sum_{t<j} k_t < D < \sum_{t\leq j} k_t \\ 0 & \text{otherwise} \end{cases}$$

$$p_i(v,w) = x_i(v,w)v_i - \int_{\underline{v}_i}^{v_i} x_i\big((s;v_{-i}),w\big)ds$$

$$q_j(v,w) = y_j(v,w)w_j + \int_{w_j}^{\overline{w}_j} y_j\big(v,(t;w_{-j})\big)dt$$

[4] The case for continuous distributions extends to discrete distributions [20].

Theorem 1. *The above mechanism is an optimal (revenue) mechanism for the single-dimensional double auction setting. Under the assumption that the integration and convex hull of f, g can be computed in polynomial time, the mechanism can be found and implemented. Moreover, the mechanism is deterministic, incentive compatible and ex-post Individual Rational.*

Proof. Let $\hat{U}_i(x, p, v_i)$ be the expected utility for buyer i when his bid is v_i and the mechanism uses allocation function x and payment function p. Similarly, we use $\hat{T}_j(y, q, w)$ to denote seller j's expected utility.

$$\hat{U}_i(x, p, v_i) = \mathrm{E}_{v_{-i},w}\big[v_i x_i(v, w) - p_i(v, w)\big]$$
$$\hat{T}_j(y, q, w_j) = \mathrm{E}_{v,w_{-j}}\big[q_j(v, w) - w_j y_j(v, w)\big]$$

Similarly, the expected utility for the auctioneer is

$$R(x, p, y, q) = \mathrm{E}_{v,w}\Big[\sum_i p_i(v, w) - \sum_j q_j(v, w)\Big] \tag{3}$$

We call a mechanism (x, p, y, q) feasible if and only if it satisfies the following constraints.

$$\begin{aligned} & x_i(v, w) \le d_i, \qquad y_j(v, w) \le k_j \\ & \sum_i x_i(v, w) \le \sum_j y_j(v, w) \\ & x_i(v, w), \qquad y_j(v, w) \ge 0 \end{aligned} \tag{4}$$

$$\hat{U}_i(p, x, v_i) \ge 0, \qquad \hat{T}_j(q, y, w_j) \ge 0 \tag{5}$$

$$\begin{aligned} & \hat{U}_i(p, x, v_i) \ge \hat{U}_i\big(p, x, v_i'\big) \\ & \hat{T}_j(q, y, w_j) \ge \hat{T}_j\big(q, y, w_j'\big) \end{aligned} \tag{6}$$

As is well known, the Incentive Compatibility is equivalent to the monotonicity. Give a mechanism (p, x, q, y), we define $H_i(x, v_i) = \mathrm{E}_{v_{-i},w}[x_i(v, w)]$ and $L_j(y, w_j) = \mathrm{E}_{v,w_{-j}}[y_j(v, w)]$. Then we have the following lemma.

Lemma 1. *A mechanism (x, p, y, q) is feasible if and only if the following conditions hold:*

$$\begin{aligned} & \text{if } v_i' \le v_i \quad \text{then } H_i\big(x, v_i'\big) \le H_i(x, v_i) \\ & \text{if } w_j' \le w_j \quad \text{then } L_j\big(y, w_j'\big) \ge L_j(y, w_j) \end{aligned} \tag{7}$$

$$\begin{aligned} & \hat{U}_i(p, x, v_i) = \hat{U}_i(p, x, \underline{v}_i) + \int_{\underline{v}_i}^{v_i} H_i\big(x, v_i'\big) dv_i' \\ & \hat{T}_j(q, y, w_j) = \hat{T}_j(q, y, \overline{w}_j) + \int_{w_j}^{\overline{w}_j} L_j\big(y, w_j'\big) dw_j' \end{aligned} \tag{8}$$

$$\hat{U}_i(p, x, \underline{v}_i) \ge 0, \qquad \hat{T}_j(q, y, \overline{w}_j) \ge 0 \tag{9}$$

and inequalities (4).

Proof. The IC constraint (6) is equivalent to

$$\begin{aligned} \hat{U}_i(p, x, v_i) &\ge \mathrm{E}_{v_{-i},w}\big[v_i x_i\big(\big(v_i'; v_{-i}\big), w\big) - p_i\big(\big(v_i'; v_{-i}\big), w\big)\big] \\ &= \mathrm{E}_{v_{-i},w}\big[\big(v_i - v_i' + v_i'\big) x_i\big(\big(v_i'; v_{-i}\big), w\big) - p_i\big(\big(v_i'; v_{-i}\big), w\big)\big] \\ &= \hat{U}_i\big(p, x, v_i'\big) + \big(v_i - v_i'\big) H_i\big(x, v_i'\big) \\ \hat{T}_j(q, y, w_j) &\ge \hat{T}_j\big(q, y, w_j'\big) + \big(w_j' - w_j\big) L_j\big(y, w_j'\big) \end{aligned} \tag{10}$$

Using (10) twice, we have (7). By integrating H_i and L_j, we have (8). The proof of the necessary part is similar. □

Now we can characterize the optimal double auction in the following lemma.

Lemma 2. *Suppose that* (x, y) *maximizes*

$$\mathrm{E}_{v,w}\left[\sum_i c_i(v_i)x_i(v,w) - \sum_j r_j(w_j)y_j(v,w)\right]$$

subject to the constraints (4) and (7). Suppose also that

$$p_i(v,w) = x_i(v,w)v_i - \int_{\underline{v}_i}^{v_i} x_i\big((v'_i; v_{-i}), w\big)dv'_i$$

$$q_j(v,w) = y_j(v,w)w_j + \int_{w_j}^{\overline{w}_j} y_j\big(v, (w'_j; w_{-j})\big)dw'_j \tag{11}$$

Then (x, p, y, q) *represents an optimal auction.*

Proof. Recalling (3), we may write the auctioneer's objective function as

$$\begin{aligned} R(p,x,q,y) &= \mathrm{E}_{v,w}\left[\sum_i p_i(v,w) - \sum_j q_j(v,w)\right] \\ &= \sum_i \mathrm{E}_{v,w}\big[x_i(v,w)v_i\big] + \sum_i \mathrm{E}_{v,w}\big[p_i(v,w) - x_i(v,w)v_i\big] \\ &\quad - \sum_j \mathrm{E}_{v,w}\big[y_j(v,w)w_j\big] - \sum_j \mathrm{E}_{v,w}\big[q_j(v,w) - y_j(v,w)w_j\big] \end{aligned} \tag{12}$$

But, using Lemma 1, we have

$$\begin{aligned} \mathrm{E}_{v,w}\big[p_i(v,w) - x_i(v,w)v_i\big] &= -\mathrm{E}_{v_i}\big[\hat{U}_i(p,x,v_i)\big] \\ &= -\mathrm{E}_{v_i}\left[\hat{U}_i(p,x,\underline{v}_i) + \int_{\underline{v}_i}^{v_i} H_i\big(x, v'_i\big)dv'_i\right] \\ &= -\hat{U}_i(p,x,\underline{v}_i) - \int_{\underline{v}_i}^{\bar{v}_i}\int_{\underline{v}_i}^{v_i} H_i\big(x, v'_i\big) f_i(v_i)dv'_i dv_i \\ &= -\hat{U}_i(p,x,\underline{v}_i) - \int_{\underline{v}_i}^{\bar{v}_i}\int_{v'_i}^{\bar{v}_i} H_i\big(x, v'_i\big) f_i(v_i)dv_i dv'_i \\ &= -\hat{U}_i(p,x,\underline{v}_i) - \int_{\underline{v}_i}^{\bar{v}_i}\int_{v'_i}^{\bar{v}_i} f_i(v_i)dv_i H_i\big(x, v'_i\big)dv'_i \\ &= -\hat{U}_i(p,x,\underline{v}_i) - \mathrm{E}_{v_{-i},w}\left[\int_{v_i}^{\bar{v}_i} f_i\big(v'_i\big)dv'_i x_i(v,w)\right] \end{aligned} \tag{13}$$

Similarly,

$$-\mathrm{E}_{v,w}\big[q_j(v,w) - y_j(v,w)w_j\big] = -\hat{T}_j(q,y,\overline{w}_j) - \mathrm{E}_{v,w_{-j}}\left[\int_{\underline{w}_j}^{w_j} g_j\big(w'_j\big)dw'_j y_i(v,w)\right]$$

Substituting (13) into (12) gives us,

$$R(p,x,q,y) = -\sum_i \hat{U}_i(p,x,\underline{v}_i) - \sum_j \hat{T}_j(p,x,\overline{w}_j) + \mathrm{E}_{v,w}\Big[\sum_i c_i(v_i)x_i(v,w)\Big] - \mathrm{E}_{v,w}\Big[\sum_j r_j(w_j)y_j(v,w)\Big] \tag{14}$$

So the auctioneer's problem is to maximize (14) subject to the constraints (4), (7), (8) and (9). In this formulation, p, q appear only in the first two terms and in the constraints (8) and (9). These two constraints may be rewritten as

$$\mathrm{E}_{v_{-i},w}\Big[x_i(v,w)v_i - \int_{\underline{v}_i}^{v_i} x_i\big((v_i'; v_{-i}), w\big)dv_i' - p_i(v,w)\Big] = \hat{U}_i(p,x,\underline{v}_i) \ge 0$$

$$\mathrm{E}_{v,w_{-j}}\Big[q_j(v,w) - y_j(v,w)w_j - \int_{w_j}^{\overline{w}_j} y_j\big(v,(w_j'; w_{-j})\big)dw_j'\Big] = \hat{T}_j(q,y,\overline{w}_j) \ge 0$$

If the seller chooses p, q according to (11), then he satisfies both (8) and (9), and he gets the best possible value for (14). So we can drop p, q from the problem entirely. This completes the proof of the lemma. □

By the above lemma, we can reduce the optimal double auction design problem to a combinatorial optimization problem. Subject to constraint (4), our greedy mechanism always maximizes $\sum_i c_i(v_i)x_i(v,w) - \sum_j r_j(w_j)x_j(v,w)$ for all v, w. If c_i is not increasing and r_j is not decreasing, we just use the standard Myerson's technique to refine c_i and r_j to $\bar{c}_i$ and $\bar{r}_j$.

Therefore, we complete the proof of Theorem 1. □

Finally, we give an example for $n = m = 1$, $v_1 \in [0,100]$, $f_1(v_1) = 1/100$, $w_1 \in [0,100]$ and $g_1(w_1) = 1/100$. We also suppose that $d_i = k_j = 1$. It is easy to see, $c_1(v_1) = 2v_1 - 100$ and $r_1(w_1) = 2w_1$. So our optimal auction is to make a trade iff $v_1 - w_1 \ge 50$. Then we charge the buyer $50 + w_1$ and pay the seller $v_1 - 50$. So our revenue is $100 - (v_1 - w_1)$. This is interesting because $(v_1 - w_1)$ is the social welfare in this game. It follows (perhaps counter-intuitively) that the revenue decreases when the social welfare increases.

4. Approximate multi-dimensional double auction

In this section, we provide a general framework for approximately reducing the double auction design problem for multiple buyers and sellers to the subproblem for a single pair of buyer and seller. As an application, we apply the framework to construct a $\frac{1}{4}$-approximate mechanism for the multi-dimensional setting. Our approach is inspired by the work of Alaei [1] which provide a general framework for the one sided auction.

Recall that all bids are drawn from public known distributions and our goal is to maximize the expected revenue for the auctioneer. It should be emphasized that, in this section, we assume the buyers' values for different items are independent, i.e. v_{ij} and $v_{ij'}$ are independent. To use Alaei's general framework, we also assume each buyer can at most buy one copy of items from one seller. Since our approximation result is $\frac{1}{4}$, we can remove this assumption by constructing k_j duplicate sellers (each with one copy item to sell) for each seller, making the problem under the above assumption and still get approximation ratio $\frac{1}{4}$.

First of all, we introduce the concept of Primary Mechanism which can be viewed as a mechanism between one buyer and one seller.

Definition 4 *(Primary Mechanism/Primary Benchmark).* A *primary mechanism* denoted by M_{ij} for buyer i and seller j is a single buyer and single seller mechanism which allows specifying an upper bound on the ex-ante expected probability $\bar{k}_{ij}$ of allocating the item j to the buyer i. A *primary benchmark* denoted by $\bar{R}_{ij}$ is a concave function such that the optimal revenue of any primary mechanism M_{ij} subject to $\bar{k}_{ij}$ is upper bounded by $\bar{R}_{ij}(\bar{k}_{ij})$.

Intuitively, for any allocation rule, define the ex-ante probability of assigning the jth seller's items to the ith buyer as $\bar{k}_{ij} = \mathrm{E}_{v_i,w_j}[x_{ij}(v_i,w_j)]$. Then we can divide the supply constraints $\sum_i x_{ij}(v,w) \le k_j$ and demand constraints $\sum_j x_{ij}(v,w) \le d_i$ to the ex-ante probability constraints, $\sum_i \bar{k}_{ij} \le k_j$ and $\sum_j \bar{k}_{ij} \le d_i$. Then we compute the optimal ex-ante probability by convex programming. Obviously, the optimal solution of the relaxed problem must be an upper bound for any original solution. Unfortunately, the solution solved by convex programming may not be a feasible solution of the original problem. To solve this problem, Alaei introduced the following rounding process to round the relaxed solution to a feasible one.

Lemma 3 *(γ-Conservative Magician). (See Theorem 2 in [1].) In the Magician problem, a magician is presented with a series of boxes one by one. He has k magic wands that can be used to open the boxes. On each box is written a probability q_i. If a wand is used on a box, it opens, but with at most probability q_i the wand breaks. Given $\sum_i q_i \le k$ and any $\gamma \le 1 - \frac{1}{\sqrt{k+3}}$, a γ-conservative magician guarantees that each box is opened with an ex-ante expected probability at least γ.*

Using the above lemma, we describe our mechanism for multi-dimensional double auction problem. Recall that in the classical auction setting, all items are sold by the auctioneer. However, in the double auction setting, items are sold by different sellers and more efforts should be taken to handle the truthfulness issue of sellers. We extend Alaei's rounding mechanism from one-dimension (considering buyers one by one) to two-dimension (considering each pair of buyer and seller sequentially) as follows.

Mechanism (Modified γ-Pre-Rounding Mechanism)

(I) Solve the following convex program and let $\bar{k}_{ij}$ denote an optimal assignment for it.

$$\begin{aligned}
\text{Maximize:}\quad & \sum_{i\in[n],\, j\in[m]} \bar{R}_{ij}(x_{ij}) \\
\text{Subject to:}\quad & \sum_{j\in[m]} x_{ij} \le d_i \quad \text{for all } i \in [n] \\
& \sum_{i\in[n]} x_{ij} \le k_j \quad \text{for all } j \in [m] \\
& x_{ij} \in [0,1] \quad \text{for all } i \in [n],\ j \in [m]
\end{aligned} \tag{CP}$$

(II) For each buyer i, create an instance of γ-conservative magician with d_i wands (this will be referred to as the buyer i's magician). For each item j create an instance of γ-conservative magician with k_j wands (this will be referred to as the seller j's magician).

(III) For each pair of buyer and seller (i, j):
  (a) Write $\bar{k}_{ij}$ on a box and present it to the buyer i's magician and the seller j's magician.
  (b) If both of them open the box, run $M_{ij}(\bar{k}_{ij})$ on buyer i and seller j otherwise consider next pair.
  (c) If the mechanism buys an item from seller j and sells it to buyer i, then break the wands of buyer i's magician and seller j's magician.

Theorem 2 *(Modified γ-Pre-Rounding Mechanism). Suppose for each buyer and seller pair (i, j), we have an α-approximate primary mechanism M_{ij} and a corresponding primary benchmark $\bar{R}_{ij}$.[5] Then for any $\gamma \in [0, 1 - \frac{1}{\sqrt{k^*+3}}]$ where $k^* = \min_{i,j}\{d_i, k_j\}$, the Modified γ-Pre-Rounding Mechanism is a $\gamma^2 \cdot \alpha$-approximation mechanism.*

Proof. The proof is similar to the proof of Theorem 7 in [1]. First, we prove that the expected revenue of any mechanism is upper bounded by $\sum_i \sum_j \bar{R}_{ij}(\bar{k}_{ij})$. For any mechanism $M = (x, p, y, q)$, let $k_{ij} = E_{v,w} x_{ij}(v, w)$. Due to the feasibility of M, k_{ij} must be a feasible solution of the convex programming (CP). So we have,

$$R(M) = \sum_i \sum_j R_{ij}(k_{ij}) \le \sum_i \sum_j \bar{R}_{ij}(k_{ij}) \le \sum_i \sum_j \bar{R}_{ij}(\bar{k}_{ij})$$

Then it suffices to show that for each pair (i, j), our mechanism can gain the revenue $\bar{R}_{ij}(\bar{k}_{ij})$ with probability at least $\gamma^2 \cdot \alpha$, i.e. each box will be opened with probability at least γ^2 (this is because the γ-conservative magician for the buyer is independent to that for the seller and each of them chooses to open the box with ex-ante probability γ, the box will be opened iff both magicians choose to open the box). This can be deduced from Lemma 3 easily. □

Then we consider the multi-dimensional double auction design problem and present a constant approximate mechanism. For each buyer and seller pair i, j, we use the mechanism in Section 3 for one-dimensional cases to be the primary mechanism M_{ij} and the expected revenue of M_{ij} to be the primary benchmark $\bar{R}_{ij}$.

Theorem 3. *Assume that all bidders' bids are drawn from continuous distributions. A $\frac{1}{4}$ approximate double auction for the multi-dimensional setting can be found and implemented in polynomial time.*

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[5] Since we require the valuations of the buyer for different items are independent, $\bar{R}_{ij}$ has a budget balanced cross monotonic cost sharing scheme defined in Definition 6 of [1].

Proof. Now we use the similar approach in Section 3 to prove that the optimal allocation rule must be the solution of the following optimization problem (denoted as SINGLE).

$$\begin{aligned} \text{Maximize:}\quad & E_{v_i,w_j}\big[x_{ij}(v_i,w_j)\big(\bar{c}_i(v_i)-\bar{r}_j(w_j)\big)\big] \\ \text{Subject to:}\quad & E_{v_i,w_j}\big[x_{ij}(v_i,w_j)\big]\le \bar{k}_{ij} \\ & x_{ij}(v_i,w_j)\in[0,1] \end{aligned}$$

We now proceed to optimally solve above stochastic optimization problem. We need the following useful claims.

Claim 1. *Let $s_1\ge s_2\ge 0$ and $p_1,p_2\ge 0$, $k\ge 0$. Then there exists an optimal solution (x_1^*,x_2^*) for the following linear programming:*

$$\begin{aligned} \max\quad & s_1x_1p_1+s_2x_2p_2 \\ \textit{s.t.}\quad & x_1p_1+x_2p_2\le k \quad x_1,x_2\in[0,1] \end{aligned}$$

such that either $x_1^*=1$ or $x_2^*=0$.

Proof. Suppose (x_1^*,x_2^*) is an optimal solution with $x_1^*<1$ and $x_2^*>0$, let $\delta_1,\ \delta_2>0$ such that $(x_1^*+\delta_1)p_1+(x_2^*-\delta_2)p_2=x_1^*p_1+x_2^*p_2$. Now the objective value of the solution $(x_1^*+\delta_1,x_2^*-\delta_2)$ is $s_1(x_1^*+\delta_1)p_1+s_2(x_2^*-\delta_2)p_2=s_1x_1^*p_1+s_2x_2^*p_2+s_1\delta_1p_1-s_2\delta_2p_2$. Since $s_1\ge s_2\ge 0$ and $\delta_1p_1-\delta_2p_2=0$, $s_1(x_1^*+\delta_1)p_1+s_2(x_2^*-\delta_2)p_2\ge s_1x_1^*p_1+s_2x_2^*p_2$. This means we can always increase x_1^* and simultaneously decrease x_2^* until either $x_1^*=1$ or $x_2^*=0$ without any loss of the objective value. □

Let $\Gamma_t=\{(v_i,w_j)\in[\underline{v}_i,\bar{v}_i]\times[\underline{w}_j,\overline{w}_j]\mid \bar{c}_i(v_i)-\bar{r}_j(w_j)=t\}$

Claim 2. *There is an optimal solution $x_{ij}^*(v_i,w_j)$ to* SINGLE *such that $x_{ij}^*(v_i,w_j)\equiv 0$ or 1 on $\Omega\subseteq[\underline{v}_i,\bar{v}_i]\times[\underline{w}_j,\overline{w}_j]$ such that $\int_{[\underline{v}_i,\bar{v}_i]\times[\underline{w}_j,\overline{w}_j]\setminus\Omega} f_i(v_i)g_j(w_j)dv_idw_j=0$.*

Proof. Let $\Omega_1=\{(v_i,w_j)\in[\underline{v}_i,\bar{v}_i]\times[\underline{w}_j,\overline{w}_j]\mid x^*(v_i,w_j)>0\}$ and $\Omega_2=\{(v_i,w_j)\in[\underline{v}_i,\bar{v}_i]\times[\underline{w}_j,\overline{w}_j]\mid \bar{c}_i(v_i)-\bar{r}_j(w_j)\ge 0\}=\bigcup_{t\ge 0}\Gamma_t$. Since $x_{ij}^*(v_i,w_j)$ is an optimal solution for SINGLE, $\Omega_1\subseteq\Omega_2$. Now suppose the claim is not true, then there exists $\Omega_3\subseteq[\underline{v}_i,\bar{v}_i]\times[\underline{w}_j,\overline{w}_j]$ such that $0<x_{ij}^*(v_i,w_j)<1$ on Ω_3 and $\int_{\Omega_3} f(v_i)g(w_j)dv_idw_j>0$. Consider $\int_{t\ge 0}\int_{\Gamma_t\cap\Omega_3}(\bar{c}_i(v_i)-\bar{r}_j(w_j))x_{ij}^*(v_i,w_j)=\sum_{t\ge 0}\int_{\Gamma_t\cap\Omega_3} tx_{ij}^*(v_i,w_j)f_i(v_i)g_j(w_j)dv_idw_j$, by Claim 1, we can reduce the value of $x_{ij}^*(v_i,w_j)$ on $\Gamma_{t_1}\cap\Omega_3$ and increase the value of $x_{ij}^*(v_i,w_j)$ on $\Gamma_{t_2}\cap\Omega_3$ where $t_2>t_1$ until $x_{ij}^*(v_i,w_j)\equiv 0$ or 1 on Ω_3. Contradiction. □

Let $\Theta_t=\bigcup_{s\ge t}\Gamma_s$ and $\Omega=\{(v_i,w_j)\in[\underline{v}_i,\bar{v}_i]\times[\underline{w}_j,\overline{w}_j]\mid x_{ij}^*(v_i,w_j)=1\}$.

Claim 3. *There exists an optimal solution $x_{ij}^*(v_i,w_j)$ and $t\ge 0$ such that $\Omega=\Theta_t$.*

Proof. There exists $t\ge 0$ such that $\int_\Omega f_i(v_i)g_j(w_j)dv_idw_j=\int_{\Theta_t} f_i(v_i)g_j(w_j)dv_idw_j$. Simply setting $x^*(v_i,w_j)=1$ on Θ_t and $x^*(v_i,w_j)=0$ on $[\underline{v}_i,\bar{v}_i]\times[\underline{w}_j,\overline{w}_j]\setminus\Theta_t$ will not decrease the objective value. □

By Claim 3, we know the optimization of SINGLE is equivalent to the following optimization problem denoted as SINGLE-1.

$$\begin{aligned} \text{Maximize:}\quad & \int_{\Theta_t}\big(\bar{c}_i(v_i)-\bar{r}_j(w_j)\big)f_i(v_i)g_j(w_j)dv_idw_j \\ \text{Subject to:}\quad & \int_{\Theta_t} f_i(v_i)g_j(w_j)dv_idw_j\le\bar{k}_{ij} \\ & t\ge 0. \end{aligned}$$

Observe that if $\int_{\Theta_0} f_i(v_i)g_j(w_j)dv_idw_j<\bar{k}_{ij}$, then the optimal solution of SINGLE-1 is without this constraint, which can be implemented by the mechanism in Theorem 1 with one single buyer and seller. Otherwise we know that $\int_{\Theta_0} f_i(v_i)g_j(w_j)dv_idw_j\ge\bar{k}_{ij}$. Since the integration $\int_{\Theta_t} f_i(v_i)g_j(w_j)dv_idw_j$ can be simply calculated in polynomial time for each t, by using binary search, we can find a t from $\int_{\Theta_t} f_i(v_i)g_j(w_j)dv_idw_j=\bar{k}_{ij}$, which is an optimal solution for SINGLE-1. The optimal objective value $\bar{R}_{ij}(\bar{k}_{ij})=\int_{\Theta_t}(\bar{c}_i(v_i)-\bar{r}_j(w_j))f_i(v_i)g_j(w_j)dv_idw_j$ can be computed efficiently when t is given. The optimal BIC mechanism with this objective value can be implemented simply by the following proceed: buy the item from seller and sell it directly to buyer if and only if $\bar{c}_i(v_i)-\bar{r}_j(w_j)\ge t$, the payment is given by the payment rule in Theorem 1.

Note M_{ij} is optimal for buyer i and seller j and $\bar{R}_{ij}$ is the expected revenue of M_{ij}, i.e. $R(M_{ij})$. Let $M'_{ij}(\lambda, x, y)$ be the randomized mechanism which runs $M_{ij}(x)$ with probability λ and $M_{ij}(y)$ with probability $1-\lambda$. Then for all $x, y, \lambda \in [0, 1]$, we have

$$\begin{aligned}
&\lambda \cdot \bar{R}_{ij}(x) + (1-\lambda) \cdot \bar{R}_{ij}(y) \\
&\quad = \lambda \cdot R\big(M_{ij}(x)\big) + (1-\lambda) \cdot R\big(M_{ij}(y)\big) \\
&\quad = R\big(M'_{ij}(\lambda, x, y)\big) \\
&\quad \le R\big(M\big(\lambda x + (1-\lambda) y\big)\big) \\
&\quad = \bar{R}_{ij}\big(\lambda x + (1-\lambda) y\big)
\end{aligned}$$

Therefore, $\bar{R}_{ij}(x)$ is a concave function. Hence, we obtain an 1-approximate primary mechanism M_{ij} and a corresponding primary benchmark $\bar{R}_{ij}$. By Theorem 2, we have a γ^2-approximation mechanism, where $\gamma = 1 - \frac{1}{\sqrt{k^*+3}} \ge \frac{1}{2}$ since $k^* = \min_{i,j}\{d_i, k_j\} \ge 1$. $\square$

For the discrete distribution case, the optimal mechanism for single buyer and single seller can be computed by Linear Programming. So we have the similar result.

Theorem 4. *Assume that all bidders' bids are drawn from discrete distributions. A $\frac{1}{4}$ approximate double auction for the multi-dimensional setting can be found and implemented in polynomial time.*

5. Optimal mechanism for discrete distributions

In this section, we consider the multi-dimensional double auction when all the bidders' value distributions are discrete. Unlike Section 4, we consider two special cases of the problem. One is the case where all buyers have unlimited demand, i.e., $d_i = +\infty$ for all buyer i and the other one is the case where all sellers have unlimited supply, i.e. $k_j = +\infty$ for all seller j. In this section, we focus on the former. The mechanism and the proof of the latter are similar.

Recall that, in the multi-dimensional setting, the auctioneer collects each buyer's bid, denoted by a vector $v_i = (v_{i1}, \dots, v_{im})$ drawn from a public known distribution V_i and seller's bid denoted by w_j drawn from W_j. Throughout this section, V_i and W_j are discrete distributions and we use f_i and g_j to denote their probability mass function, i.e. $f_i(t) = \Pr[v_i = t]$ and $g_j(t) = \Pr[w_j = t]$. It should be emphasized that, unlike Section 4, we do not need to assume that the buyer's bids for each item should be independent, i.e. v_{ij} and $v_{ij'}$ can be correlated in this section. We also add a dummy buyer 0 with only one type v_0 for buyers and seller 0 with w_0 for sellers.

Our approach is motivated by the recent results of Cai et al. [4] and Alaei et al. [2] which require a reduced form of x, y, p, q denoted by $\bar{x}$, $\bar{y}$, $\bar{p}$ and $\bar{q}$ respectively, defined as follows:

$$\begin{aligned}
&\bar{x}_{ij}(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}\big[x_{ij}(v, w)\big] \qquad \bar{y}_j(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}\big[y_j(v, w)\big] \\
&\bar{p}_i(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}\big[p_i(v, w)\big] \qquad \bar{q}_j(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}\big[q_j(v, w)\big]
\end{aligned} \tag{15}$$

Now we are ready to convert an optimization problem of x, p, y, q to a problem of $\bar{x}$, $\bar{p}$, $\bar{y}$, $\bar{q}$ which can be represented by a Linear Program with polynomial size in T, n and m where T is the maximum among all $|V_i|$ and $|W_j|$.

Then BIC constraints (1) and IR constraints (2) can be rewritten as

$$\begin{aligned}
&\mathrm{E}_{w_j}\Big[\sum_j \bar{x}_{ij}(v_i, w_j) v_{ij} - \bar{p}_i(v_i, w_j)\Big] \ge \mathrm{E}_{w_j}\Big[\sum_j \bar{x}_{ij}\big(v'_i, w_j\big) v_{ij} - \bar{p}_i\big(v'_i, w_j\big)\Big] \\
&\mathrm{E}_{v_i}\big[\bar{q}_j(v_i, w_j) - \bar{y}_j(v_i, w_j) w_j\big] \ge \mathrm{E}_{v_i}\big[\bar{q}_j\big(v_i, w'_j\big) - \bar{y}_j\big(v_i, w'_j\big) w_j\big] \\
&\mathrm{E}_{w_j}\Big[\sum_j \bar{x}_{ij}(v_i, w_j) v_{ij} - \bar{p}_i(v_i, w_j)\Big] \ge 0 \\
&\mathrm{E}_{v_i}\big[\bar{q}_j(v_i, w_j) - \bar{y}_j(v_i, w_j) w_j\big] \ge 0
\end{aligned} \tag{16}$$

Finally, all mechanism should satisfy the supply constraints, i.e., for each item j and profiles v, w, $y_j(v, w) = \sum_i x_{ij}(v, w) \le k_j$. Note that there is no demand constraint on buyers. With loss of generality, we assume that $k_j = 1$ for all j. Otherwise, we can normalize x by setting $x'_{ij}(v, w) = x_{ij}(v, w)/k_j$ and refine v, w by setting $v'_{ij} = k_j v_{ij}$ and $w'_j = k_j w_j$ such that $k'_j = 1$ for all item j.

For the single-item setting of classical auction, i.e. $m = 1$ and seller's value for his item is always 0, Alaei et al. [2] prove a sufficient and necessary condition for the supply constraint.

Lemma 4. *(See Theorem 2 in [2].) A reduced allocation rule $\bar{x}$ is feasible if and only if it can be implemented by the Stochastic Sequential Allocation (SSA) algorithm for some choice of stochastic transition table. In other words, there exists an ex-post implementation x of $\bar{x}$ such that $\bar{x}_i(v_i) = \mathrm{E}_{v_{-i}}[x_i(v)]$ and $\sum_i x_i(v) \le 1$ for all v iff there exists (s, z) such that*

$$\begin{aligned}
& s_0(v_0, 0) = 1 \\
& s_i(v_i, i) = \sum_{k=0}^{i-1} \sum_{v_k \in V_k} z_{ki}(v_k, v_i) \quad \forall i, v_i \in V_i \\
& s_k(v_k, i) = s_k(v_k, i-1) - \sum_{v_i \in V_i} z_{ki}(v_k, v_i) \quad \forall i, k < i, v_k \in V_k \\
& z_{ki}(v_k, v_i) \le s_k(v_k, i-1) f_i(v_i) \quad \forall i, k < i, v_i \in V_i, v_k \in V_k \\
& \bar{x}_i(v_i) f_i(v_i) = s_i(v_i, n) \quad \forall i, v_i \in V_i
\end{aligned} \tag{17}$$

Moreover, given any feasible reduced allocation rule $\bar{x}$, the ex-post of x can be found efficiently.

We generalize Lemma 4 to a multi-dimensional double auction setting.

Lemma 5. *Given a reduced form $\bar{x}$, there exists an ex-post implementation x such that $x_{ij}(v, w) \ge 0$, $\sum_i x_{ij}(v, w) \le 1$ and $\bar{x}_{ij}(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}[x_{ij}(v, w)]$ iff there exists (s, z) such that, for each seller j and $w_j \in W_j$*

$$\begin{aligned}
& s_0^{(j)}(v_0, w_j, 0) = 1 \\
& s_i^{(j)}(v_i, w_j, i) = \sum_{k=0}^{i-1} \sum_{v_k \in V_k} z_{ki}^{(j)}(v_k, v_i, w_j) \quad \forall i, v_i \in V_i \\
& s_k^{(j)}(v_k, w_j, i) = s_k^{(j)}(v_k, w_j, i-1) - \sum_{v_i \in V_i} z_{ki}^{(j)}(v_k, v_i, w_j) \quad \forall i, k < i, v_k \in V_k \\
& z_{ki}^{(j)}(v_k, v_i, w_j) \le s_k^{(j)}(v_k, w_j, i-1) f_i(v_i) \quad \forall i, k < i, v_i \in V_i, v_k \in V_k \\
& \bar{x}_{ij}(v_i, w_j) f_i(v_i) = s_i^{(j)}(v_i, w_j, n) \quad \forall i, v_i \in V_i
\end{aligned} \tag{18}$$

Moreover, given any feasible reduced allocation rule $\bar{x}$, the ex-post of $\bar{x}$ can be found efficiently.

Proof. First, we prove that given a reduced form $\bar{x}$, there exists an ex-post implementation x such that

$$\begin{aligned}
& x_{ij}(v, w) \ge 0 \\
& \sum_i x_{ij}(v, w) \le 1 \\
& \bar{x}_{ij}(v_i, w_j) = \mathrm{E}_{v_{-i}, w_{-j}}\big[x_{ij}(v, w)\big]
\end{aligned}$$

if and only if there exists an ex-interim implementation $\hat{x}$ such that

$$\begin{aligned}
& \hat{x}_{ij}(v, w_j) \ge 0 \\
& \sum_i \hat{x}_{ij}(v, w_j) \le 1 \\
& \bar{x}_{ij}(v_i, w_j) = \mathrm{E}_{v_{-i}}\big[\hat{x}_{ij}(v, w_j)\big]
\end{aligned}$$

The necessary part is obvious by just setting $\hat{x}_{ij}(v, w_j) = \mathrm{E}_{w_{-j}}[x_{ij}(v, w)]$. And the sufficiency can be checked by letting $x_{ij}(v, w) = \hat{x}_{ij}(v, w_j)$.

Now the lemma can be proved by straightforwardly applying Lemma 4 for all j and w_j. □

Finally, we convert the problem of multi-dimensional double auction design problem to a Linear Program with reduced form which can be solved in polynomial time in m, n, T.

Theorem 5. *Assume all bidders' bids are drawn from discrete distributions and all bidders are without demand constraints. An optimal double auction for multi-dimensional setting can be found and implemented in polynomial time.*

Proof. By Lemma 5, it suffices to prove the reduced form defined in (15) can be computed in polynomial time. Actually, it can be computed by solving the following Linear Programming.

$$\begin{aligned}
\text{Max:}\quad & \sum_{i,j} \mathrm{E}_{v_i,w_j}\left[p_i(v_i,w_j) - q_j(v_i,w_j)\right] \\
\text{s.t.}\quad & \mathrm{E}_{w_j}\left[\sum_j \bar{x}_{ij}(v_i,w_j)v_{ij} - \bar{p}_i(v_i,w_j)\right] \ge \mathrm{E}_{w_j}\left[\sum_j \bar{x}_{ij}\left(v'_i,w_j\right)v_{ij} - \bar{p}_i\left(v'_i,w_j\right)\right] \\
& \mathrm{E}_{v_i}\left[\bar{q}_j(v_i,w_j) - \bar{y}_j(v_i,w_j)w_j\right] \ge \mathrm{E}_{v_i}\left[\bar{q}_j\left(v_i,w'_j\right) - \bar{y}_j\left(v_i,w'_j\right)w_j\right] \\
& \mathrm{E}_{w_j}\left[\sum_j \bar{x}_{ij}(v_i,w_j)v_{ij} - \bar{p}_i(v_i,w_j)\right] \ge 0 \\
& \mathrm{E}_{v_i}\left[\bar{q}_j(v_i,w_j) - \bar{y}_j(v_i,w_j)w_j\right] \ge 0 \\
& \sum_i x_{ij}(v_i,w_j) \le k_j
\end{aligned} \tag{19}$$

Then by Lemma 5, we can find the ex-post allocations in polynomial time. □

Theorem 6. *Assume that all bidders' bids are drawn from discrete distributions and all sellers are without supply constraints. An optimal double auction for multi-dimensional setting can be found and implemented in polynomial time.*

The proof of above theorem is similar to Theorem 5.

6. Conclusion

In this paper, we present several optimal or approximately-optimal auctions for a double auction market. Double auction platforms have started to gain importance in electronic commerce. One possible example is the ad exchange market proposed to bring advertisers and web publishers together [13]. There are other potentials in setting up electronic platforms for sellers and buyers of other types of resources such as in the context of cloud computing.

Our results on the one hand show the power of recent significant progress in one-sided markets, and on the other hand raise new challenges in the development of mathematical and algorithmic tools for market design.

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