Commodity Auctions and Frugality Ratios

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Abstract. We study set-system auctions whereby a single buyer wants to purchase Q items of some commodity. There are multiple sellers, each of whom has some known number of items, and a private cost for supplying those items. Thus a "feasible set" of sellers (a set that is able to comprise the winning bidders) is any set of sellers whose total quantity sums to at least Q. We show that, even in a limited special case, VCG has a *frugality ratio* of at least $n-1$ (with respect to the NTUmin benchmark) and that this matches the upper bound for any set-system auction. We show a lower bound on the frugality of any truthful mechanism of $\sqrt{Q}$ in this setting and give a truthful mechanism with a frugality ratio of $2\sqrt{Q}$. However, we show that similar types of 'scaling' mechanism, in the general (integer) case, give a frugality ratio of at least $\frac{4Qe^{-2}}{\ln^2 Q}$.

1 Introduction

In this paper we examine a simple and natural type of procurement auction, whereby some central authority wishes to purchase some items from amongst a set $\mathcal{E}$ of possible sellers, or *agents*, by requesting quotes for their costs of supplying the items, then selecting and paying the winners so as to incentivise true bidding. We examine some alternative *mechanisms*, which consist of a set of rules that determine how the auction is run. We assume each seller $e \in \mathcal{E}$ provides a (sealed) bid b_e to the auction mechanism. The auctioneer then utilises a mechanism, $\mathcal{M}$, to choose a set S of winning agents (a *selection* rule) and a price p_e to pay each agent (a *payment* rule).

We focus on so-called *truthful* mechanisms. In such a mechanism each agent may maximise its profit simply by making a bid equal to the value that they have (privately) determined as their true cost — the cost the agent incurs as a result of participating in the winning set — for agent e we denote this cost by c_e. At first glance, this may appear to be somewhat restrictive, but truthful mechanisms turn out to be widespread. The first study of a truthful mechanism was by Vickrey in 1961 [11] showing how a sealed-bid second-price auction is truthful (an item is sold to the highest bidder, at a price equal to the second-highest bid). Furthermore, due to the *revelation principle* (see, e.g., [5,9]), it is possible to take any mechanism that has a *dominant strategy* and convert it into a truthful mechanism.

* Supported by EPSRC Grant EP/G069239/1 "Efficient Decentralised Approaches in Algorithmic Game Theory"

M. Serna (Ed.): SAGT 2012, LNCS 7615, pp. 180–191, 2012.

However, a truthful mechanism may not be optimal in terms of revenue. For example, if there are two sellers with very different prices, we must end up paying the larger of the prices. While accepting that some measure of overpayment is necessary, it seems reasonable to try and keep this as low as possible, particularly if we are looking for any real-world motivation. This overpayment is often described (see, e.g., [1,10,7]) in terms of a *frugality ratio*. The frugality ratio is defined as the worst-case ratio between the payments made by a given truthful mechanism and a benchmark figure for the same instance. It has been called "the price of truthfulness" [4]. When frugality was first studied [1,10], it was in the context of path auctions, and benchmark figures were described as properties of the paths. More recently, Karlin, Kempe and Tamir [7] described a benchmark figure that can be used to express a benchmark figure for any monopoly-free set-system auction (where the solutions deemed to be acceptable are described as sets of the agents). They also proposed a *scaling* mechanism for path auctions, and describe its frugality ratio. They give a lower-bound on the frugality ratio for any truthful mechanism, and show that their mechanism is within a constant factor of this lower bound. This constant factor was later improved by Yan [12] and Chen et al. [2].

Since then, Elkind, Goldberg and Goldberg [4] considered alternatives to the benchmark that was proposed in [7] (in [4] they are denoted TUmin, TUmax, NTUmin, NTUmax). Formal definitions of these are given in Definition 1. They also described a polynomial-time mechanism, based on an approximation algorithm, which gives a frugality ratio which is close to that of the well-known Vickrey-Clarke-Groves (VCG) [11,3,6] mechanism (the VCG mechanism must solve the vertex cover problem exactly, which is known to be NP-complete and hence cannot be solved in polynomial time unless P=NP). We give, in Section 2.1, a more general framework for determining the frugality ratios of similarly well-behaved approximation algorithms. (An approximation algorithm is well-behaved if it is monotonic in the bid values, i.e. an agent cannot go from being a loser to a winner by increasing its bid.) Most recently, two groups of researchers [8,2] independently proposed a more general framework of 'scaling' mechanisms that produce improved frugality ratios for a number of set-system auctions, including vertex-covers, flows and cuts. In common with the scaling mechanisms of Karlin et al. [7] they take advantage of the idea that the size of the winning set has a large influence on the overpayment made by a mechanism, and that improvements can be made when the mechanism biases the choice of winning set towards smaller winning sets (by scaling the bids). The frugality results that we present in Section 3 are slightly different, in that the feasible sets may be of similar sizes, yet the frugality ratio can still vary by a large degree.

Preliminaries

Denote a set system as a pair $(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ is the ground set of n elements and $\mathcal{F} \subseteq 2^{\mathcal{E}}$ is a collection of feasible sets.

Each element $e \in \mathcal{E}$ has cost c_e; denote the cost vector $\mathbf{c} = (c_1, \ldots, c_n)$.

Definition 1. *Let $(\mathcal{E}, \mathcal{F})$ be a set system, let $\mathbf{c}$ be a cost vector, and let S be the lowest-cost feasible set (with ties broken lexicographically) $S \in \operatorname{argmin}_{T \in \mathcal{F}} \sum_{e \in T} c_e$. Let $\mathrm{NTUmin}(\mathbf{c})$ be the solution to the problem: Minimize $B = \sum_{e \in S} b_e$ subject to the following conditions.*

(1) $b_e \geq c_e$ *for all* $e \in S$
(2) $\sum_{e \in S \setminus T} b_e \leq \sum_{e \in T \setminus S} c_e$ *for all* $T \in \mathcal{F}$
(3) for every $e \in S$*, there is* $T_e \in \mathcal{F}$ *such that* $e \notin T_e$ *and* $\sum_{e' \in S \setminus T_e} b_{e'} = \sum_{e' \in T_e \setminus S} c_{e'}$

As noted, a mechanism $\mathcal{M}$ takes a cost vector $\mathbf{c}$, selects a winning feasible set S, and pays S, incurring a price $p_{\mathcal{M}}(\mathbf{c})$. The *frugality ratio* for mechanism $\mathcal{M}$ is

$$\phi_{\mathrm{NTUmin}}(\mathcal{M}) = \sup_{\mathbf{c}} (p_{\mathcal{M}}(\mathbf{c}) / \mathrm{NTUmin}(\mathbf{c})).$$

We will also consider one of the alternative benchmarks of Elkind et al. [4]. Let $\mathrm{NTUmax}(\mathbf{c})$ be the solution to the problem: Maximize $B = \sum_{e \in S} b_e$ subject to conditions (1), (2), and (3). Let $\phi_{\mathrm{NTUmax}}(\mathcal{M}) = \sup_{\mathbf{c}} (p_{\mathcal{M}}(\mathbf{c}) / \mathrm{NTUmax}(\mathbf{c}))$.

To simplify notation, define the aggregates for a set $V \subseteq \mathcal{E}$; let $b_V = \sum_{e \in V} b_e$, $c_V = \sum_{e \in V} c_e$, and $p_V = \sum_{e \in V} p_e$.

2 Preliminary Results

Let $d(V)$ be the best feasible set (with the lowest sum of costs) using only agents in V where $V \subseteq \mathcal{E}$. We will now see a lower bound for $\mathrm{NTUmin}(\mathbf{c})$ which, informally, states that NTUmin must be at least as large as the worst-case cost of replacing one of the agents to make a feasible set without it. (The proof is omitted due to space constraints.)

Lemma 1. $\mathrm{NTUmin} \geq \max_e c_{d(\mathcal{E} \setminus \{e\})}$.

This lower bound for $\mathrm{NTUmin}(\mathbf{c})$ is a useful tool in analysing frugality ratios, and we will now see how it can be used to prove an upper bound on the frugality of mechanisms based on approximation algorithms.

2.1 Frugality of Approximation Mechanisms

Let $\mathcal{P}$ be some approximation algorithm, and let $S^{\mathcal{P}}$ be the feasible set returned by $\mathcal{P}$ (which uses the bids as an input parameter). We will assume that $\mathcal{P}$ is monotonic in the bids (that is, given fixed bids of the other agents, no agent can be chosen in the winning set when some smaller bid may result in that agent not being chosen). So if we use this algorithm as a selection rule, and use threshold payments as a payment rule, then it is well-known (e.g. [9]) that we have a resulting truthful mechanism $\mathcal{M}^{\mathcal{P}}$. (A threshold payment is the supremum of the amounts that the agent can bid and still be selected in the winning set, given the fixed bids of the other agents.) Let k be the approximation ratio of the algorithm; i.e. some k, such that for all instances of the problem $b_{S^{\mathcal{P}}} \leq k \cdot b_S$ holds. (Note that, as the mechanism is truthful, we can assume that $b_e = c_e$).

Lemma 2. *Let k be the approximation ratio of the algorithm $\mathcal{P}$. Then $\forall e \in S^{\mathcal{P}}, p_e \leq k \cdot \text{NTUmin}(\mathbf{c})$.*

Proof. We have defined $d(\mathcal{E} \setminus \{e\})$ to be a (lowest cost) feasible set, not containing e. Assume, for contradiction, that e were to make a threshold bid, $b_e > k \cdot \text{NTUmin}(\mathbf{c})$, and the winning set $S^{\mathcal{P}}$ (chosen by $\mathcal{P}$) includes e. From Lemma 1 we can observe that $b_{d(\mathcal{E}\setminus\{e\})} \leq \text{NTUmin}(\mathbf{c})$. As we have assumed that $b_e \geq k \cdot \text{NTUmin}(\mathbf{c})$, and as $e \in S^{\mathcal{P}}$ we have $b_{S^{\mathcal{P}}} > k \cdot \text{NTUmin}(\mathbf{c})$ (this holds for all choices of $S^{\mathcal{P}}$ when $e \in S^{\mathcal{P}}$). Hence, by transitivity, we have $b_{S^{\mathcal{P}}} > k \cdot b_{d(\mathcal{E}\setminus\{e\})}$. As $d(\mathcal{E} \setminus \{e\})$ is a feasible set, the approximation ratio of $\mathcal{P}$ is at least $\frac{b_{S^{\mathcal{P}}}}{b_{d(\mathcal{E}\setminus\{e\})}}$. Hence when $b_{S^{\mathcal{P}}} > k \cdot b_{d(\mathcal{E}\setminus\{e\})}$ we have $\frac{b_{S^{\mathcal{P}}}}{b_{d(\mathcal{E}\setminus\{e\})}} > k$, showing that $\mathcal{P}$ does not have an approximation ratio of k, giving a contradiction. Therefore for the threshold bid the inequality $b_e \leq k \cdot \text{NTUmin}(\mathbf{c})$ holds, and hence the payment $p_e \leq k \cdot \text{NTUmin}(\mathbf{c})$. □

Theorem 1. *Let $\mathcal{P}$ be a monotonic approximation algorithm with an approximation ratio of k. Then the resulting mechanism $\mathcal{M}^{\mathcal{P}}$ (with selection rule $\mathcal{P}$ and threshold payments) has $\phi_{\text{NTUmin}(\mathbf{c})}(\mathcal{M}^{\mathcal{P}}) \leq k(n-1)$.*

Proof. In a monopoly-free setting we have a winning set $S^{\mathcal{P}}$ such that $|S| \leq n-1$. from Lemma 2, we have upper bounds on the payment for each $e \in S$, $p_e \leq k \cdot \text{NTUmin}(\mathbf{c})$. Summing over $e \in S$ gives $p(S^{\mathcal{P}}) \leq (n-1)k \cdot \text{NTUmin}(\mathbf{c})$. □

While the approximation result is not strictly relevant to the rest of this paper, it does imply, when $k = 1$, that $\phi_{\text{NTUmin}}(VCG) \leq n-1$. (This is more precise than the observation made by Karlin et al. [7] that the frugality ratio of VCG is $O(n)$.) We will also see, in Section 3.1, that even our most restricted commodity auction has a frugality ratio that is exactly as high as this upper bound.

3 The Single-Commodity Auction

We consider a *single-commodity auction* where we have some number of identical items for sale, and a quantity Q, the number of these items the auctioneer requires. Each agent $e \in \mathcal{E}$ can provide a fixed, indivisible, quantity of these items, denoted by q_e. The private cost value of e is denoted by c_e, while the bid made to the mechanism is denoted by b_e. Again, since we focus on truthful mechanisms, we can assume $b_e = c_e$.

One could regard this more abstractly as modelling a setting where each seller has some level of capacity to assist with a task, and the buyer wants the task done, and the total capacity to be at least some amount. However, for our results to apply we would need these capacities to be small integers.

The feasible sets $\mathcal{F}$, are defined based on these quantity parameters as follows:

$$\mathcal{F} = \{T \in 2^{\mathcal{E}} : \left(\sum_{e \in T} q_e\right) \geq Q\}. \tag{4}$$

Initially in Section 3.1 we focus on the special case where each agent e only has at most 2 items for sale. We call this the {1,2} single-commodity auction. In Section 3.3 we move to the more general *integer single-commodity auctions*, where a seller's capacity may be any positive integer, not just 1 or 2.

3.1 The $\{1, 2\}$ Single-Commodity Auction

The $\{1, 2\}$ Single-Commodity Auction is a single-commodity auction with the additional restriction, that $\forall e \in \mathcal{E}, q_e \in \{1, 2\}$. While we could simply use VCG to run this auction (recall that VCG chooses the lowest-cost solution and pays each winning agent a threshold value), Table 1 shows that VCG performs poorly in terms of frugality (in fact, matching the upper bound given in Section 2.1). It is also interesting to note that this frugality ratio is as large as Q, the number of items to purchase. We can argue that measuring the frugality ratio in terms of Q seems to make sense for these types of commodity auctions, as it is more naturally a parameter of the auction than the number of agents is. Hence, we will generally consider the frugality ratio in terms of Q, although the results in terms of n are generally similar.

Table 1. In this example we see that VCG has poor frugality; we have a commodity auction for quantity Q items and observe that the number of agents $n = Q + 1$. For each agent $e \in \mathcal{E}$ the quantity q_e and cost c_e are given in the table. A value $b_e^{\min}$ for a NTUmin bid vector is also given, as is the payment made by the VCG mechanism p_e^{VCG}.

	Agent	q_e	c_e	$b_e^{\min}$	p_e^{VCG}
S	1	1	0	1	1
S	2	1	0	0	1
S	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S	$n-1$	1	0	0	1
	n	2	1		
Total				1	$n-1$

In an attempt to improve frugality, we will now look at a class of (truthful) mechanisms that choose a winning set a little more intelligently.

3.2 The $\mathcal{M}^\alpha$ Mechanism

Here we analyse a class of mechanisms, $\mathcal{M}^\alpha$, each of which is uniquely defined by its 'scaling' value $\alpha \in \mathbb{R}$; a definition for this mechanism follows. $\mathcal{M}^\alpha$ will calculate 'virtual' bids v_e for each agent e by using a scaling factor as follows:

$$v_e = \begin{cases} \alpha b_e, & \text{if } q_e = 1 \\ b_e, & \text{otherwise.} \end{cases}$$

For ease of notation, let the aggregate be $v_V = \sum_{e \in V} v_e$. Let $S^\alpha \in \text{argmin}_{T \in \mathcal{F}} v_T$ be the winning set (the lexicographically first of the feasible sets that have the lowest sum of virtual bids). The payment rule is threshold payments. It is easy to observe that this selection rule is monotonic in the bids, and recall that these are sufficient conditions for a mechanism to be truthful.

Frugality Ratio for $\mathcal{M}^\alpha$. Recall that S is the lowest-cost feasible set, and partition S into two sets, S_1 having agents with quantity 1, and S_2 for those agents having quantity 2.

As choosing both S and S^α requires that ties are broken lexicographically, there is no agent in $S^\alpha \setminus S$ that has the same quantity as an agent in $S \setminus S^\alpha$ (if it is chosen in S^α then it would have been chosen in S). For any $\alpha > 1$, then where S contains some agent e having $q_e = 2$, then S^α must also contain agent e. (If there existed $i, j \notin S$ such that $v_i + v_j \leq v_e$, then $c_i + c_j \leq c_e/\alpha$ contradicting e being chosen in S in preference to $\{i, j\}$). Therefore, where S and S^α are different, $S^\alpha \setminus S$ contains only agents with quantity 2 and $S \setminus S^\alpha$ contains only agents with quantity 1.

We now partition the winning set S^α into three sets, $S^\alpha \cap S_1$, $S^\alpha \cap S_2$, and $S^\alpha \setminus S$ then consider the payments to members of each set separately.

Lemma 3. *For every instance of $\mathcal{M}^\alpha$ when $\alpha = \sqrt{Q}$ then $p_{S^\alpha \cap S_1} \leq \sqrt{Q} \cdot$ NTUmin.*

Proof. We will examine this as two cases. Case 1. Suppose that for every $e \in S^\alpha \cap S_1$ there exists a T_e set satisfying (3) when $(T_e \setminus S) \cap \mathcal{E}_1$ is not empty. Let j be some agent in $T_e \setminus S$ with $q_j = 1$. Assume, for contradiction, that $p_e > c_j$. Hence, agent e's threshold bid $b_e = p_e > c_j$. As j would bid c_j in a truthful mechanism, but $\mathcal{M}^\alpha$ chose e then $c_j \geq p_e$ giving a contradiction. W.l.o.g., we can assume that $T_e = S \setminus \{e\} \cup \{j\}$. Observe that $T'_e \setminus \{j\} \cup \{e\}$ is also a feasible set, hence it must satisfy condition (2), giving $b^{\min}_{S \setminus (T'_e \cup \{e\})} \leq c_{T'_e \setminus (S \cup \{j\}}$, and hence $b^{\min}_e \geq c_j$ or T'_e does not satisfy condition (3), showing that $T_e = S \setminus \{e\} \cup \{j\}$ satisfies condition (3). Using $b^{\min}_e = c_j$ we have $p_{S^\alpha \cap S_1} \leq b^{\min}_{S^\alpha \cap S_1}$ and hence $p_{S^\alpha \cap S_1} \leq$ NTUmin.

Case 2. Suppose that for some $e \in S^\alpha \cap S_1$ there is some T_e set satisfying (3) when $(T_e \setminus S) \cap \mathcal{E}_1$ is empty. There is some $j \in (T_e \setminus S) \cap \mathcal{E}_2$ such that $b^{\min}_{S \setminus T_e} = c_{T_e \setminus S}$. W.l.o.g. assume that $q_{S \setminus T_e} \leq 2$. For each $e \in S^\alpha \cap S_1$ agent e's threshold bid must be $b_e \leq c_j/\alpha$. Hence $p_e = b_e \leq c_j/\alpha$. As $\alpha = \sqrt{Q}$ and Q is trivially an upper bound on the size of S_1, $p_{S^\alpha \cap S_1} \leq \sqrt{Q} \cdot c_j$, with $b^{\min}_{S^\alpha \cap S_1} \geq c_j$ (from $S \setminus T_e \subseteq S^\alpha \cap S_1$), this gives $p_{S^\alpha \cap S_1} \leq \sqrt{Q} \cdot b^{\min}_{S^\alpha \cap S_1} \leq \sqrt{Q} \cdot$NTUmin.

Similar proofs for the other two sets are omitted due to space constraints.

Lemma 4. *For every instance of $\mathcal{M}^\alpha$ when $\alpha = \sqrt{Q}$ then $p_{S^\alpha \cap S_2} \leq \sqrt{Q} \cdot b^{\min}_{S^\alpha \cap S_2}$.*

Lemma 5. *For every instance of $\mathcal{M}^\alpha$ having $\alpha = \sqrt{Q}$ then $p_{S^\alpha \setminus S} \leq \sqrt{Q} \cdot b^{\min}_{S \setminus S^\alpha}$.*

Theorem 2. *For $\{1, 2\}$ Single-Commodity Auctions with quantity Q, the $\mathcal{M}^\alpha$ scaling mechanism when $\alpha = \sqrt{Q}$, gives $\phi_{\text{NTUmin}}(\alpha\mathcal{M}) \leq 2\sqrt{Q}$.*

Proof. From Lemmas 3,4, and 5, the inequalities $p_{S^\alpha \cap S_2} \leq \sqrt{Q} \cdot b^{\min}_{S^\alpha \cap S_2}$, $p_{S^\alpha \setminus S} \leq \sqrt{Q} \cdot c_{S \setminus S^\alpha}$, and $p_{S^\alpha \cap S_1} \leq \sqrt{Q} \cdot \mathrm{NTUmin}$. hold. As $S \setminus S^\alpha$ and $S^\alpha \cap S_2$ are disjoint sets within S, then $b^{\min}_{S^\alpha \cap S_2} + c_{S \setminus S^\alpha} \leq b^{\min}_S \leq \mathrm{NTUmin}$. Therefore, we have $p_{S^\alpha \cap S_2} + p_{S^\alpha \setminus S} \leq \sqrt{Q} \cdot \mathrm{NTUmin}$, and add to give $p_S \leq 2\sqrt{Q} \cdot \mathrm{NTUmin}$ and hence $\phi_{\mathbf{NTUmin}}(\mathcal{M}^\alpha) \leq 2\sqrt{Q}$. □

A Lower Bound on Frugality. Here, we see that any truthful mechanism must pay at least $\sqrt{Q} \cdot \mathrm{NTUmin}$, showing that the $\mathcal{M}^\alpha$ mechanism with $\alpha = \sqrt{Q}$ is within at most a factor of two of optimal.

Theorem 3. *There exists a $\{1,2\}$ single-commodity auction for Q items such that any truthful mechanism $\mathcal{M}$, must pay at least $\sqrt{Q} \cdot \mathrm{NTUmin}$.*

Proof. For any quantity Q, let I be an instance of a set-system auction having $\mathcal{E} = \{1, \ldots, Q+1\}$ and $\mathbf{q} = \{1, \ldots, 1, 2\}$. Suppose that $\mathcal{M}$ is some truthful mechanism. Consider each $e \in \{1, \ldots, Q\}$ and suppose an instance such that $b_e = 1$, $b_{Q+1} = \sqrt{Q}$ and all other agents bid 0. We are interested in two cases, either every $e \in \{1, \ldots, Q\}$ would be chosen in the winning set by $\mathcal{M}$, or else there is some such e for which $Q+1$ would be chosen instead.

Case 1. Suppose that every $e \in \{1, \ldots, Q\}$ is chosen in preference to $Q+1$. Let $\mathbf{b} = (0, \ldots, 0, \sqrt{Q})$ be a bid vector. Observe that $S = \{1, \ldots, Q\}$ and that $\mathbf{b^{min}} = (\sqrt{Q}, 0, \ldots, 0)$ denotes a bid vector satisfying conditions (1),(2) and (3), hence $\mathrm{NTUmin} \leq \sqrt{Q}$. As every agent in S, would have been chosen by $\mathcal{M}$ with a bid of 1 then $S^{\mathcal{M}} = S$ and each threshold bid must be at least 1, hence $p_{\mathcal{E}} \geq Q$ and $p_{\mathcal{E}}/\mathrm{NTUmin} \geq \sqrt{Q}$.

Case 2. Suppose (w.l.o.g) that agent $Q+1$ is chosen in preference to agent 1. Let $\mathbf{b} = (1, 0, \ldots, 0, 1)$ be a bid vector. Observe that $S = \{1, \ldots, Q\}$ (with the tie broken lexicographically) and that $\mathbf{b^{min}} = (1, 0, \ldots, 0)$ denotes a bid vector satisfying conditions (1),(2) and (3), hence $\mathrm{NTUmin} \leq 1$. As mechanism $\mathcal{M}$ will choose agent $Q+1$ with bid $\sqrt{Q}$ in preference to 1, being truthful implies that $\mathcal{M}$ will still choose $Q+1$ with a lower bid of 1, hence $Q+1 \in S^{\mathcal{M}}$. As agent $Q+1$ would still have been chosen had it bid $\sqrt{Q}$, its threshold bid is at least $\sqrt{Q}$, and hence $p_{Q+1} \geq \sqrt{Q}$. This gives $p_{\mathcal{E}} \geq \sqrt{Q}$ and hence $p_{\mathcal{E}}/\mathrm{NTUmin} \geq \sqrt{Q}$.

For every truthful mechanism $\mathcal{M}$, either Case 1 or Case 2 applies, hence the frugality ratio $\phi_{\mathbf{NTUmin}}(\mathcal{M}) \geq \sqrt{Q}$. □

3.3 Integer Single-Commodity Auctions

We consider improvements to frugality bounds in the more general setting, where the restriction on the quantity of each agent to 1 or 2 is relaxed. We have a lower bound on frugality of $\sqrt{Q}$ from the $\{1,2\}$ single commodity auction, but we may believe that there is a stronger lower bound in the integer case. While we do not have a result for all truthful mechanisms, we obtain an asymptotically stronger lower bound on frugality that applies to a natural class of scaling mechanisms, of at least $\frac{4Qe^{-2}}{\ln^2 Q}$, for all mechanisms in this class.

Preliminaries. Let k be a 'maximum quantity' parameter such that $\forall e \in \mathcal{E}, q_e \leq k$ holds and assume that $k \leq \sqrt{Q}$. Let β be a scaling function, returning a linear scaling vector, $\mathbf{a} = \beta(Q, k)$ (with $a_e \in \mathbb{R}$). Let $\mathcal{M}^\beta$ be the mechanism that uses the scaling vector $\mathbf{a} = (a_1, \ldots, a_k)$ returned by β, as follows. Compute a 'virtual' bid v_e for each agent e as $v_e = b_e a_{q_e}$. Let $S \in \operatorname{argmin}_{T \in \mathcal{F}} v_T$ be the winning set. Each agent e will be paid its threshold value, p_e. If we consider every scaling function β, and the resulting class of mechanisms, then we can think of $\mathcal{M}^\beta$ as the class of all 'blind-scaling' mechanisms; where the mechanism must choose a scaling factor for each possible quantity, based only on the quantity required Q and the maximum quantity parameter k.

A Lower Bound for Blind-Scaling Mechanisms. The proof will examine a series of example instances given, and show that at least one of them must cause a payment ratio that satisfies the lower bound. We can generalise the example given in Table 1, and will show this in Table 2. For each $j \in \{1, \ldots, k-1\}$ let Table 2 describe instance I_j. Observe the assumption that $j < k \leq \sqrt{Q}$ implies that $m \geq j$ which is required by the structure of the example (m is defined in the example as $m = \lceil \frac{Q}{j} \rceil$).

We can see that there are j agents in S that can have a (NTUmin) bid value $b_e^{\min} = 1$. We can show that there can be no more than j agents that can each bid 1 as follows; $j+1$ agents, each with quantity j, could be 'replaced' by the j agents outside S, each with quantity $j+1$, so no set of $j+1$ agents in S can bid a sum of more than j.

More formally, $\forall e \in S$, let $T_e = S \setminus \{1, \ldots, j, e\} \cup \{(m+1), \ldots, (m+j+1)\}$. Observe that $\left(\sum_{i=1}^{j} q_i\right) + q_e = j(j+1)$ and $\left(\sum_{i=1}^{j+1} q_{m+i}\right) = j(j+1)$ hence $q_S =$

Table 2. Instance I_j: In this example we have a $\{j, j+1\}$ commodity auction for quantity Q items. Let $m = \lceil \frac{Q}{j} \rceil$ and observe that the winning set is given by $S = \{1, \ldots, m\}$. For each agent $e \in \mathcal{E}$ the quantity q_e and cost c_e are given in the table. A value $b_e^{\min}$ for a NTUmin bid vector is also given, giving NTUmin$\leq j$. The payment made by the $\mathcal{M}^\beta$ mechanism is also given in Table 2 as p_e.

	Agent	q_e	c_e	$b_e^{\min}$	p_e
	1	j	0	1	a_{j+1}/a_j
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	j	j	0	1	a_{j+1}/a_j
S	$j+1$	j	0	0	a_{j+1}/a_j
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	m	j	0	0	a_{j+1}/a_j
	$m+1$	$j+1$	1		
	$\vdots$	$\vdots$	$\vdots$		
	$m+j+1$	$j+1$	1		
Total				j	ma_{j+1}/a_j

q_T and T_e is a feasible set. Using this T_e in condition (3) for all $e \in \{j+1,\dots,m\}$ gives $\left(\sum_{i=1}^{j} b_i^{\min}\right) + b_e^{\min} = \sum_{i=1}^{j+1} c_{m+1}^{j}$. As $\sum_{i=1}^{j} b_i^{\min} = j$ and $\sum_{i=1}^{j+1} c_{m+i} = j$ then we have $b_e^{\min} = 0$, which shows that for all $e \in \{j+1,\dots,m\}$ then vector $\mathbf{b}^{\min}$ has some T_e satisfying condition (3) of Definition 1. For all $e \in \{1,\dots,j\}$, let $T_e = S \setminus \{e\} \cup \{m+1\}$ which gives $b_e^{\min} = 1$, showing that the bid vector $\mathbf{b}^{\min}$ has, for all $e \in S$, some T_e satisfying condition (3) and as we can observe $\mathbf{b}^{\min}$ satisfies conditions (1) and (2) then this shows NTUmin $\leq b_S^{\min}$ and hence NTUmin $\leq j$.

We can also generalise the payment to each $e \in S$. For each agent $e \in S$, if $v_e > v_{m+1}$ then agent e would not be chosen, as the winning set could become $S \setminus \{e\} \cup \{m+1\}$. Where $v_e = v_{m+1}$, then agent e may still be chosen, hence when agent e can submit a threshold bid b_e such that $v_e = v_{m+1}$ and this gives the threshold payment.

If we assume for all $e \in S$, that $b_e = \frac{a_{j+1}}{a_j}$ then as $v_e = b_e a_j$ we have $v_e = \frac{a_{j+1}}{a_j} a_j = a_{j+1} = v_{m+1}$. This shows that $b_e = \frac{a_{j+1}}{a_j}$ is a threshold bid for all $e \in S$, hence the payment is given by $p_e = \frac{a_{j+1}}{a_j}$.

Let $\mathbf{c}$ be a cost vector for instance I_j and let $p_{\mathcal{E}}$ be the sum of payments. We examine the payment ratio $\frac{p_{\mathcal{E}}}{\text{NTUmin}}$ as follows. There are at least $\frac{Q}{j}$ agents in S, each is paid $\frac{a_{j+1}}{a_j}$, and NTUmin $\leq j$; hence the payment ratio satisfies the inequality $\frac{p_{\mathcal{E}}}{\text{NTUmin}} \geq \frac{Q a_{j+1}}{j^2 a_j}$. We can then use this as we move onto the first part of the proof. We will use the 'maximum quantity' parameter, k, and will examine a series of instances where all agents have quantity at most k. We give a certain ratio, $\frac{Q^{\frac{k-1}{k}}}{k^2}$, and we will show (from these instances) that a minimum separation is needed between any consecutive scaling values (a_j, a_{j+1}) (where $j < k$) in order to satisfy this ratio. We will then show how having this minimum separation between consecutive scaling values implies a large separation between the first and k-th value, and give a further instance where a large separation will result in a frugality ratio larger than $\frac{Q^{\frac{k-1}{k}}}{k^2}$.

Finally we will show how to compute a value for k that gives a lower-bound for any given Q.

Proposition 1. *For instance I_j of $\mathcal{M}^\beta$ with $j \leq k-1$ and $\frac{a_j}{a_{j+1}} \leq Q^{\frac{1}{k}}$ the inequality $\frac{p_{\mathcal{E}}}{\text{NTUmin}} \geq \frac{Q^{\frac{k-1}{k}}}{k^2}$ holds.*

Proof. As $j \leq k$ implies $\frac{1}{j^2} \geq \frac{1}{k^2}$, then $\frac{Q a_{j+1}}{j^2 a_j} \geq \frac{Q a_{j+1}}{k^2 a_j}$. It follows, due to transitivity with $\frac{p_{\mathcal{E}}}{\text{NTUmin}} \geq \frac{Q a_{j+1}}{j^2 a_j}$ that $\frac{p_{\mathcal{E}}}{\text{NTUmin}} \geq \frac{Q a_{j+1}}{k^2 a_j}$. Also $\frac{a_j}{a_{j+1}} \leq Q^{\frac{1}{k}}$ can be be expressed as $\frac{a_{j+1}}{a_j} \geq Q^{\frac{-1}{k}}$ therefore, by transitivity $\frac{p_{\mathcal{E}}}{\text{NTUmin}} \geq \frac{Q}{k^2}\frac{a_{j+1}}{a_j} \geq \frac{Q Q^{\frac{-1}{k}}}{k^2}$. This can be simplified to state $\frac{p_{\mathcal{E}}}{\text{NTUmin}} \geq \frac{Q^{\frac{k-1}{k}}}{k^2}$, completing the proof. □

This minimum separation required between every a_j and a_{j+1} implies that there is large separation between a_1 and a_k. We will see, in Table 3, that such a large separation then results in a similarly large frugality ratio.

Table 3. Instance I_k: In this example we have a commodity auction for quantity Q items with the parameter k. Let $m = \lceil \frac{Q}{k} \rceil$ and observe that the winning set is given by $S = \{1, \ldots, m\}$. For each agent $e \in \mathcal{E}$ the quantity q_e and cost c_e are given in the table. A value $b_e^{\min}$ for a NTUmin bid vector is also given, showing NTUmin $\leq mk$. The payment made by the $\mathcal{M}^\beta$ mechanism is also given in the table as p_e.

	Agent	q_e	c_e	$b_e^{\min}$	p_e
S	1	k	0	k	ka_1/a_k
S	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
S	m	k	0	k	ka_1/a_k
	$m+1$	1	1		
	$\vdots$	$\vdots$	$\vdots$		
	$m+k$	1	1		
Total				mk	mka_1/a_k

Proposition 2. *For instance I_k of $\mathcal{M}^\beta$ the inequality $\frac{p_\mathcal{E}}{\text{NTUmin}} \geq \frac{a_1}{a_k}$ holds.*

Proof. For each $e \in S$, there is exactly one feasible set not containing e — that is $\mathcal{E} \setminus \{e\}$. Therefore the only bid vector that could satisfy NTUmin must satisfy condition (3) of Definition 1 with $T_e = \mathcal{E} \setminus \{e\}$. Therefore the NTUmin bid for each $e \in S$ must be given by $b_e^{\min} = c_{T_e \setminus S} = c_{\{m+1,\ldots,m+k+1\}} = k$. As there are m agents in S, each having a bid $b_e^{\min} = k$, we have NTUmin $\leq mk$. Similarly, the threshold bid for e must be where $v_e = v_{\{m+1,\ldots,m+k\}}$. Assuming $b_e = \frac{ka_1}{a_k}$ multiplying by the scaling factor a_k gives $v_e = \frac{ka_1}{a_k} a_k = ka_1$. The virtual bids of the competing agents $i \in \{m+1, \ldots, m+k+1\}$ are $v_i = a_1$, hence $v_{\{m+1,\ldots,m+k\}} = ka_1$ showing that $b_e = \frac{ka_1}{a_k}$ is a threshold bid, and hence the payment $p_e = \frac{ka_1}{a_k}$.

Therefore, in Instance I_k, there are m agents in S; each is paid $\frac{ka_1}{a_k}$ giving a total payment of $\frac{mka_1}{a_k}$. As we have seen NTUmin $\leq mk$ hence $\frac{p_\mathcal{E}}{\text{NTUmin}} \geq \frac{a_1}{a_k}$. □

We now see there is always some instance which implies a lower bound on the payment ratio, for any possible scaling vector of the mechanism.

Proposition 3. *For any scaling vector* **a** *given by $\mathcal{M}^\beta$ there is either some Instance I_j for $j \in \{1, \ldots, k-1\}$ or Instance I_k such that the inequality $\frac{p_\mathcal{E}}{\text{NTUmin}} \geq \frac{Q^{\frac{k-1}{k}}}{k^2}$ holds.*

Proof. If there existed some $j \in \{1, \ldots, k-1\}$ such that $\frac{a_j}{a_{j+1}} \leq Q^{\frac{1}{k}}$ then Proposition 1 implies that $\frac{p_\mathcal{E}}{\text{NTUmin}} \geq \frac{Q^{\frac{k-1}{k}}}{k^2}$. So, suppose that the expression $\forall j \in \{1, \ldots, k-1\}, \frac{a_j}{a_{j+1}} > Q^{\frac{1}{k}}$ holds. We can see this implies that the consecutive scaling values must have a certain separation. By way of example, this gives $\frac{a_1}{a_2} > Q^{\frac{1}{k}}$, $\frac{a_2}{a_3} > Q^{\frac{1}{k}}$ etc. By transitivity we would have $\frac{a_1}{a_3} > Q^{\frac{2}{k}}, \frac{a_1}{a_4} > Q^{\frac{3}{k}}$ etc. This can then be generalised, for $j \in \{1, \ldots, k-1\}$ to give $\frac{a_1}{a_{j+1}} > Q^{\frac{j}{k}}$.

For $j = k - 1$, then we have $\frac{a_1}{a_k} > Q^{\frac{k-1}{k}}$. Referring back to Proposition 2, Instance I_k gives $\frac{p\varepsilon}{\mathrm{NTUmin}} \geq \frac{a_1}{a_k}$ and, by transitivity, $\frac{p\varepsilon}{\mathrm{NTUmin}} > Q^{\frac{k-1}{k}}$.

Hence there is some instance, either I_j for $j \in \{1, \ldots, k-1\}$ or I_k that satisfies the proposition. □

Now that we have seen that there is always some instance that gives at least this payment ratio in terms of k, we can use this to prove a lemma that shows a lower bound on the frugality ratio for all Integer Single-Commodity Auctions.

Lemma 6. *For all Integer Single-Commodity Auctions with quantity Q and maximum quantity parameter $k \leq \sqrt{Q}$, for every blind-scaling scaling mechanisms $\mathcal{M}^\beta$ the inequality $\phi_{\mathrm{NTUmin}}(\mathcal{M}^\beta) \geq \frac{Q^{\frac{k-1}{k}}}{k^2}$ holds.*

Proof. The blind-scaling mechanism $\mathcal{M}^\beta$ must, by definition, calculate its scaling vector $\mathbf{a}$ for use on any instance that it may be given with these parameters. Once this scaling vector is fixed the mechanism may possibly be given either Instance I_k or Instance I_j for any $j \in \{1, \ldots, k-1\}$. Proposition 3 shows that at least one of these instances gives $\frac{p\varepsilon}{\mathrm{NTUmin}} \geq \frac{Q^{\frac{k-1}{k}}}{k^2}$. The existence of such an instance proves $\phi_{\mathrm{NTUmin}}(\mathcal{M}^\beta) \geq \frac{Q^{\frac{k-1}{k}}}{k^2}$. □

Now that we have shown a lower bound on frugality for values of Q in terms of the parameter k, we can specify a value of k such as to give a lower bound entirely in terms of Q. To that end, suppose $k = \frac{\ln Q}{2}$, and we will see this implies a lower bound of $\frac{4Qe^{-2}}{\ln^2 Q}$ for $\mathcal{M}^\beta$ mechanisms.

Theorem 4. *Given any Integer Single-Commodity Auction having quantity Q, for every blind-scaling mechanism $\mathcal{M}^\beta$ the inequality $\phi_{\mathrm{NTUmin}}(\mathcal{M}^\beta) \geq \frac{4Qe^{-2}}{\ln^2 Q}$ holds.*

Proof. Considering the proof of Lemma 6, suppose $k = \ln Q/2$. The expression given in Lemma 6 implies $\frac{Q^{\frac{k-1}{k}}}{k^2} = \frac{4Qe^{-2}}{\ln^2 Q}$, and hence, $\phi_{\mathrm{NTUmin}}(\mathcal{M}^\beta) \geq \frac{4Qe^{-2}}{\ln^2 Q}$. □

4 Conclusion

While single-commodity auctions are quite simple, they show surprisingly high frugality ratios. Particularly in the $\{1, 2\}$ case, a lower bound on the frugality ratio for every truthful mechanism of $\sqrt{Q}$ seems unreasonably high. This result could also seem to call into question the suitability of NTUmin as a reasonable benchmark. Our scaling mechanism is shown to be within a factor of 2 of optimal; it may be that this factor of 2 could be reduced with a stronger analysis.

While we have shown a fairly large lower bound on the frugality of 'blind-scaling' mechanisms in the more general case of integer single-commodity auctions, it is not known if some other form of mechanism would result in better frugality. Also, we have not presented any mechanism that would give a frugality ratio of better than Q in this case, although it seems that some form of scaling mechanisms should, at least, give some slightly better result. Choosing to measure frugality in terms of Q or n makes little difference in the $\{1, 2\}$ case, but the difference is more pronounced in the integer case, and showing good frugality results in terms of n may be an interesting goal.

We have only considered frugality in this setting with respect to NTUmin. More recently (see, e.g., [2,8]) we have seen frugality ratios analysed with respect to NTUmax. It is likely that we will get more satisfactory frugality ratios with respect to NTUmax, particularly in the $\{1, 2\}$ case. Although, in the integer case, we may still get reasonably large frugality ratios. Take, for example, Theorem 3 and amend the quantity vector to be $\mathbf{q} = (1, \ldots, 1, Q)$. This would give NTUmax $= 1$ (as $T_e = \{Q + 1\}$ is the only alternative feasible set, and so must satisfy condition (3)). The rest of the proof could then be applied, with the obvious minor changes, to show that $\phi_{\mathrm{NTUmax}}(\mathcal{M}) \geq \sqrt{Q}$.

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