Distributed Selfish Load Balancing*

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Abstract

Suppose that a set of m tasks are to be shared as equally as possible amongst a set of n resources. A game-theoretic mechanism to find a suitable allocation is to associate each task with a "selfish agent", and require each agent to select a resource, with the cost of a resource being the number of agents to select it. Agents would then be expected to migrate from overloaded to underloaded resources, until the allocation becomes balanced.

Recent work has studied the question of how this can take place within a distributed setting in which agents migrate selfishly without any centralized control. In this paper we discuss a natural protocol for the agents which combines the following desirable features: It can be implemented in a strongly distributed setting, uses no central control, and has good convergence properties. For $m \gg n$, the system becomes approximately balanced (an ϵ-Nash equilibrium) in expected time $O(\log \log m)$. We show using a martingale technique that the process converges to a perfectly balanced allocation in expected time $O(\log \log m + n^4)$. We also give a lower bound of $\Omega(\max\{\log \log m, n\})$ for the convergence time.

1 Introduction

Suppose that a consumer learns the price she would be charged by some domestic power supplier other than the one she is currently using. It is plausible that if the alternative price is lower than the price she is currently paying, then there is some possibility that she will switch to the new power supplier. Furthermore, she is more likely to switch if the ratio of current price to new price is large. If there is only a small saving, then it becomes unattractive to make the switch, since an influx of new business (oneself and other consumers) may drive up the price of the new power supplier and make it no longer competitive.

We study a simple mathematical model of the above natural rule, in the context of a load balancing (or task allocation) scenario that has received a lot of recent attention. We assume the presence of many individual users, who may assign their tasks to chosen resources. The users are *selfish* in the sense that they attempt to optimize their own situation, i.e., try to assign their tasks to minimally loaded resources, without trying to optimize the global situation. In general, a *Nash equilibrium* among a set of selfish users is a state in which no user has the incentive to change her current decision. In our setting, this corresponds to no user having an incentive to reallocate their task to some other resource. An ϵ-Nash equilibria is a standard notion of approximate Nash equilibrium, and is a state where no user can change her cost by a multiplicative factor of less than $1 - \epsilon$ by changing action. Here we do not focus on the *quality* of equilibria, but rather on the (perhaps more algorithmic) question of convergence time to such a state.

We assume a strongly distributed and concurrent setting, i.e., there is no centralized control mechanism whatsoever, and all users may choose to reallocate their tasks at the same time. Thus, we do not (and cannot) use the traditional *Elementary Step System*, where the assumption is that at most one user may reallocate her task at any given stage [8, 5].

Throughout we let m denote the number of tasks (in the above discussion, customers) and n the number of resources (power suppliers). As hinted in the above discussion, we assume that typically $m \gg n$. In a single time step (or round) each task does the following. Let i be the resource currently being used by the task. Select j uniformly at random from $\{1, \ldots, n\}$ and find the load of resource j. Let X_i and X_j be the loads of resources i and j respectively. If $X_j < X_i$, migrate from i to j with a probability of $1 - X_j/X_i$; the transition from round t to round $t + 1$ is given in Figure 1. Notice that if we had unconditional migrations, i.e., without an additional coin flip (move only with probability $1 - X_j(t)/X_i(t)$), then this may lead to an unstable system; consider for example the case $m = 2$ with initially most tasks assigned to one of the resources: the overload would oscillate between the two resources, with a

*This work was partially supported by the EPSRC grants "Discontinuous Behaviour in the Complexity of Randomized Algorithms" and "Algorithmics of Network-sharing Games", and by the Natural Sciences and Engineering Research Council of Canada (NSERC) discovery grant 250284-2002.

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load ratio tending towards 2:1.

> **For** each task b **do** in parallel
> Let i_b be the current resource of task b
> Choose resource j_b uniformly at random
> Let $X_{i_b}(t)$ be the current load of resource i
> Let $X_{j_b}(t)$ be the current load of resource j
> **If** $X_{i_b}(t) > X_{j_b}(t)$ **then**
> Move task b from resource i_b to j_b with probability $1 - X_{j_b}(t)/X_{i_b}(t)$

Figure 1: The protocol with "neutral moves" allowed.

It can easily be seen that, if all tasks use the above policy, then the expected load of every resource at the next step is m/n. This provides a compelling motivation for the policy, which is that as a result, no task has an incentive to deviate unilaterally from this policy. In the terminology of [6] this is a *Nash rerouting policy*. Although the above rule is very natural and has the nice property as described above, we show that it may take a long time to converge to an perfectly balanced allocation of tasks to resources. Define a *neutral move* to be a task migration from a resource with load ℓ at time t to a resource with load $\ell - 1$ at time t (so, if no other task migrates, then the cost to the migrating task is unchanged.) We consider a modification in which neutral moves are specifically disallowed (see Figure 2). That seemingly-minor change is necessary to ensure fast convergence from an almost balanced state to a perfectly-balanced state. To summarize, here are the most important features of the modified protocol:

- We do not need any global information whatsoever (apart from the number of available resources); in particular, a task does not need to know the total number of tasks in the system. Also, it is strongly distributed and concurrent. If additional tasks were to enter the system, it would rapidly converge once again, with no outside intervention.
- A migrating task needs to query the load of only one other resource (thus, doing a constant amount of work in each round).
- When a task finds a resource with a significantly smaller load (that is, a load that is smaller by at least two), the migration policy is *exactly* the same as that used by the Nash rerouting policy of Figure 1, so the incentive is to use that probability.
- When a task finds a resource with a load that is smaller by exactly one unit, the migration policy is sufficiently close to the Nash rerouting policy that the difference in expected load is at most one, and there is little incentive to deviate.
- The protocol is simple (as well as provably efficient) enough to convince users to actually stick to it.

1.1 Related Work The papers [5, 8, 6] are most closely related to this work. Even-Dar et al. [5] introduce the idea of using a potential function to measure closeness to a balanced allocation, and use it to show convergence for sequences of randomly-selected "best response" moves in a more general setting in which tasks may have variable weights and resources may have variable capacities. (A "best response" move is one where a task migrates to a resource with smallest cost. In order to ensure that a move is best response, it is necessary to do them consecutively rather than concurrently.) Goldberg [8] considered a protocol in which tasks select alternative resources at random and migrate if the alternative load is lower. The protocol may be implemented in a weakly distributed sense, requiring that migration events take place one at a time, and costs are updated immediately.

Most recently, Even-Dar and Mansour allow concurrent, independent rerouting decisions where tasks are allowed to migrate from overloaded to underloaded resources. Their rerouting process terminates in expected $O(\log\log m + \log n)$ rounds when the system reaches a Nash equilibrium. This faster convergence (in terms of the number n of resources) is attained using a certain amount of global knowledge. A task is required to know whether its link is overloaded (having above average load) and tasks on underloaded links do not migrate at all. Our rerouting policy does not require that agents know anything other that their current link load, and the load of a randomly-chosen alternative. Even-Dar and Mansour also present a general framework that can be used to show a logarithmic convergence rate for a wide class of rerouting strategies. Our protocol does not fall into that class, since we do not require migrations to occur only from overloaded links. Furthermore, our reassignment strategy has a non-logarithmic lower bound on the convergence time.

Our rerouting strategy is also related to reallocation processes for balls into bins games. The goal of a balls into bins game is to allocate m balls as evenly as possible into n bins. It is well-known that a fairly even distribution can be achieved if every ball is allowed to randomly choose d bins and then the ball is allocated to the least loaded among the chosen bin (see [13] for an overview). Czumaj, Riley, and Scheideler [3] consider such an allocation where each ball initially chooses two bins. They show that, in a polynomial number of steps, the reallocation process ends up in a state with maximum load at most $\lceil m/n \rceil + 1$. In [16] Sanders, Egner, and Korst show that a maximum load of $\lceil m/n \rceil + 1$ is optimal if every ball is restricted to two random choices.

Leaving aside the distributed framework, our scenario becomes a special case of one introduced by Koutsoupias and Papadimitriou [10]. They consider n parallel links (resources) and a set T of m tasks. (In [10] and subsequent papers, usually m denotes the number of resources and n the number of tasks. Here we are using m and n as they are used the balls-into-bins literature.) Each link ℓ_i comes with delay d_i, and each task with a weight w_i. (In this paper, all weights and delays are the same.) An *assignment* is a vector $A = (A_1, \ldots, A_m)$ which assigns the i-th task to resource A_i. In the language of game theory, an assignment associates each task with a *pure strategy*. The *load* of resource ℓ in assignment A is defined to be $L(\ell, A) = d_\ell \sum_{i \in T: A_i = \ell} w_i$. The load of task i in assignment A is $L(A_i, A)$. A is said to be a *Nash* assignment if, for every task i and every resource ℓ, we have $L(A_i, A) \leq L(\ell, A')$, where the assignment A' is derived from A by re-assigning task i to resource ℓ, and making no other change. Koutsoupias and Papadimitriou worked in the more general setting of *mixed strategies*. In a mixed strategy, instead of choosing a resource A_i, task i chooses a vector $(p_{i,1}, \ldots, p_{i,m})$ in which $p_{i,j}$ denotes the probability with which task i will use resource j. A collection of mixed strategies (one strategy for each task) is a Nash equilibrium if no task can reduce its expected cost by modifying its own probability vector. Much research has been done since then (for example [12, 4, 7, 15, 2]) in different contexts and with different goals.

1.2 Overview of our results

Section 3 deals with upper bounds on convergence time. The main result, Theorem 3.1, is that the protocol of Figure 2 converges to a Nash equilibrium within expected time $O(\log \log m + n^4)$.

The proof of Theorem 3.1 shows that the system becomes *approximately* balanced very rapidly. Specifically, Corollary 3.4 shows that if $n \leq m^{1/3}$, then for all ϵ, either version of the distributed protocol (with or without neutral moves allowed) attains ϵ-Nash equilibrium (where all load ratios are within $[1-\epsilon, 1+\epsilon]$) in expected $O(\log \log m)$ rounds. The rest of Section 3 analyses the protocol of Figure 2. It is shown that within an additional $O(n^4)$ rounds the system becomes optimally balanced.

In Section 4, we provide two lower bound results. The first one, Theorem 4.1, shows that the first protocol (of Figure 1, including moves that do not necessarily yield a strict improvement for an individual task but allow for simply "neutral" moves as well, results in exponential (in n) expected convergence time. Finally, in Theorem 4.2 we provide a general lower bound (regardless of which of the two protocols is being used) on the expected convergence time of $\Omega(\log \log m)$. This lower bound matches the upper bound as a function of m.

2 Notation

There are m tasks and n resources. An assignment of tasks to resources is represented as a vector $(x_1, \ldots, x_n)$ in which x_i denotes the number of tasks that are assigned to resource i. In the remainder of this paper, $[n]$ denotes $\{1, \ldots, n\}$. The assignment is a Nash equilibrium if for all $i \in [n]$ and $j \in [n]$, $|x_i - x_j| \leq 1$. We study a distributed process for constructing a Nash equilibrium. The states of the process, $X(0), X(1), \ldots$, are assignments. The transition from state $X(t) = (X_1(t), \ldots, X_n(t))$ to state $X(t+1)$ is given by the greedy distributed protocol in Figure 2.

> **For** each task b **do** in parallel
>   Let i_b be the current resource of task b
>   Choose resource j_b uniformly at random
>   Let $X_{i_b}(t)$ be the current load of resource i
>   Let $X_{j_b}(t)$ be the current load of resource j
>   **If** $X_{i_b}(t) > X_{j_b}(t)$ **then**
>   Move task b from resource i_b to j_b with
>   probability $1 - X_{j_b}(t)/X_{i_b}(t)$

Figure 2: The modified protocol, with "neutral moves" disallowed.

Note that if $X(t)$ is a Nash equilibrium, then $X(t+1) = X(t)$ so the assignment stops changing. Here is a formal description of the transition from a state $X(t) = x$. Independently, for every $i \in [n]$, let $(Y_{i,1}(x), \ldots, Y_{i,n}(x))$ be a random variable drawn from a multinomial distribution with the constraint $\sum_{j=1}^{n} Y_{i,j}(x) = x_i$. ($Y_{ij}$ represents the number of migrations from i to j in a round.) The corresponding probabilities $(p_{i,1}(x), \ldots, p_{i,n}(x))$ are given by

$$p_{i,j}(x) = \begin{cases} \frac{1}{n}\left(1 - \frac{x_j}{x_i}\right) & \text{if } x_i > x_j + 1, \\ 0 & \text{if } i \neq j \text{ but } x_i \leq x_j + 1, \\ 1 - \sum_{j \neq i} p_{i,j}(x) & \text{if } i = j. \end{cases}$$

Then $X_i(t+1) = \sum_{\ell=1}^{n} Y_{\ell,i}(x)$.

For any assignment $x = (x_1, \ldots, x_n)$, let $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. Similar to [5, 8, 6] we define potential function $\Phi(x) = \sum_{i=1}^{n} (x_i - \overline{x})^2$. Note that $\Phi(x) = \sum_{i=1}^{n} x_i^2 - n\overline{x}^2$.

3 Upper bound on convergence time

Our main result is the following

THEOREM 3.1. *Let T be the number of rounds taken by the protocol of Figure 2 to reach a Nash equilibrium for the first time. Then $\mathbb{E}[T] = O(\log \log m + n^4)$.*

The proof of this theorem proceeds as follows. First (Lemma 3.1) we give an upper bound on $\mathbb{E}[\Phi(X(t))]$ which implies (Corollary 3.3) that there is a $\tau = O(\log \log m)$ such that, with high probability, $\Phi(X(\tau)) = O(n)$. We also show (Observation 3.4 and Corollary 3.5) that $\Phi(X(t))$ is a super-martingale and (Lemma 3.5) that it has enough variance. Using these facts, we obtain the upper bound on the convergence time.

Definition: Let $S_i(x) = \{j \mid x_j < x_i - 1\}$. $S_i(x)$ is the set of resources that are significantly smaller than resource i in state x. Similarly, let $L_i(x) = \{j \mid x_j > x_i + 1\}$ and let $d_i(x) = \frac{1}{n}\sum_{j:|x_i - x_j| \le 1}(x_i - x_j)$.

OBSERVATION 3.1. $\mathbb{E}[X_i(t+1) \mid X(t) = x] = \overline{x} + d_i(x)$.

Proof.

$$\mathbb{E}[X_i(t+1) \mid X(t) = x] = \sum_{\ell=1}^{n} \mathbb{E}[Y_{\ell,i}(x)] = \sum_{\ell=1}^{n} x_\ell p_{\ell,i}(x)$$

$$= \sum_{\ell \in L_i(x)} x_\ell \frac{1}{n}\left(1 - \frac{x_i}{x_\ell}\right) + x_i\left(1 - \sum_{j \in S_i(x)} \frac{1}{n}\left(1 - \frac{x_j}{x_i}\right)\right),$$

which can be simplified into the required form. □

OBSERVATION 3.2. $\sum_{i=1}^{n} \left(\mathbb{E}[X_i(t+1) \mid X(t) = x]\right)^2 = n\overline{x}^2 + \sum_{i=1}^{n} d_i(x)^2$.

Proof. Using Observation 3.1,

$$\begin{aligned}
&\sum_{i=1}^{n} \left(\mathbb{E}[X_i(t+1) \mid X(t) = x]\right)^2 \\
&\quad = \sum_{i=1}^{n} \left(\overline{x} + d_i(x)\right)^2 \\
&\quad = n\overline{x}^2 + 2\overline{x}\sum_{i=1}^{n} d_i(x) + \sum_{i=1}^{n} d_i(x)^2,
\end{aligned}$$

and the second term is zero since $d_i(x) = \mathbb{E}[X_i(t+1) \mid X(t) = x] - \overline{x}$. □

OBSERVATION 3.3. $\mathrm{var}[X_i(t+1) \mid X(t) = x] \le \frac{1}{n}\sum_{\ell \in L_i(x)} (x_\ell - x_i) + \frac{1}{n}\sum_{j \in S_i(x)} (x_i - x_j)$.

Proof.

$$\begin{aligned}
&\mathrm{var}(X_i(t+1) \mid X(t) = x) \\
&= \sum_{\ell=1}^{n} \mathrm{var}(Y_{\ell,i}(x)) = \sum_{\ell=1}^{n} x_\ell p_{\ell,i}(x)(1 - p_{\ell,i}(x)) \\
&= \sum_{\ell \in L_i(x)} x_\ell \frac{1}{n}\left(1 - \frac{x_i}{x_\ell}\right)(1 - p_{\ell,i}(x)) + x_i p_{i,i}(x) \times \\
&\qquad \times \left(\sum_{j \in S_i(x)} \frac{1}{n}\left(1 - \frac{x_j}{x_i}\right)\right).
\end{aligned}$$

Using $1 - p_{\ell,i}(x) \le 1$ and $p_{i,i}(x) \le 1$ and simplifying, we get the result. □

Definition: For any assignment x, let $s_i(x) = |\{j \mid x_j = x_i - 1\}|$ and $l_i(x) = |\{j \mid x_j = x_i + 1\}|$. Let $u_1(x) = \sum_{i=1}^{n}\sum_{j \in [n]:|x_i - x_j| > 1} |x_i - x_j|$ and $u_2(x) = \sum_{i=1}^{n} \left(s_i(x) - l_i(x)\right)^2$. Let $u(x) = u_1(x)/n + u_2(x)/n^2$.

OBSERVATION 3.4. $\mathbb{E}[\Phi(X(t+1)) \mid X(t) = x] \le u(x)$.

Proof.

$$\begin{aligned}
&\mathbb{E}[\Phi(X(t+1)) \mid X(t) = x] + n\overline{x}^2 \\
&\quad = \sum_{i=1}^{n} \mathbb{E}[X_i(t+1)^2 \mid X(t) = x] \\
&\quad = \sum_{i=1}^{n} \left(\mathbb{E}[X_i(t+1) \mid X(t) = x]\right)^2 + \\
&\qquad + \sum_{i=1}^{n} \mathrm{var}(X_i(t+1) \mid X(t) = x).
\end{aligned}$$

Using Observations 3.2 and 3.3, this is at most $n\overline{x}^2 + \sum_{i=1}^{n} d_i(x)^2 + u_1(x)/n$. But

$$d_i(x) = \frac{1}{n}\sum_{j:|x_i - x_j| \le 1} (x_i - x_j) = \frac{1}{n}(s_i(x) - \ell_i(x)),$$

so the result follows. □

LEMMA 3.1.

$$\mathbb{E}[\Phi(X(t+1)) \mid X(t) = x] \le n + 2n^{1/2}\Phi(x)^{1/2}.$$

Proof. In the proof of Observation 3.4, we established that $\mathbb{E}[\Phi(X(t+1)) \mid X(t) = x] \le \sum_{i=1}^{n} d_i(x)^2 + u_1(x)/n$. Upper-bounding $u_1(x)$ and using $d_i(x) \le 1$, we have

$$\mathbb{E}[\Phi(X(t+1)) \mid X(t) = x] \le n + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - x_j|,$$

and since $|x_i - x_j| \le |x_i - \overline{x}| + |x_j - \overline{x}|$, this is at most $n + 2\sum_{i=1}^{n} |x_i - \overline{x}|$. By Cauchy-Schwarz, $\left(\sum_i |x_i - \overline{x}| \cdot 1\right)^2 \le \sum_i |x_i - \overline{x}|^2 \sum_i 1$ so

$$\mathbb{E}[\Phi(X(t+1)) \mid X(t) = x] \le n + 2(n\sum_{i=1}^{n} |x_i - \overline{x}|^2)^{1/2}.$$

□

COROLLARY 3.1.

$$\mathbb{E}[\Phi(X(t+1))] \le n + 2n^{1/2}(\mathbb{E}[\Phi(X(t))])^{1/2}.$$

Proof. Using Lemma 3.1, $\mathbb{E}[\Phi(X(t+1))] \le n + 2n^{1/2}\mathbb{E}[\Phi(x)^{1/2}]$. Now use Jensen's inequality. □

LEMMA 3.2. *Either there is a $t' < t$ s.t. $\mathbb{E}[\Phi(X(t'))] \le 18n$ or $\mathbb{E}[\Phi(X(t))] \le 9^{1-2^{-t}} n^{1-2^{-t}} \Phi(X(0))^{2^{-t}}$.*

Proof. The proof is by induction on t. The base case is $t = 0$. For the inductive step, note that $1-2^{-t} = \sum_{k=1}^{t} 2^{-k}$. Suppose that for all $t' < t$, $\mathbb{E}[\Phi(X(t'))] > 18n$ (otherwise we are finished). Then by Corollary 3.1,

$$\begin{aligned} &\mathbb{E}[\Phi(X(t))] \\ &\quad\le\quad n + 2n^{1/2}(\mathbb{E}[\Phi(X(t-1))])^{1/2} \\ &\quad\le\quad 3n^{1/2}(\mathbb{E}[\Phi(X(t-1))])^{1/2}. \end{aligned}$$

Applying the inductive hypothesis,

$$\begin{aligned} &\mathbb{E}[\Phi(X(t))] \\ &\quad\le\quad 3n^{1/2}\left(3^{2(1-2^{-(t-1)})} n^{1-2^{-(t-1)}} \Phi(X(0))^{2^{-(t-1)}}\right)^{1/2}. \end{aligned}$$

□

COROLLARY 3.2. *There is a $\tau \le \lceil \lg\lg \Phi(X(0)) \rceil$ such that $\mathbb{E}[\Phi(X(\tau))] \le 18n$.*

Proof. Take $t = \lceil \lg\lg \Phi(X(0)) \rceil$. Either there is a $\tau < t$ with $\mathbb{E}[\Phi(X(\tau))] \le 18n$ or, by the lemma,

$$\mathbb{E}[\Phi(X(t))] \le 9n\Phi(X(0))^{2^{-t}} \le 18n.$$

□

COROLLARY 3.3. *There is a $\tau \le \lceil \lg\lg \Phi(X(0)) \rceil$ such that $\Pr(\Phi(X(\tau)) > 720n) \le 1/40$.*

Proof. Use Markov's inequality. □

COROLLARY 3.4. *For all $\epsilon > 0$, provided that $n < m^{1/3}$, the expected time to reach ϵ-Nash equilibrium is $O(\log\log m)$.*

Proof. (sketch) For the asymptotic bound we can assume without loss of generality that $m > (60/\epsilon)^2$. We show that for any starting assignment $X(0)$, there exists $\tau \le \log\log(m^2)$ such that $\Pr(X(\tau)$ is $\epsilon-$Nash$) > \frac{39}{40}$. This implies the statement of the result.

Suppose assignment x is not ϵ-Nash. If $X(t) = x$ there exist resources i, j with $X_i(t) - X_j(t) > \epsilon m/n$. If $X(t+1)$ is obtained from $X(t)$ by transferring $\epsilon m/2n$ tasks from i to j, then it can be shown that $\Phi(X(t)) - \Phi(X(t+1)) \ge (\epsilon m/2n)^2$. It follows that $\Phi(X(t)) \ge (\epsilon m/2n)^2$.

From Corollary 3.3, $\Pr(\Phi(X(\tau)) < 720n) > \frac{39}{40}$, for $\tau = \log\log(\Phi(0)) = O(\log\log m)$.

An assignment $X(\tau)$ with $\Phi(X(\tau)) \le 720n$ must be ϵ-Nash if $(\epsilon m/2n)^2 > 720n$. Note that $m > n^3$ and $m > (60/\epsilon)^2$. Hence, from $\epsilon^2(60/\epsilon)^2 n^3 > 4.720.n^3$, we can deduce $\epsilon^2 m^2 > 4.720.n^3$, hence $(\epsilon m/2n)^2 > 720n$. □

Corollary 3.3 tells us that $\Phi(X(\tau))$ is likely to be $O(n)$. We want to show that $\Phi(X(t))$ quickly gets even smaller (all the way to a Nash equilibrium) and to this end, we show that $\Phi(X(t))$ is a super-martingale. By Observation 3.4, it suffices to show $u(x) \le \Phi(x)$, and we proceed with this. In the following, we shall consider the cases $|x_i - \overline{x}|$ for all $i \in [n]$ (Lemma 3.3) and $\exists i \in [n] : |x_i - \overline{x}| \ge 2.5$ (Lemma 3.4) separately.

LEMMA 3.3. *Suppose that assignment $x = (x_1, \ldots, x_n)$ satisfies $|x_i - \overline{x}| < 2.5$ for all $i \in [n]$. Then $u(x) \le \Phi(x)$.*

Proof. For all $i \in [n]$ and $j \in [n]$ we have $|x_i - x_j| \le |x_i - \overline{x}| + |x_j - \overline{x}| < 5$. Let $z = \min_i x_i$ so every $x_i \in \{z, \ldots, z+4\}$. Let $n_i = |\{j \mid x_j = z+i\}|$. Then

$$\begin{aligned} n^2\Phi(x) &= n^2 \sum_{i=1}^{n} x_i^2 - n\left(\sum_{i=1}^{n} x_i\right)^2 \\ &= n^2\left(\sum_{j=0}^{4} n_j (z+j)^2\right) - n\left(\sum_{j=0}^{4} n_j(z+j)\right)^2. \end{aligned}$$

Also, $n^2 u(x) = n u_1(x) + u_2(x)$, where

$$\begin{aligned} u_1(x) \quad = \quad & n_0(2n_2 + 3n_3 + 4n_4) + n_1(2n_3 + 3n_4) + \\ & + n_2(2n_0 + 2n_4) + n_3(3n_0 + 2n_1) + \\ & + n_4(4n_0 + 3n_1 + 2n_2) \end{aligned}$$

and

$$\begin{aligned} u_2(x) \quad = \quad & n_0 n_1^2 + n_1(n_0 - n_2)^2 + n_2(n_1 - n_3)^2 + \\ & + n_3(n_2 - n_4)^2 + n_4 n_3^2. \end{aligned}$$

Plugging in these expressions and simplifying, we get

$$\begin{aligned} & n^2\Phi(x) - n^2 u(x) = \\ & \quad 4n_0n_1n_2 + 3n_0^2n_3 + 4n_0n_1n_3 + 4n_0n_2n_3 + \\ & \quad + 4n_1n_2n_3 + 3n_0n_3^2 + 8n_0^2n_4 + 12n_0n_1n_4 + \\ & \quad + 3n_1^2n_4 + 8n_0n_2n_4 + 4n_1n_2n_4 + 12n_0n_3n_4 + \\ & \quad + 4n_1n_3n_4 + 4n_2n_3n_4 + 8n_0n_4^2 + 3n_1n_4^2, \end{aligned}$$

which is clearly non-negative since all coefficients are positive. □

LEMMA 3.4. *Suppose that assignment $x = (x_1, \ldots, x_n)$ satisfies $|x_n - \overline{x}| \ge 2.5$ and, for all $i \in [n]$, $|x_i - \overline{x}| \le |x_n - \overline{x}|$. Let $w = (w_1, \ldots, w_{n-1})$ be the assignment with $w_i = x_i$ for $i \in [n-1]$. Then $\Phi(x) - u(x) \ge \Phi(w) - u(w)$, i.e., the potential drop for x is at least as big as that for w.*

Proof. Let $k = |x_n - \overline{x}|$. We will show

(1) $\Phi(x) - \Phi(w) \ge k^2$, and

(2) $u(x) - u(w) \le 2k+1$.

Then

$$\Phi(x) - u(x) - (\Phi(w) - u(w)) \ge k^2 - (2k+1),$$

which is non-negative since $k \ge 2.5 \ge 1+\sqrt{2}$.

First, we prove (1). Let $f(z) = \sum_{i=1}^{n-1}(x_i - z)^2$. Note that the derivative of $f(z)$ is

$$f'(z) = 2(n-1)z - 2\sum_{i=1}^{n-1} x_i = 2(n-1)z - 2(n-1)\overline{w}.$$

Furthermore the second derivative is $f''(z) = 2(n-1) \ge 0$. Thus, $f(z)$ is minimized at $z = \overline{w}$. Now note that

$$\Phi(x) - \Phi(w) = k^2 + \sum_{i=1}^{n-1}(x_i - \overline{x})^2 - \sum_{i=1}^{n-1}(x_i - \overline{w})^2 \ge k^2.$$

Now we finish the proof by proving (2). Assume first that $x_n = \overline{x} + k$. Then

$$\begin{aligned} & u_1(x) - u_1(w) \\ &= 2\sum_{i\in[n]:|x_i - x_n|>1} |x_i - x_n| \le 2\sum_{i=1}^{n}|x_i - x_n| \\ &= 2\sum_{i=1}^{n}(x_n - x_i) = 2nk. \end{aligned}$$

Let $z_j = |\{\ell \mid x_\ell = j\}|$. Clearly $z_j = 0$ for $j > x_n$. Let $M = \lceil x_n - 2k \rceil$ For $\ell \in [n]$ we have $x_\ell \ge \overline{x} - k = x_n - 2k$ so $z_j = 0$ for $j < M$. Now $u_2(x) = \sum_{j=M}^{x_n} z_j(z_{j-1} - z_{j+1})^2$. The representation of w in terms of z_js is the same as the representation of x except that z_{x_n} is reduced by one. Therefore,

$$\begin{aligned} & u_2(x) - u_2(w) \\ &= z_{x_n-1}\left((z_{x_n-2} - z_{x_n})^2 - (z_{x_n-2} - z_{x_n} + 1)^2\right) + \\ &\quad + (z_{x_n-1} - z_{x_n+1})^2 \\ &= z_{x_n-1}(-2z_{x_n-2} + 2z_{x_n} + z_{x_n-1} - 1) \\ &\le z_{x_n-1}(2z_{x_n} + z_{x_n-1}). \end{aligned}$$

But since $z_{x_n} \le n - z_{x_n-1}$, the upper bound on the right-hand side is at most

$$z_{x_n-1}(2n - 2z_{x_n-1} + z_{x_n-1}) = 2z_{x_n-1}(n - z_{x_n-1}/2),$$

which is at most n^2 since the right-hand side is maximized at $z_{x_n-1} = n$. To finish the proof of (2), use the definition of u to deduce that

$$u(x) - u(w) \le \frac{u_1(x) - u_1(w)}{n} + \frac{u_2(x) - u_2(w)}{n^2}.$$

The proof of (2) when $x_n = \overline{x} - k$ is similar. □

Letting $k = |x_n - \overline{x}|$, the idea is to show that $\Phi(x) - \Phi(w) \ge k^2$ and $u(x) - u(w) \le 2k+1$, because then $\Phi(x) - u(x) - (\Phi(w) - u(w)) \ge k^2 - (2k+1)$, which is non-negative since $k \ge 2.5 \ge 1+\sqrt{2}$.

COROLLARY 3.5. *For any assignment $x = (x_1, \ldots, x_n)$, $\Phi(x) - u(x) \ge 0$.*

Proof. The proof is by induction on n. The base case, $n=1$, follows from Lemma 3.3. Suppose $n > 1$. Neither $\Phi(x)$ nor $u(x)$ depends upon the order of the components in x, so assume without loss of generality that $|x_i - \overline{x}| \le |x_n - \overline{x}|$ for all i. If $|x_n - \overline{x}| < 2.5$ then apply Lemma 3.3. Otherwise, use Lemma 3.4 to find an assignment $w = (w_1, \ldots, w_{n-1})$ such that $\Phi(x) - u(x) \ge \Phi(w) - u(w)$. By the inductive hypothesis, $\Phi(w) - u(w) \ge 0$. □

Together, Observation 3.4 and Corollary 3.5 tell us that $\mathbb{E}[\Phi(X(t+1)) \mid X(t) = x] \le \Phi(x)$. The next lemma will be used to give a lower bound on the variance of the process. Let $V = 0.4n^{-2}$.

LEMMA 3.5. *Suppose that $X(t) = x$ and that x is not a Nash equilibrium. Then*

$$\Pr(\Phi(X(t+1) \ne \Phi(x) \mid X(t) = x) \ge V.$$

Proof. Choose s and ℓ such that for all $i \in [n]$, $x_s \le x_i \le x_\ell$. Since x is not a Nash equilibrium, $x_\ell > x_s + 1$. Assuming $X(t) = x$, consider the following experiment for choosing $X(t+1)$.

Independently, for every $i \ne \ell$, choose $(Y_{i,1}(x), \ldots, Y_{i,n}(x))$ from the multinomial distribution described in Section 2. For every task $b \in x_\ell$, let $z_b = 1$ with probability $1 - x_s/x_\ell$ and $z_b = 0$ otherwise. Let x_ℓ^+ be the number of tasks b with $z_b = 1$ and let x_ℓ^- be the number of tasks b with $z_b = 0$. Choose $(Y_{\ell,1}^+(x), \ldots, Y_{\ell,n}^+(x))$ from a multinomial distribution with the constraint $\sum_{j=1}^{n} Y_{\ell,j}^+(x) = x_\ell^+$ and probabilities given by

$$p_{\ell,j}^+(x) = \begin{cases} \frac{1}{n} & \text{if } j = s, \\ \frac{1}{n}\left(1 - \frac{x_j}{x_\ell}\right) & \text{if } j \ne s \text{ and } x_\ell > x_j + 1, \\ 0 & \text{if } \ell \ne j \text{ but } x_\ell \le x_j + 1, \\ 1 - \sum_{j\ne\ell} p_{\ell,j}(x) & \text{if } \ell = j. \end{cases}$$

Similarly, choose $(Y_{\ell,1}^-(x), \ldots, Y_{\ell,n}^-(x))$ from a multinomial distribution with the constraint $\sum_{j=1}^{n} Y_{\ell,j}^-(x) = x_\ell^-$ and probabilities given by

$$p_{\ell,j}^-(x) = \begin{cases} 0 & \text{if } j = s, \\ \frac{1}{n}\left(1 - \frac{x_j}{x_\ell}\right) & \text{if } j \ne s \text{ and } x_\ell > x_j + 1, \\ 0 & \text{if } \ell \ne j \text{ but } x_\ell \le x_j + 1, \\ 1 - \sum_{j\ne\ell} p_{\ell,j}(x) & \text{if } \ell = j. \end{cases}$$

For all j, let $Y_{\ell,j}(x) = Y^+_{\ell,j}(x) + Y^-\ell, j(x)$. The reader may verify that $X(t+1)$ is chosen from the correct distribution.

Now, consider the transition from x to $X(t+1)$. Condition on the choice for $(Y_{i,1}(x), \ldots, Y_{i,n}(x))$ for all $i \neq \ell$. Suppose $x^+_\ell > 2$. Condition on the choice for $(Y^-_{\ell,1}(x), \ldots, Y^-_{\ell,n}(x))$. Flip a coin for each of the first $x^+_b - 2$ tasks with $z_b = 1$ to determine which of $Y^+_{\ell,1}(x), \ldots, Y^+_{\ell,n}(x)$ the task contributes to. Condition on these choices. Consider the following options:

(1) Let x_1 be the resulting value of $X(t+1)$ when we add both of the last two tasks to $Y^+_{\ell,\ell}(x)$.

(2) Let x_2 be the resulting value of $X(t+1)$ when we add one of the last two tasks to $Y^+_{\ell,\ell}(x)$ and the other to $Y^+_{\ell,s}(x)$.

(3) Let x_3 be the resulting value of $X(t+1)$ when we add both of the last two tasks to $Y^+_{s,s}(x)$.

Note that, given the conditioning, each of these choices occurs with probability at least n^{-2}. Also, $\Phi(x_1)$, $\Phi(x_2)$ and $\Phi(x_3)$ are not all the same. Thus, $\Pr(\Phi(X(t+1) \neq \Phi(x) \mid X(t) = x, x^+_\ell > 2) \geq n^{-2}$. Also,

$$\Pr(x^+_\ell > 2) = 1 - \left(\frac{x_s}{x_\ell}\right)^{x_\ell} - x_\ell \left(1 - \frac{x_s}{x_\ell}\right)\left(\frac{x_s}{x_\ell}\right)^{x_\ell - 1}.$$

Since the derivative with respect to x_s is negative, this is minimized by taking x_s as large as possible, namely $x_\ell - 2$, so $\Pr(x^+_\ell > 2) \geq 1 - 7e^{-2} \geq 0.4$, and the result follows. □

In order to finish our proof of convergence, we need the following observation about $\Phi(x)$.

OBSERVATION 3.5. *For any assignment x, $\Phi(x) \leq m^2$. Let $r = m \bmod n$. Then $\Phi(x) \geq r(1 - r/n)$, with equality if and only if x is a Nash equilibrium.*

Proof. Suppose that in assignment x there are resources i and j such that $x_i - x_j \geq 2$. Let x' be the assignment constructed from x by transferring a task from resource i to resource j. Then

$$\begin{aligned}
&\Phi(x) - \Phi(x') \\
&\quad= x_i^2 - {x'_i}^2 + x_j^2 - {x'_j}^2 \\
&\quad= x_i^2 - (x_i^2 - 2x_i + 1) + x_j^2 - (x_j^2 + 2x_j + 1) \\
&\quad= 2x_i - 2x_j - 2 = 2(x_i - x_j) - 2 > 0
\end{aligned}$$

Now suppose that, in some assignment x', resources i and j satisfy $x'_i \geq x'_j > 0$. Let x be the assignment constructed from x' by transferring a task from resource j to resource i. Since $(x'_i + 1) - (x'_j - 1) \geq 2$, the above argument gives $\Phi(x) > \Phi(x')$. We conclude that an assignment x with maximum $\Phi(x)$ must have all of the tasks in the same resource, with $\Phi(x) = m^2$.

Furthermore, an assignment x with minimum $\Phi(x)$ must have $|x_i - x_j| \leq 1$ for all i, j. In this case there must be r resources with loads of $q+1$ and $n-r$ resources with loads of q, where $m = qn + r$. So

$$\begin{aligned}
\Phi(x) &= r(q + 1 - \bar{x})^2 + (n-r)(q - \bar{x})^2 \\
&= r\left(1 - \frac{r}{n}\right)^2 + (n-r)\left(\frac{r}{n}\right)^2 \\
&= r\left(1 - \frac{r}{n}\right).
\end{aligned}$$

Note that x is a Nash assignment if and only if $|x_i - x_j| \leq 1$ for all i and j. □

Combining Observation 3.5 and Corollary 3.3 we find that there is a $\tau \leq \lceil \lg\lg m^2 \rceil$ such that $\Pr(\Phi(X(\tau)) > 720n) \leq 1/40$. Let $B = 7200n + \left\lceil \frac{m^2}{n} \right\rceil - \frac{m^2}{n}$. Let $t' = \tau + \lceil 10B^2/V \rceil$.

LEMMA 3.6. *Given any starting state $X(0) = x$, the probability that $X(t')$ is a Nash equilibrium is at least $3/4$.*

Proof. The proof is based on a standard martingale argument, see [11]. Suppose that $\Phi(X(\tau)) \leq 720n$. Let $W_t = \Phi(X(t+\tau)) - r(1 - r/n)$ and let $D_t = \min(W_t, B)$. Note that $D_0 \leq 720n$. Together, Observation 3.4 and Corollary 3.5 tell us that W_t is a supermartingale. This implies that D_t is also a supermartingale since

$$\begin{aligned}
&\mathbb{E}[D_{t+1} \mid D_t = x < B] \\
&\quad\leq \mathbb{E}[W_{t+1} \mid W_t = x < B] \leq W_t = D_t,
\end{aligned}$$

and

$$\mathbb{E}[D_{t+1} \mid D_t = B] \leq B = D_t.$$

Together, Lemma 3.5 and Observation 3.5 tell us that if $x > 0$, $\Pr(W_{t+1} \neq W_t \mid W_t = x) \geq V$. Thus, if $0 < x < B$,

$$\begin{aligned}
&\Pr(D_{t+1} \neq D_t \mid D_t = x) \\
&\quad= \Pr(\min(W_{t+1}, B) \neq W_t \mid W_t = x) \\
&\quad\geq \Pr(W_{t+1} \neq W_t \wedge B \neq W_t \mid W_t = x) \\
&\quad= \Pr(W_{t+1} \neq W_t \mid W_t = x) \geq V.
\end{aligned}$$

Since $D_{t+1} - D_t$ is an integer, $\mathbb{E}[(D_{t+1} - D_t)^2 \mid 0 < D_t < B] \geq V$. Let T be the first time at which either (a) $D_t = 0$ (i.e., $X(t+\tau)$ is a Nash equilibrium), or (b) $D_t = B$. Note that T is a stopping time. Define $Z_t = (B - D_t)^2 - Vt$, and observe that $Z_{t \wedge T}$ is a sub-martingale, where $t \wedge T$ denotes the minimum of t and T. Let p be the probability that (a) occurs. By the optional stopping theorem $\mathbb{E}[D_T] \leq D_0$, so

$(1-p)B = \mathbb{E}[D_T] \le D_0$ and $p \ge 1 - D_0/B \ge \frac{9}{10}$. Also, by the optional stopping theorem

$$\begin{aligned} & pB^2 - V\mathbb{E}[T] \\ = \quad & \mathbb{E}[(B-D_T)^2] - V\mathbb{E}[T] = \mathbb{E}[Z_T] \ge Z_0 \\ = \quad & (B-D_0)^2 > 0, \end{aligned}$$

so $\mathbb{E}[T] \le pB^2/V$. Conditioning on (a) occurring, it follows that $\mathbb{E}[T \mid D_T = 0] \le B^2/V$. Hence $\Pr(T > 10B^2/V \mid D_T = 0) \le \frac{1}{10}$. So, if we now run for $10B^2/V$ steps, then the probability that we do not reach a Nash equilibrium is at most $\frac{1}{40} + 2 \times \frac{1}{10} < 1/4$. □

Now we can give the proof of Theorem 3.1.

Proof. Subdivide time into intervals of t' steps. The probability that the process has not reached a Nash equilibrium before the $(j+1)$st interval is at most $(1/4)^{-j}$. □

4 Lower Bounds

THEOREM 4.1. *Let $X(t)$ be the process in Figure 1 with $m = n$. Let $X(0)$ be the assignment given by $X(0) = (n, 0, \ldots, 0)$. Let T be the first time at which $X(t)$ is a Nash equilibrium. Then $E[T] = \exp(\Theta(\sqrt{n}))$.*

Proof. For an assignment x, let $n_0(x)$ denote the number of resources i with $x_i = 0$. Thus, $n_0(X(0)) = n-1$. The (unique) Nash equilibrium x assigns one task to each resource, so $n_0(x) = 0$. Let $k = \lfloor \sqrt{n} \rfloor$. We will show that for any assignment x with $n_0(x) \ge k$,

$$\Pr(n_0(X(t)) < k \mid X(t-1) = x) \le \exp(-\Theta(\sqrt{n})).$$

This implies the result.

Suppose $X(t-1) = x$ with $n_0(x) \ge k$. For convenience, let n_0 denote $n_0(x)$. Let x' denote $X(t)$, and let n_0' denote $n_0(x')$. We will use the phrase "with very high probability" to mean with probability at least $1 - \exp(-\Theta(\sqrt{n}))$. We will show that, with very high probability, $n_0' \ge k$.

During the course of the proof, we will assume, where necessary, that n is sufficiently large. This is without loss of generality given the Θ notation in the statement of the result.

Case 1.$n_0 > 8k$.

Consider the protocol in Figure 1. Let $U = \{b \mid x_{j_b} = 0\}$. $E[|U|] = n_0$, so by a Chernoff bound, $|U| \ge \lceil n_0/2 \rceil + \lceil 3n_0/8 \rceil$ with very high probability. Suppose this is the case. Partition U into U_1 and U_2 with $|U_1| = \lceil n_0/2 \rceil$. Let $W = \cup_{b \in U_1} \{j_b\}$. First, suppose $|W| \le \frac{3}{8} n_0$. In that case

$$\begin{aligned} |\{j \mid x_j' > 0\}| &\le n - |U_1| + \frac{3}{8} n_0 \\ &= n - \lceil n_0/2 \rceil + \frac{3}{8} n_0 \le n - k, \end{aligned}$$

so $n_0' \ge k$. Otherwise, let $U' = \{b \in U_2 \mid j_b \in W\}$.

$$E[|U'|] = |U_2| \frac{|W|}{n_0} \ge \frac{9}{64} n_0 > \frac{9}{8} k,$$

so by a Chernoff bound, with very high probability, $|U'| \ge k$, which implies $n_0' \ge k$.

Case 2.$k \le n_0 \le 8k$.

Consider the protocol in Figure 1. Let L be the set of "loners" defined by $L = \{i \mid x_i = 1\}$ and let $\ell = |L|$. The number of resources i with $x_i > 1$ is $n - n_0 - \ell$ and this is at most half as many as the number of tasks assigned to such resources (which is $n - \ell$), so $\ell \ge n - 2n_0$. Let $U = \{b \mid i_b \in L \text{ and } x_{j_b} = 0\}$. $E[|U|] = \ell \frac{n_0}{n} \ge \frac{(n-2n_0)n_0}{n}$, so by a Chernoff bound, $|U| \ge 2\lceil \frac{1}{4} \ell \frac{n_0}{n} \rceil$ with very high probability. Suppose this is the case. Let U_1 and U_2 be disjoint subsets of U of size $\lceil \frac{1}{4} \ell \frac{n_0}{n} \rceil$. Order tasks in U arbitrarily and let $S = \{b \in U \mid \text{for some } b' \in U \text{ with } b' < b, j_{b'} = j_b.\}$. (Note that $|S|$ does not depend on the ordering.) Let $W = \cup_{b \in U_1} \{j_b\}$.

Note that if $|W| \le \frac{1}{5} \ell \frac{n_0}{n}$ then $|S| \ge \frac{1}{20} \ell \frac{n_0}{n} > \frac{n_0}{40} \left(\frac{\ell}{n}\right)^2$. Otherwise, let $U' = \{b \in U_2 \mid j_b \in W\}$.

$$E[|U'|] = |U_2| \frac{|W|}{n_0} \ge \frac{n_0}{20} \left(\frac{\ell}{n}\right)^2,$$

so, with very high probability, $|U'| \ge \frac{n_0}{40} \left(\frac{\ell}{n}\right)^2$, so $|S| \ge \frac{n_0}{40} \left(\frac{\ell}{n}\right)^2$.

Suppose then that $|S| \ge \frac{n_0}{40} \left(\frac{\ell}{n}\right)^2$. Assuming that n is sufficiently large $|S| \ge k/41$. Let $B_0 = \cup_{b \in U} \{j_b\}$ and $B_1 = \cup_{b \in L - U} \{i_b\}$. Note that every resource in $B_0 \cup B_1$ is used in x' for some task $b \in L$. Thus, $|B_0 \cup B_1| \le \ell - |S|$. Let $R = \{i \mid x_i = 0\} \cup L - B_0 - B_1$. Then $|R| \ge n_0 + \ell - (\ell - |S|) \ge n_0 + |S| \ge (1 + \frac{1}{41})k$.

Let $T = \{b \mid i_b \notin L, j_b \in R\}$. $E[T] = (n - \ell) \frac{|R|}{n}$ and

$$\begin{aligned} & \Pr\left(T \ge \frac{|R|}{100}\right) \\ \le \quad & \binom{n-\ell}{\frac{|R|}{100}} \left(\frac{|R|}{n}\right)^{|R|/100} \le \left(\frac{2n_0 e 100}{n}\right)^{|R|/100}, \end{aligned}$$

so with very high probability, $T < |R|/100$. In that case, $n_0' \ge |R|(1 - \frac{1}{100}) \ge k$. □

The following theorem provides a lower bound on the expected convergence time regardless of which of the two protocols is being used.

THEOREM 4.2. *Suppose that m is even. Let $X(t)$ be the process in Figure 2 with $n = 2$. Let $X(0)$ be the assignment given by $X(0) = (m, 0)$. Let T be the first time at which $X(t)$ is a Nash equilibrium. Then $E[T] = \Omega(\log \log m)$. The same result holds for the process in Figure 1.*

Proof. Note that both protocols have the same behaviour since m is even and, therefore, the situation $x_1 = x_2 + 1$ cannot arise. For concreteness, focus on the protocol in Figure 2.

Let $y(x) = \max_i x_i - m/2$ and let $y_t = y(X(t))$ so $y_0 = m/2$ and, for a Nash equilibrium x, $y(x) = 0$. We will show that for any assignment x, $\Pr(y_{t+1} > y(x)^{1/10} \mid X(t) = x) \geq 1 - y_t^{-1/4}$.

Suppose $X(t) = x$ is an assignment with $x_1 \geq x_2$. As we have seen in Section 2, $Y_{1,2}(x)$ (the number of migrations from resource 1 to resource 2 in the round) is a binomial random variable

$$B\left(x_1, \frac{1}{2}\left(1 - \frac{x_2}{x_1}\right)\right) = B\left(\frac{m}{2} + y_t, \frac{2y_t}{m + 2y_t}\right).$$

In general, let T_t be the number of migrations from the most-loaded resource in $X(t)$ to the least-loaded resource and note that the distribution of T_t is $B\left(\frac{m}{2} + y_t, \frac{2y_t}{m+2y_t}\right)$ with mean y_t. If $T_t = y_t + \ell$ or $T_t = y_t - \ell$ then $y_{t+1} = \ell$. Thus $\Pr(y_{t+1} > y_t^{1/10}) = \Pr(|T_t - E[T_t]| > y_t^{1/10})$.

Note that the mode of a binomial distribution is one or both of the integers closest to the expectation, and the distribution is monotonically decreasing as you move away from the mode.

$$\begin{aligned}
&\Pr(T_t = y_t) \\
&\quad= \binom{\frac{1}{2}m + y_t}{y_t}\left(\frac{2y_t}{m + 2y_t}\right)^{y_t}\left(\frac{m}{m + 2y_t}\right)^{\frac{1}{2}m}
\end{aligned}$$

$$\begin{aligned}
&\Pr(T_t = y_t + j) \\
&\quad= \binom{\frac{1}{2}m + y_t}{y_t + j}\left(\frac{2y_t}{m + 2y_t}\right)^{y_t + j}\left(\frac{m}{m + 2y_t}\right)^{\frac{1}{2}m - j}
\end{aligned}$$

Suppose $j > 0$.

$$\begin{aligned}
&\frac{\Pr(T_t = y_t + j)}{\Pr(T_t = y_t)} \\
&\quad= \left(\frac{2y_t}{m + 2y_t}\right)^{j}\left(\frac{m}{m + 2y_t}\right)^{-j} \times \\
&\qquad \times \left(\frac{y_t!(\frac{1}{2}m)!}{(y_t + j)!(\frac{1}{2}m + y_t - (y_t + j))!}\right) \\
&\quad= \left(\frac{2y_t}{m}\right)^{j}\left(\prod_{\ell=1}^{j}\frac{\frac{1}{2}m + 1 - \ell}{y_t + \ell}\right) \\
&\quad= \left(\frac{2y_t}{m}\right)^{j}\left(\prod_{\ell=1}^{j}\frac{m + 2 - 2\ell}{2y_t + 2\ell}\right) \\
&\quad> \left(\frac{2y_t}{m}\right)^{j}\left(\prod_{\ell=1}^{j}\frac{m - 2j}{2y_t + 2j}\right) \\
&\quad= \left[\left(\frac{2y_t}{m}\right)\left(\frac{m - 2j}{2y_t + 2j}\right)\right]^{j}.
\end{aligned}$$

Similarly, for $j < 0$,

$$\begin{aligned}
&\frac{\Pr(T_t = y_t + j)}{\Pr(T_t = y_t)} \\
&\quad= \left(\frac{2y_t}{m}\right)^{j}\left(\prod_{\ell=1}^{|j|}\frac{y_t + 1 - \ell}{\frac{1}{2}m + \ell}\right) \\
&\quad= \left(\frac{m}{2y_t}\right)^{|j|}\left(\prod_{\ell=1}^{|j|}\frac{2y_t + 2 - 2\ell}{m + 2\ell}\right) \\
&\quad> \left(\frac{m}{2y_t}\right)^{|j|}\left(\frac{2y_t - 2|j|}{m + 2|j|}\right)^{|j|} \\
&\quad= \left[\left(\frac{m}{2y_t}\right)\left(\frac{2y_t - 2|j|}{m + 2|j|}\right)\right]^{|j|} \\
&\quad= \left[\left(\frac{2y_t}{m}\right)\left(\frac{m - 2j}{2y_t + 2j}\right)\right]^{j}.
\end{aligned}$$

So for all j,

$$\begin{aligned}
&\frac{\Pr(T_t = y_t + j)}{\Pr(T_t = y_t)} \\
&\quad> \left[\left(\frac{2y_t}{m}\right)\left(\frac{m - 2j}{2y_t + 2j}\right)\right]^{j} = \left[\left(\frac{y_t}{y_t + j}\right)\left(\frac{m - 2j}{m}\right)\right]^{j}.
\end{aligned}$$

So, for all j with $|j| \leq y_t^{1/4}$, where $y_t^{1/4}$ is the positive fourth root of y_t, this is at least

$$\begin{aligned}
&\left(\frac{y_t}{y_t + y_t^{1/4}}\right)^{y_t^{1/4}}\left(\frac{m - 2y_t^{1/4}}{m}\right)^{y_t^{1/4}} \\
&\quad\geq \left(\frac{y_t}{y_t + y_t^{1/4}}\right)^{y_t^{1/4}}\left(\frac{2y_t - 2y_t^{1/4}}{2y_t}\right)^{y_t^{1/4}} \\
&\quad= \left(\frac{y_t - y_t^{1/4}}{y_t + y_t^{1/4}}\right)^{y_t^{1/4}} = \left(\frac{y_t + y_t^{1/4} - 2y_t^{1/4}}{y_t + y_t^{1/4}}\right)^{y_t^{1/4}} \\
&\quad= \left(1 - \frac{2y_t^{1/4}}{y_t + y_t^{1/4}}\right)^{y_t^{1/4}} \geq \left(1 - \frac{2y_t^{1/4}}{y_t}\right)^{y_t^{1/4}} \\
&\quad= \left(1 - 2y_t^{-3/4}\right)^{y_t^{1/4}} \geq 1 - 2y_t^{-3/4}y_t^{1/4} \\
&\quad= 1 - 2y_t^{-1/2} \geq \frac{1}{2}
\end{aligned}$$

where the last inequality just requires $y_t \geq 16$.

For $|j| \leq y_t^{1/4}$, $\Pr(T_t = y_t + j) \geq \frac{1}{2}\Pr(T_t = y_t)$, hence $\Pr(T_t = y_t) \leq 2/(1 + 2y_t^{1/4})$.

Since $\Pr(T_t = y_t + j) \leq \Pr(T_t = y_t)$, it follows that

$$\begin{aligned}
&\Pr(T_t \in [y_t - y_t^{1/10}, y_t + y_t^{1/10}]) \\
&\quad\leq (2y_t^{1/10} + 1)\Pr(T_t = y_t) < 3y_t^{-3/20}.
\end{aligned}$$

We say that the transition from y_t to y_{t+1} is a "fast round" if $y_{t+1} \leq y_t^{1/10}$ (equivalently, it is a fast round if

$T_t \in [y_t - y_t^{1/10}, y_t + y_t^{1/10}]$). Otherwise it is a slow round. Recall that $y_0 = m/2$. Let

$$r = \left\lfloor \log_{10}\left(\frac{\log(y_0)}{\log(12^{20/3})}\right)\right\rfloor.$$

If the first j rounds are slow then $y_j \geq y_0^{10^{-j}}$. If $j \leq r$ then $y_0^{10^{-j}} \geq 12^{20/3}$ so the probability that the transition from y_j to y_{j+1} is the first fast round is at most $3\left(y_0^{10^{-j}}\right)^{-3/20} \leq 1/4$.

Also, if $j < r$ then these probabilities increase geometrically so that the ratio of the probability that the transition to y_{j+1} is the first fast round and the probability that the transition to y_j is the first fast round is

$$\begin{aligned}\frac{3\left(y_0^{10^{-(j+1)}}\right)^{-3/20}}{3\left(y_0^{10^{-j}}\right)^{-3/20}} &= \left(y_0^{10^{-j}-10^{-(j+1)}}\right)^{3/20} \\ &\geq \left(y_0^{10^{-(j+1)}}\right)^{3/20} \geq 12 \geq 2,\end{aligned}$$

so $\sum_{j=0}^{r-1} \Pr(\text{trans. from } y_j \text{ to } y_{j+1} \text{ is the 1st fast round}) \leq 2 \cdot 1/4 = \frac{1}{2}$. Therefore, with probability at least $1/2$, all of the first r rounds are slow. In this case, $\arg\min_t(y_t \leq 16) = \Omega(\log\log(m))$, which proves the theorem. $\square$

We also have the following observation.

OBSERVATION 4.1. *Let $X(t)$ be the process in Figure 2 with $m = n$. Let $X(0)$ be the assignment given by $X(0) = (2, 0, 1, \ldots, 1)$. Let T be the first time at which $X(t)$ is a Nash equilibrium. Then $E[T] = \Omega(n)$.*

The observation follows from the fact that the state does not change until one of the two tasks assigned to the first resource chooses the second resource.

5 Conclusions

We have analyzed a very simple, strongly distributed rerouting protocol for m tasks on n resources. We have proved an upper bound of $(\log\log m + n^4)$ on the expected convergence time (convergence to a Nash equilibrium), and for $m > n^3$ an upper bound of $O(\log\log m)$ on approximate convergence time. Our lower bound of $\Omega(\log\log m + n)$ matches the upper bound as function of m. We have also shown an exponential lower bound on the convergence time for a related protocol that allows "neutral moves".

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