Introduction to total search problems

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Total search problems (in NP)

in P

$$\forall x \exists y \phi(x, y)$$

Total search problems (in NP)

Radon's theorem, Tverberg's theorem, colourful Carathéodory theorem, Lagrange's four-squares theorem, existence of solutions to parity games, mean payoff games, discounted payoff games, simple stochastic games, P-matrix linear complementarity problems, Banach's fixpoint theorem

Existence of a stable configuration in a Hopfield network, pure NE of congestion games

Existence of mixed Nash equilibria, Arrow-Debreu market equilibra, envy-free partition of a line into connected pieces, Hairy Ball theorem, Sperner's Lemma, Brouwer's fixpoint theorem, Kakutani's fixpoint theorem, fundamental theorem of arithmetic, Ramsey's theorem

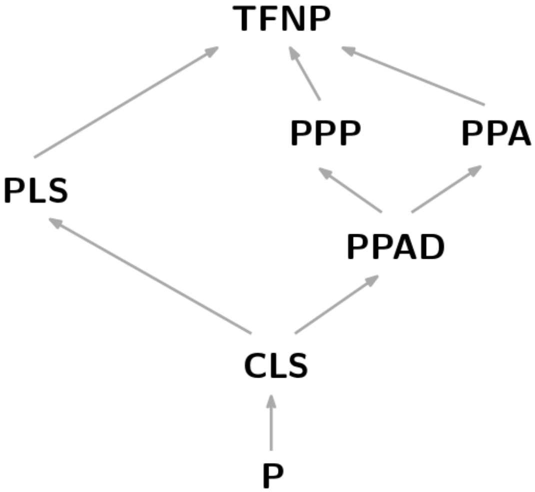
$$\forall x \exists y \phi(x, y)$$

Tucker's Lemma, Borsuk-Ulam theorem, Ham sandwich theorem, spicy chicken theorem[1], Smith's theorem, Kneser-Lovász theorem, existence of a second "room partitioning" in a triangulated surface, Chévalley-Warning theorem

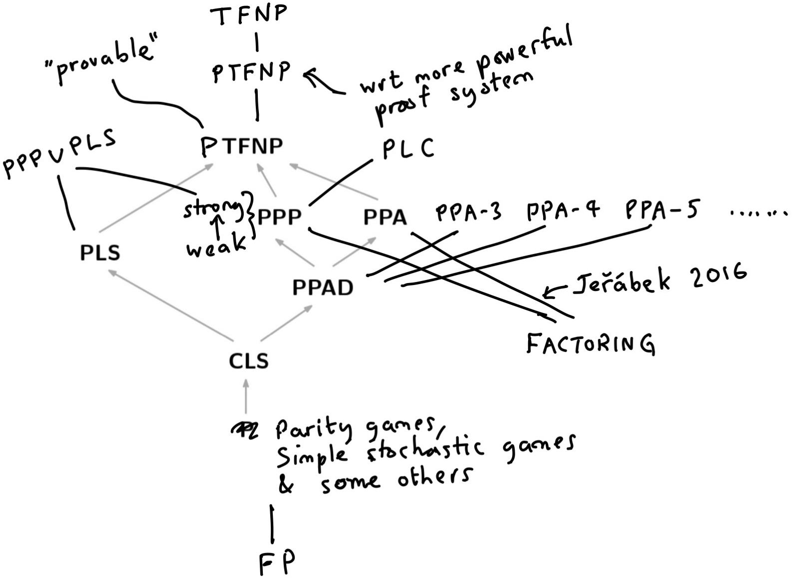
Goldbach's conjecture, Legendre's conjecture

[1] this is a real theorem

TFNP syntactic subclasses



TFNP syntactic subclasses



TFNP problems are "NP-intermediate"

not NP-hard unless NP=co-NP

(Megiddo & Papadimitriou '91) (Proof: consider what happens when you try to reduce SAT to (say) NASH, or indeed any other "total" search problem)

☞ 2 reasons to study its computational complexity:

- inherent interest of the problem, applicability of algorithms
- potential to shed light on P versus NP.

TFNP: problems like NASH for which all instances have easy-to-check solutions; not NP-complete.
Are there any other hard[1] **TFNP** problems?

[1] seemingly hard

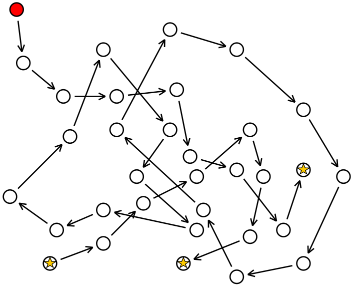
Inefficient proofs of existence

We like computationally inefficient proofs of existence
Syntactic TFNP subclasses correspond to non-(efficiently)-constructive existence proof principles.

General note: sometimes, some work needed to convert a theorem into a computational total search problem

"Nash equilibrium computation belongs to PPAD" highlights the existence principle used to proof existence. *Completeness* for PPAD indicates you *need* to invoke that principle.

"**P**olynomial **P**arity **A**rgument on a **D**irected graph", turns out to capture Brouwer's fixpoint theorem (or at any rate, approximate fixpoints)

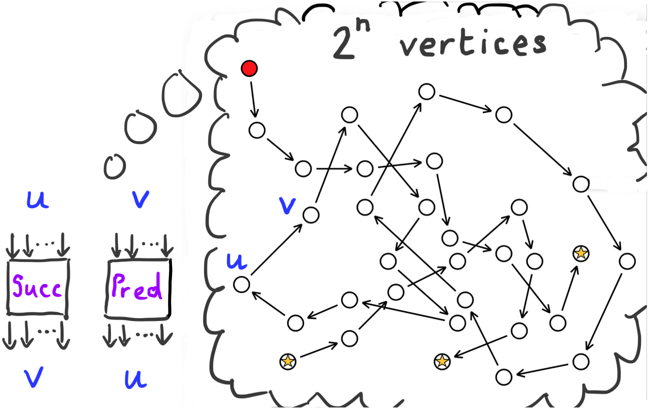


PPAD-complete problem: apply this principle as generally as possible...

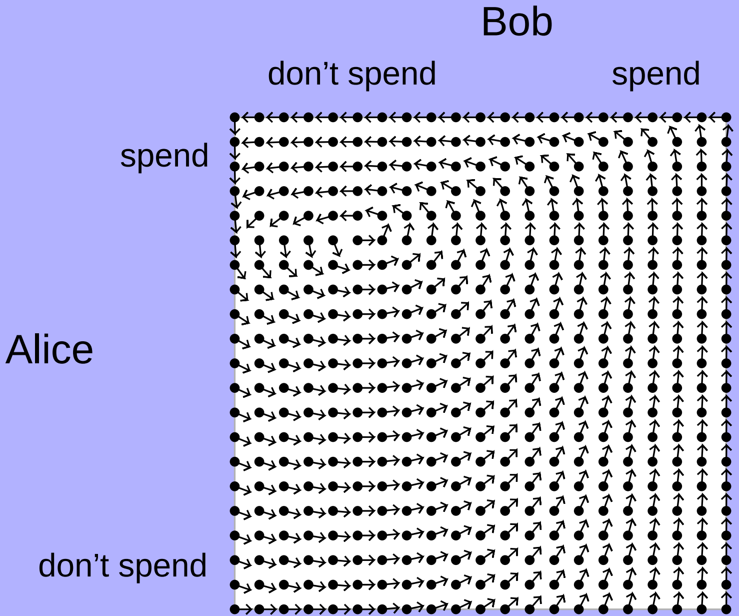
Papadimitriou (1994): On the complexity of the parity argument and other inefficient proofs of existence

The END-OF-LINE problem

Boolean circuits *Succ*, *Pred*, n inputs and n outputs.
Directed graph G on $\{0, 1, 2, \ldots, 2^n - 1\}$ — edge (u, v) is present iff $Succ(u) = v$ and $Pred(v) = u$.
Stipulate that 0 has an outgoing edge but no incoming edge.
Problem is to find any other degree-1 vertex.



“Incentive direction” of players in some game



We are reducing the search for NE to search for a Brouwer fixpoint...

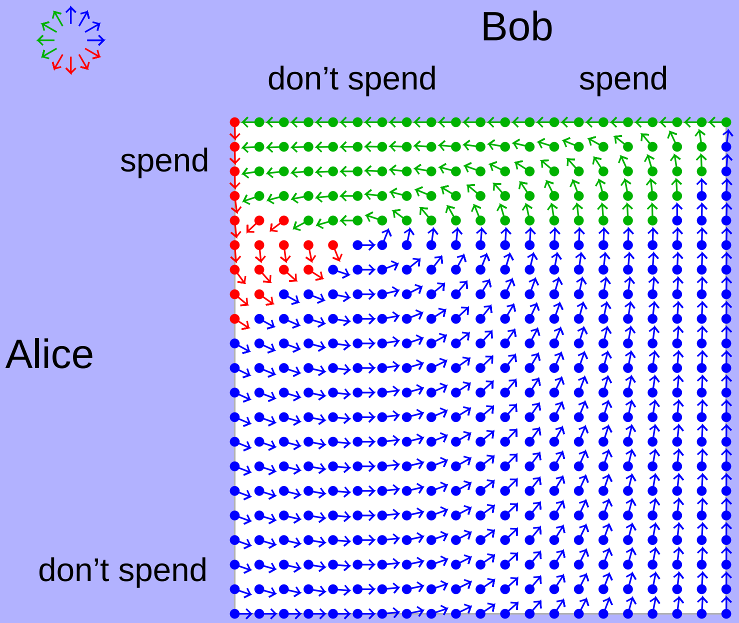
Brouwer's fixpoint theorem

continuous functions from a compact domain to itself, have fixpoints.

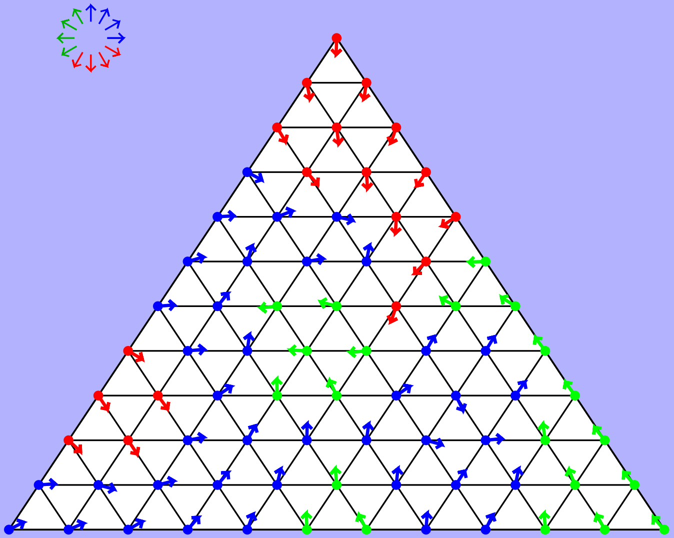
proof. construct approximate fixpoints (in a computationally inefficient manner)
...in a way that reduces computation of approx fixpoints to search on large graphs...

L.E.J. Brouwer (1881-1966)

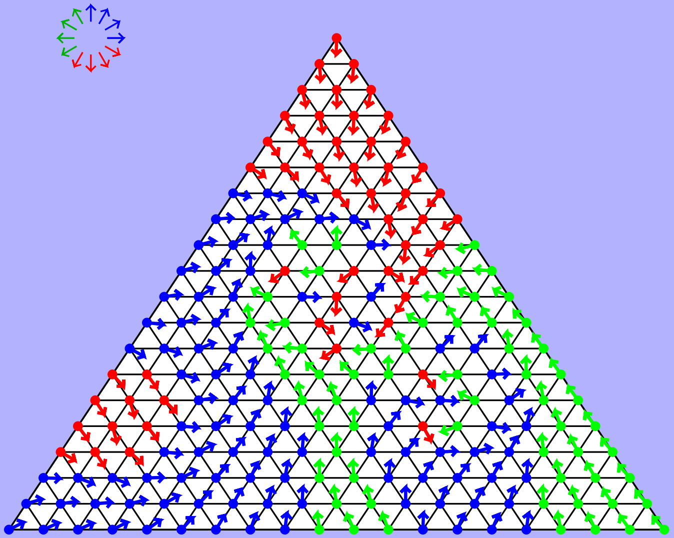
"Incentive direction", colour-coded



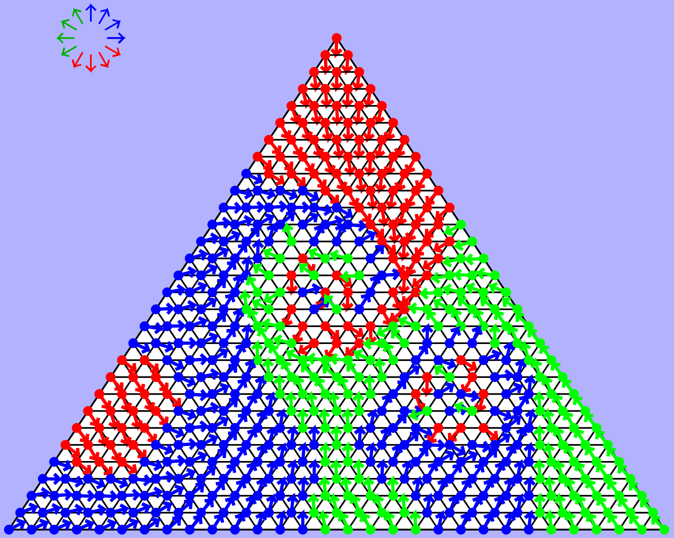
Now, pretend this triangle is high-dimension domain



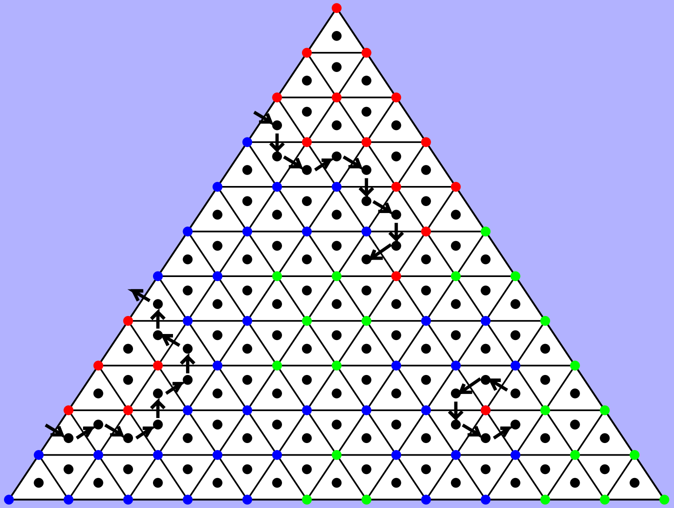
Search for "trichromatic triangles" at higher resolution...



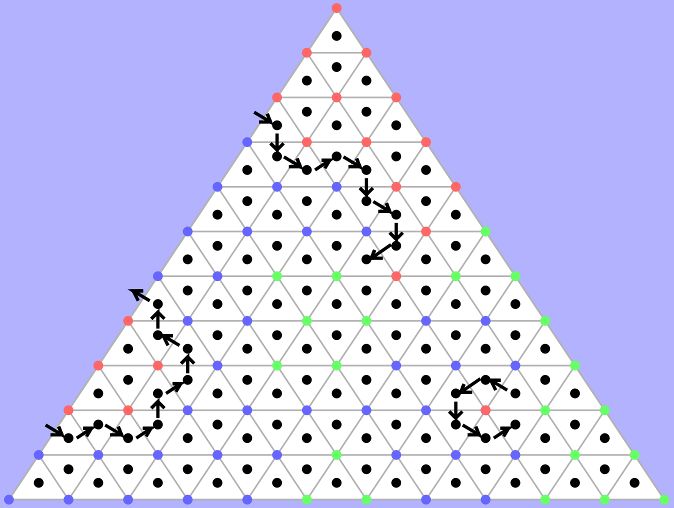
...converges to Brouwer fixpoint



The corresponding graph



The corresponding graph



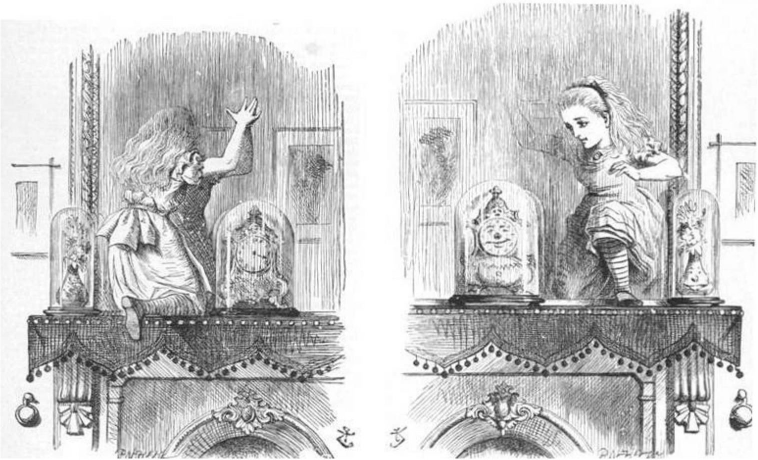
From PPAD to PPA

END OF LINE: most general problem that uses the end of line principle in a directed graph.

PPA: undirected graph. Circuit C, n inputs, $2n$ outputs, edge (v_1, v_2) is present iff v_2 is one of the outputs of C on input v_1 and vice versa.

PPA contains PPAD, easy to reduce END OF LINE to above.

Really a modulo-2 counting argument. A much nicer complexity class definition than PPAD. But PPAD happens to be the relevant one for NASH etc.



Understanding PPA-Completeness, Deng et al, 2016. Search for fixpoint on a Mobius band, Klein bottle, projective plane

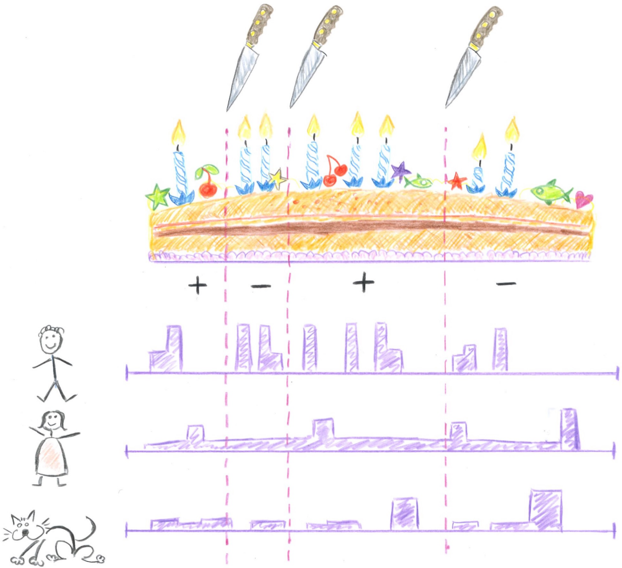
2D TUCKER (Tucker's Lemma in 2 dimensions)

1	1	1	1	1	1	1	1	1	1
2									
-1									
-2				$\vdots$					$-x$
-2			$\cdots$	$*$	$\cdots$				
				$\vdots$					2
x									2
									1
									-2
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

entries in $\{\pm 1, \pm 2\}$ generated by circuit

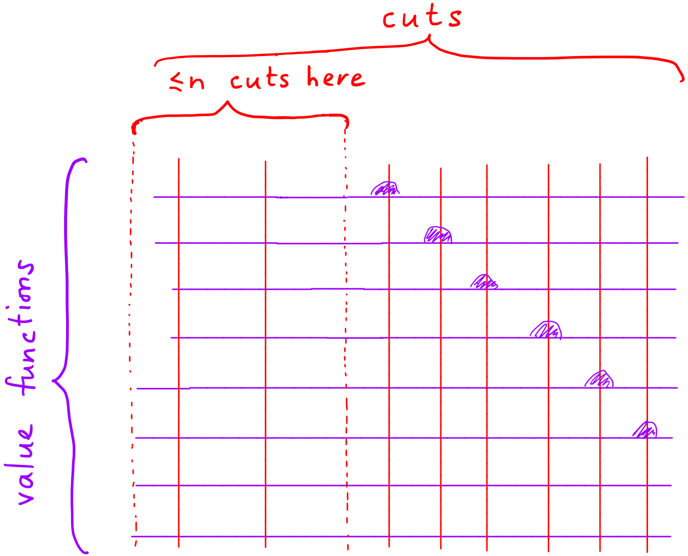
find contact-point of z and $-z$; **PPA**-complete

CONSENSUS-HALVING (Hobby-Rice theorem, 1965))

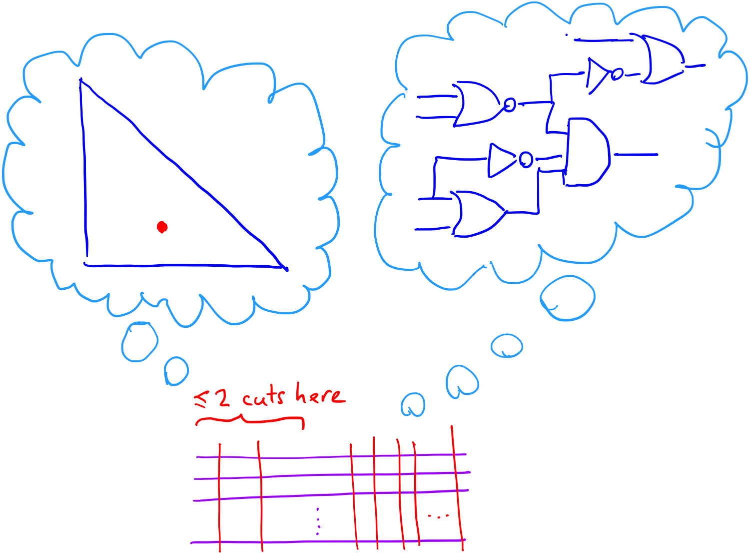


F. Simmons and F. Su: Consensus-halving via theorems of Borsuk-Ulam and Tucker Mathematical Social Sciences (2003)

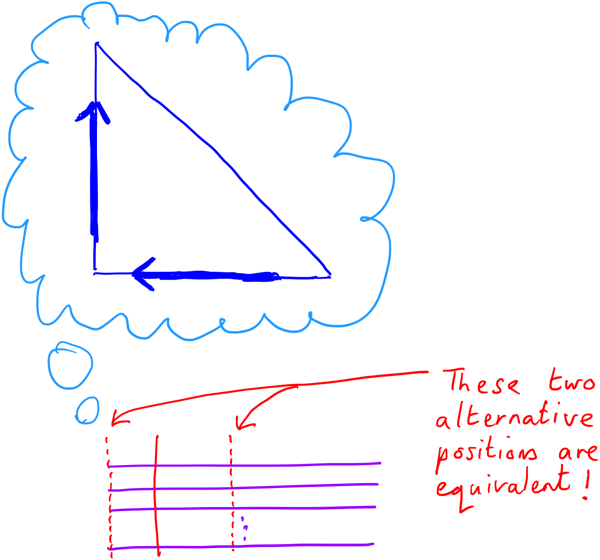
basic structure of instances of CONSENSUS-HALVING



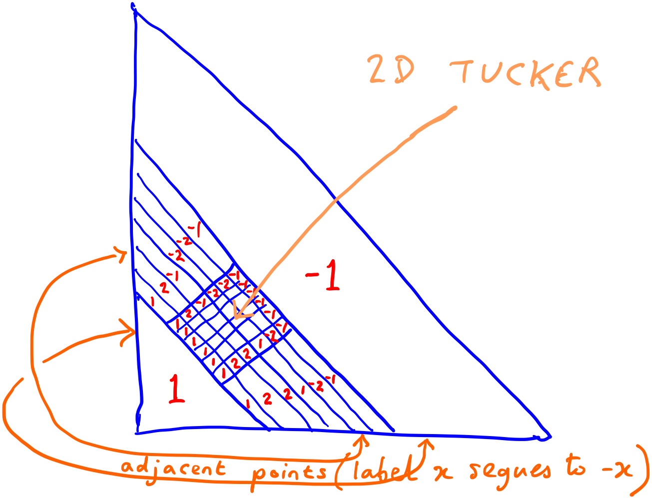
representing a point in a 2-simplex



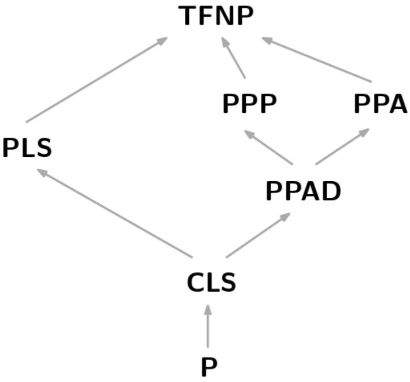
The 2-simplex embeds a Möbius strip!



Embed 2D TUCKER

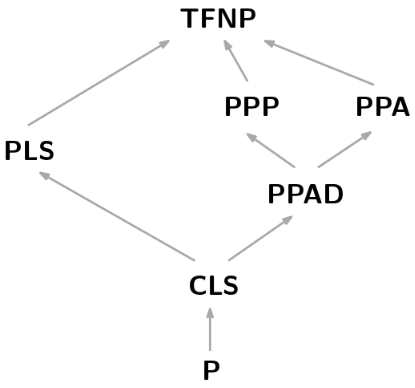


Open problem



Is that it? (Papadimitriou, a few years ago)

Open problem



Is that it? (Papadimitriou, a few years ago)

(apart from the classes shown in my more detailed diagram, and classes formed from unions/intersections of all these)

Open problem (more well-posed version)

class	principle
PPP	$\forall f \exists x, y \Big(f(x) = 0 \vee (x \neq y \wedge f(x) = f(y)) \Big)$
PPA	$\forall f \exists x \Big(f(0) \neq 0 \vee f(f(x)) \neq x \vee f(x) = x \Big)$
PLS	$\forall f, g \exists x \Big(g(f(x)) \leq g(x) \Big)$
PPAD	$\forall f, g \exists x \Big((f(g(0)) \neq 0 \wedge g(f(0)) = 0) \Rightarrow (x \neq 0 \wedge (f(g(x)) \neq x \vee g(f(x)) \neq x)) \Big)$
PPADS	$\forall f, g \exists x \Big((f(g(0)) \neq 0 \wedge g(f(0)) = 0) \Rightarrow (x \neq 0 \wedge g(f(x)) \neq x) \Big)$

Any other fundamentally different theorems in the above logic?