DAG-width: Cops and Robbers on Directed Graphs

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Joint work with
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GAMES Workshop, 2005
Motivation

- Tree-width introduced by Robertson and Seymour
  - Tree decompositions provide for recursive algorithms
  - Bounding tree-width gives polynomial time execution
- Directed tree-width by Johnson, Robertson, Seymour and Thomas
  - Not an obvious extension of tree-width
  - Complicated definition does not lend itself to algorithms

Aim

Find a natural extension of tree-width to directed graphs that is algorithmically useful.
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Overview

- Review tree-width
- Cops and robber game
- DAG-decompositions and DAG-width
- An algorithm for parity games
- Further work
Recall...

The tree-width of a graph measures its similarity to a tree.

A graph has tree-width $\leq k$ if it can be covered by sub-graphs of size $\leq (k + 1)$ in a tree-like fashion.
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The tree-width of a graph measures its similarity to a tree.

A graph has \textit{tree-width} \( \leq k \) if it can be covered by sub-graphs of size \( \leq (k + 1) \) in a tree-like fashion.
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Tree-width

A **tree decomposition** of a graph $G$ is a tuple $(\mathcal{T}, (X_t)_{t \in V(\mathcal{T})})$ such that:

- $\mathcal{T}$ is a tree
- $X_t$ cover $V(G)$
- For every edge $(u, v) \in E(G)$, there is a $t \in V(\mathcal{T})$ with $\{u, v\} \subseteq X_t$
- For every $t'$ on the path from $t$ to $t''$, $X_t \cap X_{t''} \subseteq X_{t'}$

The width of a tree decomposition is $\max_{t \in V(\mathcal{T})}|X_t| - 1$.

The tree-width of a graph is the minimal width of all its tree decompositions.

Tree-width can be characterised by a **cops and robber** game.
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Cops and robber game

Theorem (Seymour and Thomas 1993)

\[ G \text{ has tree-width } k \text{ if, and only if } k + 1 \text{ cops have a winning strategy} \]

Question

What about directed graphs?

Paul Hunter (HU-Berlin)

DAG-width

GAMES 2005
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Observations

Let game-width($G$) be the minimal number of cops required to catch a robber on $G$.

- directed tree-width($G$) ≤ game-width($G$) ≤ tree-width($G$)
- game-width($G$) = 1 iff $G$ is a DAG
- game-width of directed union is maximum width of components
- game-width is not preserved under edge reversal

Problem

Find a decomposition that corresponds to game-width
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Problem

Find a decomposition that corresponds to game-width
Another observation...

In a tree decomposition, an edge only leaves a subtree through its connection with the rest of the tree.
The DAG-width of a directed graph measures its similarity to a DAG.

A graph has **DAG-width** \( \leq k \) if it can be covered by subsets of size \( \leq k \) in a DAG-like fashion such that an edge only leaves a sub-DAG through its (root’s) connection with the rest of the DAG.
DAG-width

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![Diagram of DAG-width concept]
DAG-width

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Guarding

Definition

If $\mathcal{G}$ is a directed graph, $W, X \subseteq V(\mathcal{G})$, we say $X$ guards $W$ if every edge which leaves $W$ ends in $X$. 
**Guarding**

**Definition**

If $\mathcal{G}$ is a directed graph, $W, X \subseteq V(\mathcal{G})$, we say $X$ guards $W$ if every edge which leaves $W$ ends in $X$. 

[Diagram showing a directed graph with sets $W$ and $X$, and edges leaving $W$ to $X$.]
A **DAG-decomposition** of a directed graph $\mathcal{G}$ is a tuple $(\mathcal{D}, (X_d)_{d \in V(\mathcal{D})})$ such that:

- $\mathcal{D}$ is a DAG
- $X_d$ cover $V(\mathcal{G})$
- For every $d'$ on the path from $d$ to $d''$ ($d \preceq_D d' \preceq_D d''$), $X_d \cap X_{d''} \subseteq X_{d'}$
- For every $(c, d) \in E(\mathcal{D})$, $X_c \cap X_d$ guards $\left( \bigcup_{d \preceq_D d'} X_{d'} \right) \setminus X_c$. If $d$ is a root of $\mathcal{D}$, we replace $X_c$ with $\emptyset$.

The width of a DAG-decomposition is $\max_{d \in V(\mathcal{D})} |X_d|$. The DAG-width of a directed graph is the minimal width of all its DAG-decompositions.
A DAG-decomposition of a directed graph $G$ is a tuple $(D, (X_d)_{d \in V(D)})$ such that:

- $D$ is a DAG
- $X_d$ cover $V(G)$
- For every $d'$ on the path from $d$ to $d''$ ($d \preceq_D d' \preceq_D d''$), $X_d \cap X_{d''} \subseteq X_{d'}$
- For every $(c, d) \in E(D)$, $X_c \cap X_d$ guards $\left( \bigcup_{d \preceq_D d'} X_{d'} \right) \setminus X_c$. If $d$ is a root of $D$, we replace $X_c$ with $\emptyset$.

The **width** of a DAG-decomposition is $\max_{d \in V(D)} |X_d|$.

The DAG-width of a directed graph is the minimal width of all its DAG-decompositions.
DAG-decompositions and DAG-width

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Theorem

$G$ has DAG-width $k$ if and only if $k$ cops have a monotone winning strategy on $G$.

A monotone strategy is one where every vertex is visited by a cop at most once.

Theorem (Complexity Issues)

- For fixed $k$, deciding if $G$ has DAG-width $\leq k$ is in $\text{Ptime}$.
- Given $G$ and $k$, deciding if $G$ has DAG-width $\leq k$ is $\text{NP}$-complete.
Results

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More results...

- Results from game-width carry over to DAG-width
  - $\text{dtw}(G) \leq \text{game-width}(G) \leq \text{DAG-width}(G) \leq \text{tw}(G)$
  - Directed unions
- $\text{DAG-width}(G) \leq \text{entanglement}(G) + 1$
- $\text{DAG-width}(G) \leq \text{directed path-width}(G)$

Theorem

Parity games on graphs of bounded DAG-width can be decided in polynomial time
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**Theorem**

*Parity games on graphs of bounded DAG-width can be decided in polynomial time*
Parity games algorithm

Similar to Obdržálek’s algorithm for bounded tree-width

1. An edge (or node) of a tree decomposition separates the graph
2. Positional strategies can then be represented as functions from the interface to itself (border)
3. Compute borders in a bottom-up manner
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Extension to DAG-decompositions

- Interface covers edges **leaving** sub-DAG
- Problem 1: Handling edges entering sub-DAG
  - Solution: Use functions from sub-DAG to interface (frontier)
- Problem 2: Adding vertices to frontiers
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Conclusions and further work

- Introduced a natural extension of tree-width to directed graphs.
- Provided a polynomial-time algorithm for parity games on graphs of bounded DAG-width – subsuming results on bounded tree-width and entanglement.

- Are monotone strategies sufficient?
- Generalisation of havens, brambles, minors, separators?
- Generalisation of Courcelle’s theorem?
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