Complexity Bounds for Regular Games

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Overview

- Regular games: History and Motivation
- Condition types: Translations and Succinctness
- Completeness results
- Consequences and further work

Entrée: Definition – Game

An infinite game (V, E, Win) consists of:

- Two players – Player 0 and Player 1
- An arena (V, E), and
- A winning condition $\mathrm{Win} \subseteq V^\omega$.

Player 0 and Player 1 alternately move a token around (V, E) for an infinite number of moves generating an infinite sequence of vertices $\pi \in V^\omega$.

Player 0 wins if and only if $\pi \in \mathrm{Win}$.

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A game (V, E, Win) is regular if there exists $\mathcal{F} \subseteq 2^V$ such that

$$\pi \in \mathrm{Win} \iff \mathsf{inf}(\pi) \in \mathcal{F}$$

where $\mathsf{inf}(\pi)$ is the set of vertices occurring infinitely often in π.

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- Muller games
- Parity games

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Close connections with infinite automata

- Equivalent to infinite alternating automata
- Used to show equivalence of Muller, Rabin and Parity tree automata, giving
  - Complementation of languages defined by Rabin tree automata
  - Decidability of $S2S$, SnS, Muller acceptance, ...

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A regular game is:

- *Determined (Martin, 1975)*
- *Decidable (McNaughton, 1993)*

Decision Problem

Given a regular game (V, E, Win) and a starting position $v \in V$ does Player 0 win from v?

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Soufflé: Types of winning conditions

A winning condition type is a method to describe a winning condition.

$$\langle \mathcal{I}_{(V,E)}, \mathrm{Acc}_{(V,E)} \rangle \text{ where } \mathrm{Acc}_{(V,E)} \subseteq \mathcal{I}_{(V,E)} \times V^{\omega}$$

For regular games $\mathrm{Acc}_{(V,E)} \subseteq \mathcal{I}_{(V,E)} \times 2^{V}$

Example

The Explicit winning condition type is one which explicity lists a family of subsets of V

Instance: $\mathcal{F}$ where $\mathcal{F} \subseteq 2^{V}$

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Instance: $(C, \chi, \mathcal{F})$ where $\chi : V \to C$ and $\mathcal{F} \subseteq 2^C$
Acceptance: $\chi\big(\mathrm{inf}(\pi)\big) \in \mathcal{F}$

Parity: Assigns each vertex a priority and accepts sets with even minimal priority
Instance: χ where $\chi : V \to \omega$
Acceptance: $\min\Big(\chi\big(\mathrm{inf}(\pi)\big)\Big)$ is even

Win-set: Explicitly lists a family of subsets of $W \subseteq V$ and only considers the vertices in W
Instance: $(W, \mathcal{F})$ where $W \subseteq V$ and $\mathcal{F} \subseteq 2^W$
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Soufflé: Translations

Condition type $\mathfrak{A}$ is translatable to condition type $\mathfrak{B}$ if there is a polynomial time algorithm which, for *every* game of type $\mathfrak{A}$, produces a condition of type $\mathfrak{B}$ that describes the same game.

Example

An explicitly presented game can be translated to a Muller game by taking the identity function as a colouring and using the same list of sets. Thus the Explicit condition type is translatable to the Muller condition type.

Example

Suppose we have a Muller game where half the vertices are coloured blue, half are red, and the list of sets is $\{\{\mathit{red}\}\}$. The explicit game equivalent to this requires exponentially more space to describe. Thus the Muller condition type is not translatable to the Explicit condition type.

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Soufflé: Succinctness

If $\mathfrak{A}$ is translatable to $\mathfrak{B}$ but $\mathfrak{B}$ is not translatable to $\mathfrak{A}$, we say $\mathfrak{B}$ is more succinct than $\mathfrak{A}$.

Theorem

- *The Emerson-Lei type is more succinct than the Muller type*
- *The Muller type is more succinct than the Win-set type*
- *The Win-set type is more succinct than the Explicit type*

Corollary

- *Deciding Muller games reduces to deciding Emerson-Lei games*
- *Deciding Win-set games reduces to deciding Muller games*
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Main course: Completeness

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Deciding Win-set games is PSPACE complete

Proof (Sketch).

- Membership in PSPACE follows from PSPACE algorithm for Emerson-Lei games (Nerode, Remmel, Yakhnis 1996)
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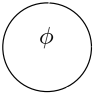
Main course: Reduction from QSAT

$$\Phi = \ldots \forall x_1 \exists x_0 . \phi$$



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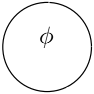

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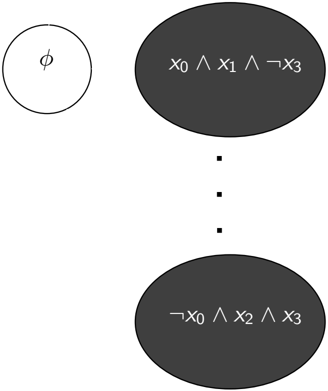

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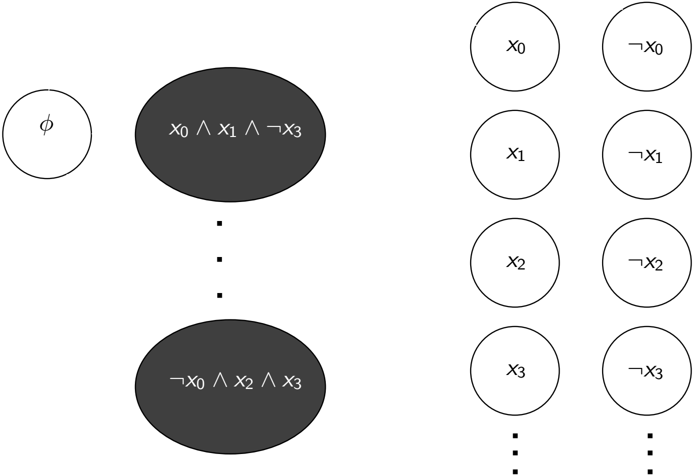

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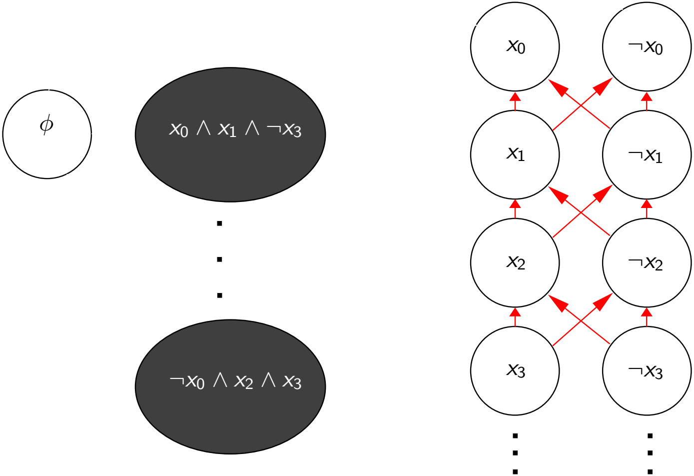

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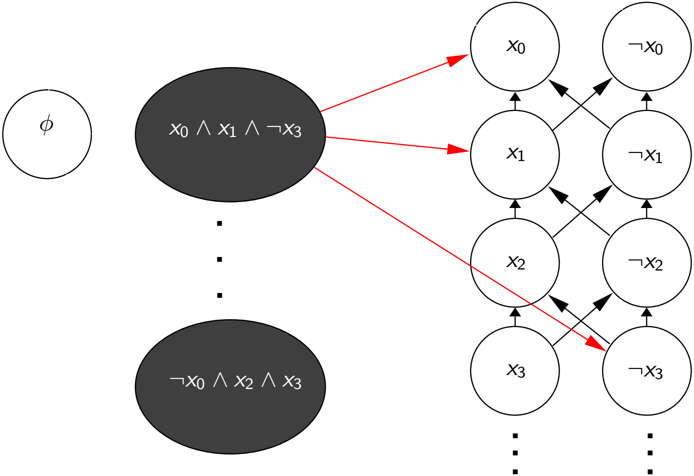

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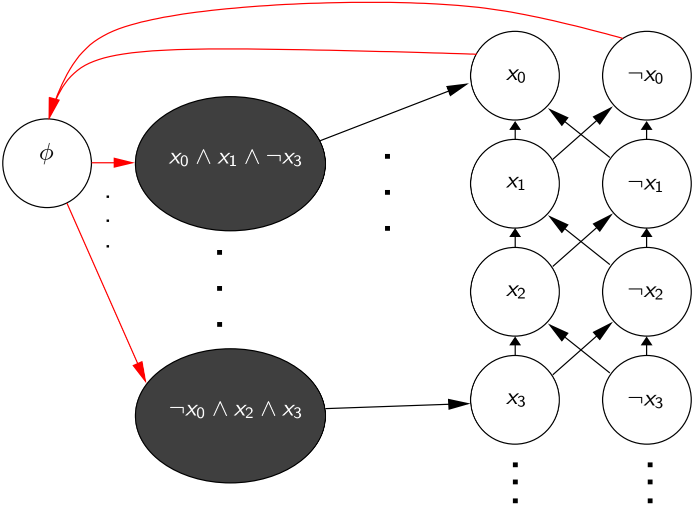

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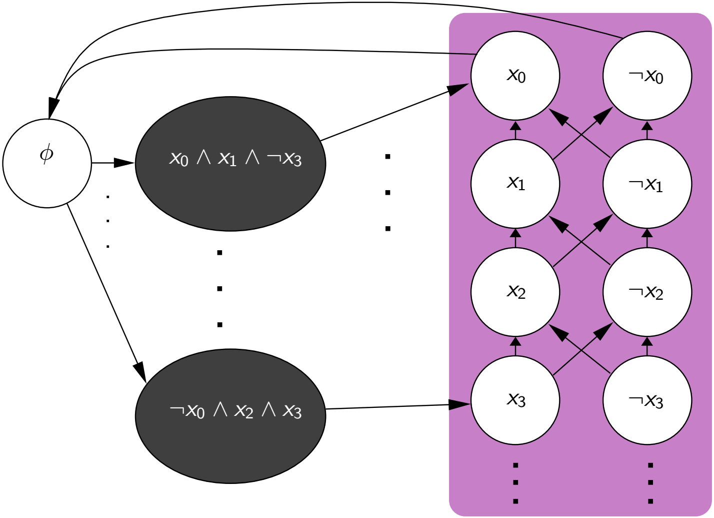
$$W = \{x_i, \neg x_i \ : \ 0 \leq i \leq n\}$$



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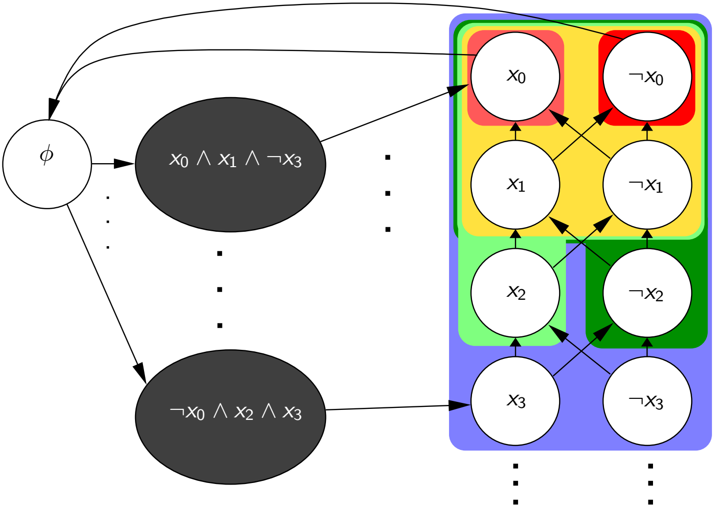
$$\mathcal{F} = \{S_i, S_i \cup \{x_i\}, S_i \cup \{\neg x_i\} : i \text{ even}\} \text{ where } S_i = \{x_j, \neg x_j : j < i\}$$



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Deciding Emerson-Lei games is PSPACE complete

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The non-emptiness and model-checking problems for Muller tree automata are PSPACE complete

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We have:

- Examined winning conditions in isolation – introducing some new conditions and a notion of reduction
- Shown PSPACE completeness for Win-set games – a result which extends to decision problems associated with Muller tree automata.

Open problems:

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