Isotropic problems

We shall see later how stress is a function of strain E, or equivalently, of C, say $\sigma = \sigma(C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33})$.

If can be shown that for *isotropic* problems, the stress is just a function of the *principal invariants*[1] of C

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\begin{aligned}
I_1 &= \operatorname{tr}(C) \\
I_2 &= \frac{1}{2}\left(\operatorname{tr}(C)^2 - \operatorname{tr}(C^2)\right) \\
I_3 &= \det(C)
\end{aligned}
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[1] To complicate matters even more, compressible problems often use the *deviatoric invariants*: $\bar{I}_1 = I_1 I_3^{-\frac{1}{3}}$, and $\bar{I}_2 = I_2 I_3^{-\frac{2}{3}}$. These are the invariants of C after it has been scaled to have determinant 1—see [Horgan and Saccomandi, Journal of Elasticity, 2004] for a discussion.

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Linear elasticity

Linearise E, removing terms that are quadratic in the displacement:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \mathcal{O}(u^2)$$

so define

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$$

This is the *infinitesimal strain tensor*

Note: normally in linear elasticity $\mathbf{x}$ represents undeformed position, so ϵ_{ij} is defined to be $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

Note also that linearising the incompressibility constraint $\det F = 1$ gives:

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Balance laws

Stresses

There are various definitions of stress in nonlinear elasticity:

Cauchy stress, σ, the force per unit deformed area acting on surfaces on the deformed body (i.e. the true stress) (symmetric)

1st Piola-Kirchhoff stress, S, the force per unit *undeformed area* acting on surfaces on the deformed body (not symmetric)

2nd Piola-Kirchhoff stress, T, the force per unit undeformed area acting on surfaces on the undeformed body (symmetric)

Relationships:

$$S = JF^{-1}\sigma \qquad T = SF^{-\mathrm{T}} \qquad \sigma = \frac{1}{J}FTF^{\mathrm{T}}$$

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Equilibrium equation

Let ρ_0 and ρ be the density in the undeformed and deformed bodies, and let $\mathbf{b}$ be the *body force density* (e.g. gravity, for which $\mathbf{b} = [0, 0, -9.81]$)

For a body in *static equilibrium*, the equilibrium equation is

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0 \qquad \text{in } \Omega$$

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We can transform to the undeformed state, for which the 1st Piola-Kirchhoff stress arises

$$\frac{\partial S_{Mi}}{\partial X_M} + \rho_0 b_i = 0 \qquad \text{in } \Omega_0$$

The two equations can be written as

$$\begin{aligned} \mathrm{div}(\sigma) + \rho \mathbf{b} = 0 \qquad & \text{in } \Omega \\ \mathrm{Div}(S) + \rho_0 \mathbf{b} = 0 \qquad & \text{in } \Omega_0 \end{aligned}$$

Replacing the 1st PK stress with the 2nd PK stress, we obtain

$$\frac{\partial}{\partial X_M}\left(T_{MN} \frac{\partial x_i}{\partial X_N} \right) + \rho_0 b_i = 0 \qquad \text{in } \Omega_0$$

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Boundary conditions

Mixed Dirichlet-Neumann boundary conditions are the specification of

- deformation/displacement on one part of the boundary (Dirichlet BCs)
- *tractions* the rest of the boundary (Neumann BCs)

The Neumann boundary condition is

$$\sigma_{ij} n_j = s_i \qquad \text{on deformed surface}$$

where $\mathbf{s}$ is the prescribed traction, which again has to be transformed back to the undeformed body

Splitting $\partial\Omega_0$ into Γ_1 and Γ_2, overall the boundary conditions are

$$\mathbf{x} = \mathbf{x}^* \qquad \text{on } \Gamma_1$$
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Other balance equations

Linear elasticity

Use $\frac{\partial}{\partial x_i} \approx \frac{\partial}{\partial X_i}$ (which means all 3 types of stress are equal to lowest order); work with

$$\frac{\partial \sigma_{ij}}{\partial X_j} + \rho_0 b_i = 0 \qquad \text{in } \Omega_0$$

Time-dependent problems

Defining the acceleration $\mathbf{a} = \frac{\partial^2 \mathbf{x}}{\partial t^2}$.

$$\rho_0 a_i = \frac{\partial}{\partial X_M}\left(T_{MN} \frac{\partial x_i}{\partial X_N} \right) + \rho_0 b_i \qquad \text{in } \Omega_0$$

Material laws

The strain energy function

An *elastic* material is one where stress is a function of strain: $T \equiv T(E)$ say

A **hyper-elastic** material is an elastic material for which there exists a **strain energy function** whose *derivative with respect to strain gives the stress*.

Specifically, there exists $W \equiv W(E)$ such that[2]

$$T_{MN} = \frac{\partial W}{\partial E_{MN}}$$

W must be determined experimentally (propose a law and experimentally determine parameters)

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It is often simpler to just work with C

$$W \equiv W(C) \qquad \text{such that} \qquad T_{MN} = 2\frac{\partial W}{\partial C_{MN}}$$

Isotropic materials

- In general, W is a function of the six independent components of C (recall that C is symmetric)
- However, for (compressible) *isotropic* materials, it can be shown that

$$W \equiv W(I_1, I_2, I_3)$$

only

Incompressible strain energy functions

Recall that for incompressible materials we have the constraint

$$\det F = 1$$

(everywhere), i.e. $I_3 = 1$. This introduces a Lagrange multiplier $p \equiv p(\mathbf{X})$, which must be computed together with the deformation.

The material law becomes, for an isotropic material

$$W(C) = W^{\mathrm{mat}}(I_1, I_2) - \frac{p}{2}(I_3 - 1)$$

This gives: $T_{MN} = 2\frac{\partial W^{\mathrm{mat}}}{\partial C_{MN}} - p\left(C^{-1}\right)_{MN}$, or equivalently

$$\sigma_{ij} = \sigma_{ij}^{\mathrm{mat}} - p\delta_{ij}$$

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Example strain energy functions

Incompressible strain energies:

- Neo-Hookean: $W^{\mathrm{mat}}(I_1, I_2) = c_1(I_1 - 3)$
- Mooney-Rivlin: $W^{\mathrm{mat}}(I_1, I_2) = c_1(I_1 - 3) + c_2(I_2 - 3)$
- Veronda-Westman: $W^{\mathrm{mat}}(I_1, I_2) = c_1 e^{\alpha(I_1 - 3)} + c_2(I_2 - 3)$
- Similar exponential laws are often used in biology

$$W^{\mathrm{mat}}(C) = c_1 e^{\alpha(Q(C)-1)}$$

where $Q(C)$ is a quadratic in the entries of C

A compressible strain energy: the compressible Neo-Hookean law

$$W(I_1, I_2, I_3) = c_1(\bar{I}_1 - 3) + c_3(J - 1)^2$$

Material law for (compressible) linear elasticity

Stress σ_{ij} is linearly related to strain ϵ_{ij}:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

For *isotropic materials*, it can be shown that this relationship must be of the form

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk}$$

where material parameters λ and μ are the *Lamé coefficients*

This relationship is often re-written using derived parameters E (Young's modulus) and ν (Poisson's ratio)

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Overall governing equations:

Overall governing equations: static, incompressible nonlinear elasticity

Given a material relationship $W \equiv W(C, p)$, $T \equiv 2\frac{\partial W}{\partial C}$:

Find $\mathbf{x} \equiv \mathbf{x}(\mathbf{X})$ and $p \equiv p(\mathbf{X})$ satisfying

$$\begin{aligned} \frac{\partial}{\partial X_M}\left(T_{MN}(\mathbf{x}, p)\frac{\partial x_i}{\partial X_M}\right) + \rho_0 b_i &= 0 \\ \det F(\mathbf{x}) &= 1 \end{aligned}$$

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Sometimes this is expanded and expressed explicitly in terms of $\mathbf{u}$

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u} + \rho_0\mathbf{b} = 0$$

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Overall governing equations: fluids

For fluids, $\mathbf{u}$ is used to denote *flow rather than displacement* and $\mathbf{x}$ is the independent variable (i.e. an Eulerian formulation is used).

For incompressible flow:

- **Kinematics:** Incompressibility $\Rightarrow \nabla \cdot \mathbf{u} = 0$ again, ...
- **Balance law:** use time-dependent Eulerian, i.e. $\rho \frac{D}{Dt} u_i = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i$
- **Material law:** Stress is a function of strain-rate, one material parameter, μ, the viscosity, and of pressure, as before.

Overall, the Navier-Stokes equations are: find $\mathbf{u}$ and p satisfying

$$\begin{aligned} \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) &= -\boldsymbol{\nabla} p + \mu\nabla^2\mathbf{u} + \rho\mathbf{b} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

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Numerical methods (incompressible nonlinear elasticity only)

Weak form

We cam compute the weak form as before

- multiplying (inner product) of the first equation with a test function $\mathbf{v} \in \mathcal{V}$, integrate, use divergence theorem.
- multiply second equation with $q \in \mathcal{W}$, integrate

Find $\mathbf{x} \in \mathcal{V}$ and $p \in \mathcal{W}$ such $\mathbf{x} = \mathbf{x}^*$ on Γ_1 and

$$\int_{\Omega_0} T_{MN}(\mathbf{x}, p) \frac{\partial x_i}{\partial X_N} \frac{\partial v_i}{\partial X_M} \,\mathrm{d}V_0 - \int_{\Omega_0} \rho_0 \mathbf{b} \cdot \mathbf{v} \,\mathrm{d}V_0 - \int_{\Gamma_2} \mathbf{s} \cdot \mathbf{v} \,\mathrm{d}S_0$$

$$+ \int_{\Omega_0} q \left(\det F(\mathbf{x}) - 1\right) \mathrm{d}V_0 = 0$$

$$\forall \mathbf{v} \in \mathcal{V}_0, q \in \mathcal{W}$$

Newton's method

Using $\mathbf{u}$ as the unknown instead of $\mathbf{x}$, write weak problem as:

Find $\mathbf{u}^h \in \mathcal{V}^h$, $p^h \in \mathcal{W}^h$ such that $\mathbf{u} = \mathbf{u}^*$ on Γ_1 and:

$$\mathcal{F}(\mathbf{u}^h, p^h, \mathbf{v}, q) = 0 \qquad \forall \mathbf{v} \in \mathcal{V}_0{}^h, q \in \mathcal{W}^h$$

Use **quadratic basis functions for displacement, linear for pressure**. This is necessary for a 'stable' scheme (accuracy).

Suppose there are

- N quadratic bases, $\phi_1, \ldots, \phi_N$
- M linear bases, $\psi_1, \ldots, \psi_M$:

Let $\mathbf{v} = \begin{bmatrix} \phi_i \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \phi_i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \phi_i \end{bmatrix}$ and $q = \psi_i \qquad \Rightarrow 3N + M$ nonlinear eqns

Solve using Newton's method as described in lecture 4.

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Find $\mathbf{u}^h \in \mathcal{V}^h$, $p^h \in \mathcal{W}^h$ such that $\mathbf{u} = \mathbf{u}^*$ on Γ_1 and:

$$\mathcal{F}(\mathbf{u}^h, p^h, \mathbf{v}, q) = 0 \qquad \forall \mathbf{v} \in \mathcal{V}_0{}^h, q \in \mathcal{W}^h$$

Use **quadratic basis functions for displacement, linear for pressure**. This is necessary for a 'stable' scheme (accuracy).

Suppose there are

- N quadratic bases, $\phi_1, \ldots, \phi_N$
- M linear bases, $\psi_1, \ldots, \psi_M$:

Let $\mathbf{v} = \begin{bmatrix} \phi_i \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \phi_i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \phi_i \end{bmatrix}$ and $q = \psi_i \qquad \Rightarrow 3N + M$ nonlinear eqns

Solve using Newton's method as described in lecture 4.

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Solve using Newton's method as described in lecture 4.

Object-oriented design

```
SolidMechanicsProblemDefinition
    mBodyForce
    mFixedNodes
    mFixedNodeLocations
    mNeumannBoundaryElements
    mTractions
```

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```
IncompressibleNonlinearElasticitySolver
    Solve(mesh, solidMechProblemDefn, absIncompMaterialLaw)
```

▷ *Use Newton's method to solve the given problem*

Object-oriented design

`AbstractIncompressibleMaterialLaw`

Abs. method: `GetStrainEnergyValue(C)`

▷ *Take in C, return $W(C)$*

This doesn't work as code needs to use $T = 2\frac{\partial W}{\partial C}$ (and also $\frac{\partial^2 W}{\partial C_{MN} \partial C_{PQ}}$)

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Object-oriented design

`AbstractIncompressibleMaterialLaw`

Abs. method: `ComputeStressAndStressDerivative(C,p)`

- ▷ *Take in C, p, return $2\frac{\partial W}{\partial C}$ and $4\frac{\partial^2 W}{\partial C_{MN}\partial C_{PQ}}$*

`AbstractIsotropicIncompressibleMaterialLaw` *inherits from* `(above)`:

Method: `ComputeStressAndStressDerivative(C,p)`

Abs. method: `Get_dW_dI1(I1,I2)` Get $\frac{\partial W}{\partial I_1}$

Abs. method: `Get_dW_dI2(I1,I2)` Get $\frac{\partial W}{\partial I_2}$

Abs. method: `Get_d2W_dI1(I1,I2)` Get $\frac{\partial^2 W}{\partial {I_1}^2}$

Abs. method: `Get_d2W_dI2(I1,I2)`

Abs. method: `Get_d2W_dI1dI2(I1,I2)`

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Object-oriented design

The Mooney-Rivlin law is

$$W^{\mathrm{mat}}(I_1, I_2) = c_1(I_1 - 3) + c_2(I_2 - 3)$$

`MooneyRivlinMaterialLaw` *inherits from* `AbsIsotropicIncompMaterialLaw`:

Implemented method: `Get_dW_dI1(I1,I2)`

- ▷ *return* c_1

Implemented method: `Get_dW_dI2(I1,I2)`

- ▷ *return* c_2

Implemented method: `Get_d2W_dI1(I1,I2)`

- ▷ *return 0, etc.*

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