An Abstract Approach to Entanglement

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Why Abstract?

- *How* are things entangled? Not *how much*!
- Make structure more obvious
- How much quantum computation can we get from the algebra alone?
- Non standard models ae interesting for practical as well as philosophical reasons

Towards a type theory for quantum computation.

Compact Closed Categories

A *compact closed category* is a symmetric monoidal category where every object A has a chosen dual A^* and *unit* and *counit* maps

$$\eta_A : I \rightarrow A^* \otimes A$$
$$\epsilon_A : A \otimes A^* \rightarrow I$$

such that

$$\begin{array}{ccccc}
A & \xrightarrow{\cong} & A \otimes I & \xrightarrow{\mathrm{id}_A \otimes \eta_A} & A \otimes (A^* \otimes A) \\
\downarrow{\scriptstyle \mathrm{id}_A} & & & & \downarrow{\scriptstyle \alpha} \\
A & \xleftarrow[\cong]{} & I \otimes A & \xleftarrow[\epsilon_A \otimes \mathrm{id}_A]{} & (A \otimes A^*) \otimes A
\end{array}$$

and the same diagram for the dual.

Example : FDHilb

Let **FDHilb** be the category whose objects are finite dimensional Hilbert spaces, and whose arrows are linear maps; **FDHilb** is compact closed with the following structure:

1. $A^* = [A \to \mathbb{C}]$

2. Let $\{a_i\}_i$ be any orthonormal basis for A; then η_A and ϵ_A are the linear maps defined by

$$\eta_A : 1 \mapsto \sum_i \overline{a_i} \otimes a_i$$

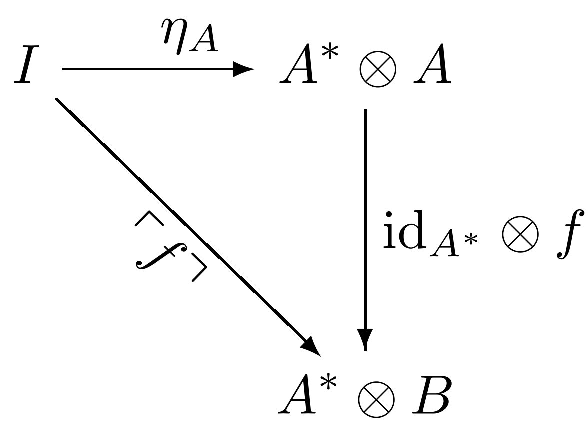
$$\epsilon_A : a_i \otimes \overline{a_j} \mapsto \delta_{ij}$$

Names

In any compact closed category we have

$$[A, B] \cong [I, A^* \otimes B]$$

via the *name* $\ulcorner f \urcorner$ of $f : A \to B$.



Strong Compact Closure

Suppose that $\mathcal{C}$ is equipped with a contravariant, involutive strict monoidal functor $(\cdot)^\dagger$ which is the identity on objects. Call $f^\dagger$ the *adjoint* of f.

Say that that $\mathcal{C}$ is *strongly compact closed* if

$$\epsilon_A = \sigma_{A^*,A} \circ \eta_A^\dagger .$$

Now suppose $\psi, \phi : I \to A$, we can define *abstract inner product*

$$\langle \psi \mid \phi \rangle := \psi^\dagger \circ \phi$$

Example : FDHilb

FDHilb is strongly compact closed.

- Let $f^\dagger$ be the unique linear map defined by $\langle f^\dagger\phi \mid \psi\rangle = \langle \phi \mid f\psi\rangle$; note that this coincides with the usual adjoint given by the conjugate transpose of matrices.

NB: when working with qubits we'll identify A^* and A and hence also f^* and $f^\dagger$. The isomorphism is not natural, but relative to the standard basis. Hence we take

$$\eta_Q = 1 \mapsto |00\rangle + |11\rangle \, .$$

1. Polycategories and Abstract Entanglement

Free Compact Closure on a Category

Given a category $\mathcal{A}$ of basic maps we can construct the free compact closed category generated by it.

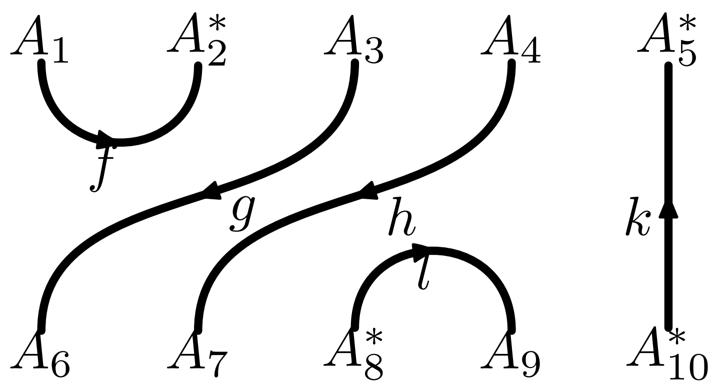
Objects: signed vectors of objects from $\mathcal{A}$, i.e. maps $\{A_1, \ldots, A_n\} \to \{+, -\}$.

Arrows: $f : A \to B$

- an involution θ on $A^* \otimes B$
- a functor $p : \theta \to \mathcal{A}$
- some scalars

If $\mathcal{A}$ has a suitable endofunctor $(\cdot)^\dagger$, then this can be lifted to get the free strongly compact closed category.

Free Compact Closure on a Category



Problem!

Consider a category with one object Q and some collection of (unitary) maps $Q \to Q$.

Its free compact closure is an interesting category of quantum states and maps: suffices for many simple protocols such as teleportation and swapping.

But:

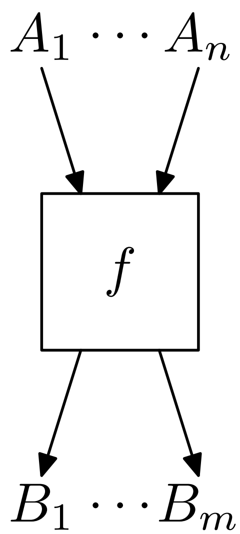
From the structure of the maps we can immediately see that there are only bipartite entangled states!

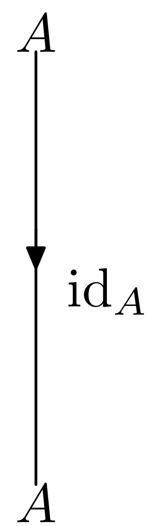
Polycategories

Introduced by Lambek (1969) and used to give categorical models for classical logic (among other things).

A *symmetric compact polycategory*, $\mathcal{P}$, consists of

- Objects $\mathrm{Obj}_{\mathcal{P}}$;
- Polyarrows $f : \Gamma \to \Delta$ between vectors of objects Γ, Δ;
- Identities $\mathrm{id}_A : A \to A$ for each 1-vector A;


A1 ··· An
f
B1 ··· Bm


A
idA
A

Polycategories (cont.)

If $|\Theta| > 0$ then given

$$\Gamma \xrightarrow{f} \Delta_1, \Theta, \Delta_2 \qquad \text{and} \qquad \Gamma_1, \Theta, \Gamma_2 \xrightarrow{g} \Delta$$

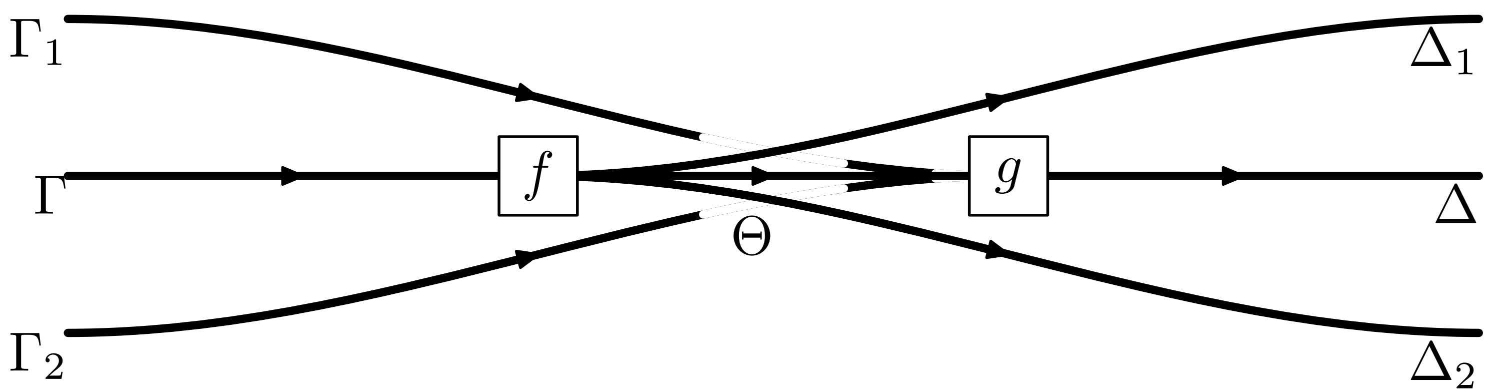
we may form the composition

$$\Gamma_1, \Gamma, \Gamma_2 \xrightarrow{g \ {}_i\overset{k}{\circ}_j f} \Delta_1, \Delta, \Delta_2$$

where $|\Delta_1| = i$, $|\Gamma_1| = j$ and $|\Theta| = k > 0$

Polycategories (cont.)

Easier to understand composition from a diagram:

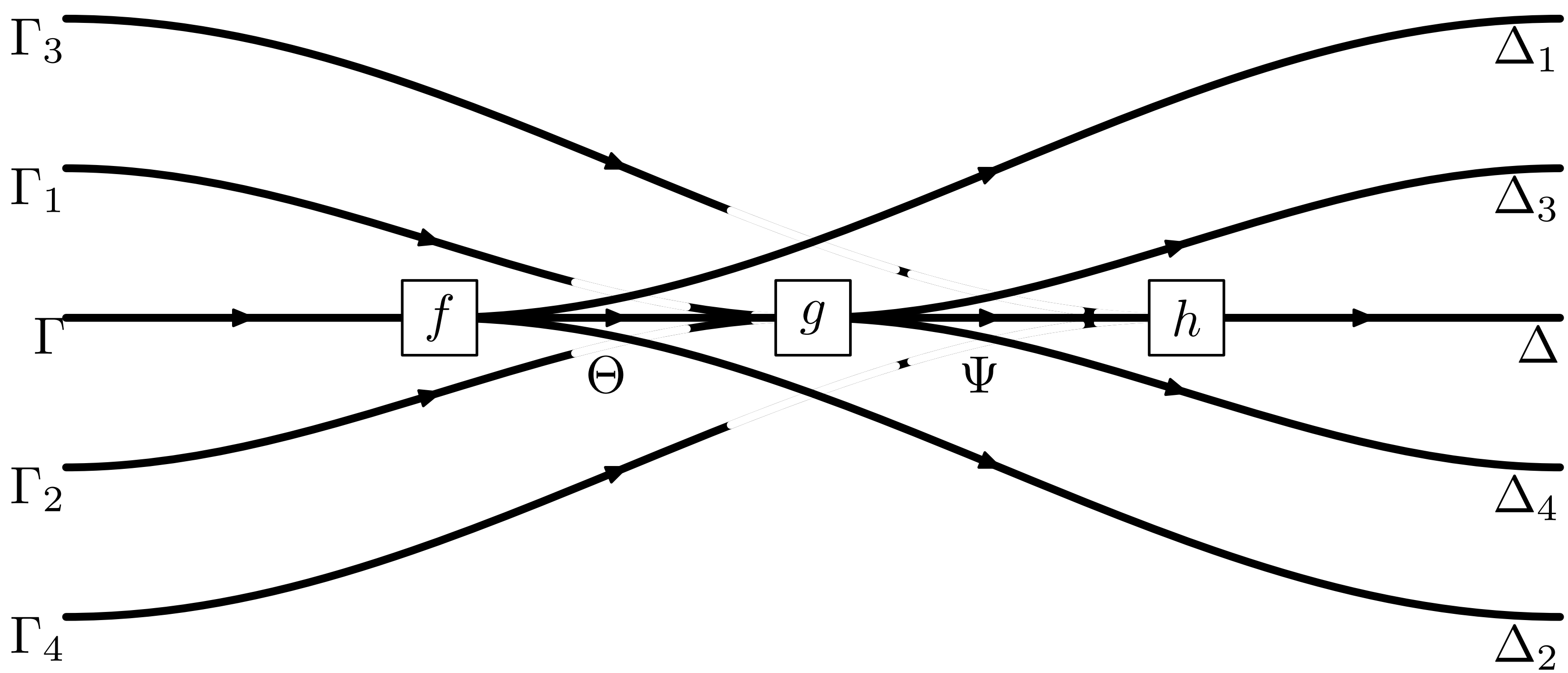


Identities:

$$\mathrm{id} \circ f = f = f \circ \mathrm{id}$$

Polycategories (cont.)

Composition is associative, so this diagram is unambiguous:



Example

let $\mathcal{Q}$ be the the polycategory whose only object is Q, and which is generated by the following non-identity poly-arrows.

$$\begin{aligned} |0\rangle\,, |1\rangle &\quad : \quad - \to Q \\ \langle 0|\,, \langle 1| &\quad : \quad Q \to - \\ H, X, Y, Z &\quad : \quad Q \to Q \\ CZ &\quad : \quad Q, Q \to Q, Q \end{aligned}$$

Why Polycategories?

Polycategories are a bit strange. Why use them?

- Suited for many-input, many-out protocols
- **No trivial composites**.

Disadvantages:

- No identities at compound maps means can't have all the equations we might want, e.g. $CZ \circ CZ = \mathrm{id}_{Q,Q}$.
  ...but we can get around this.

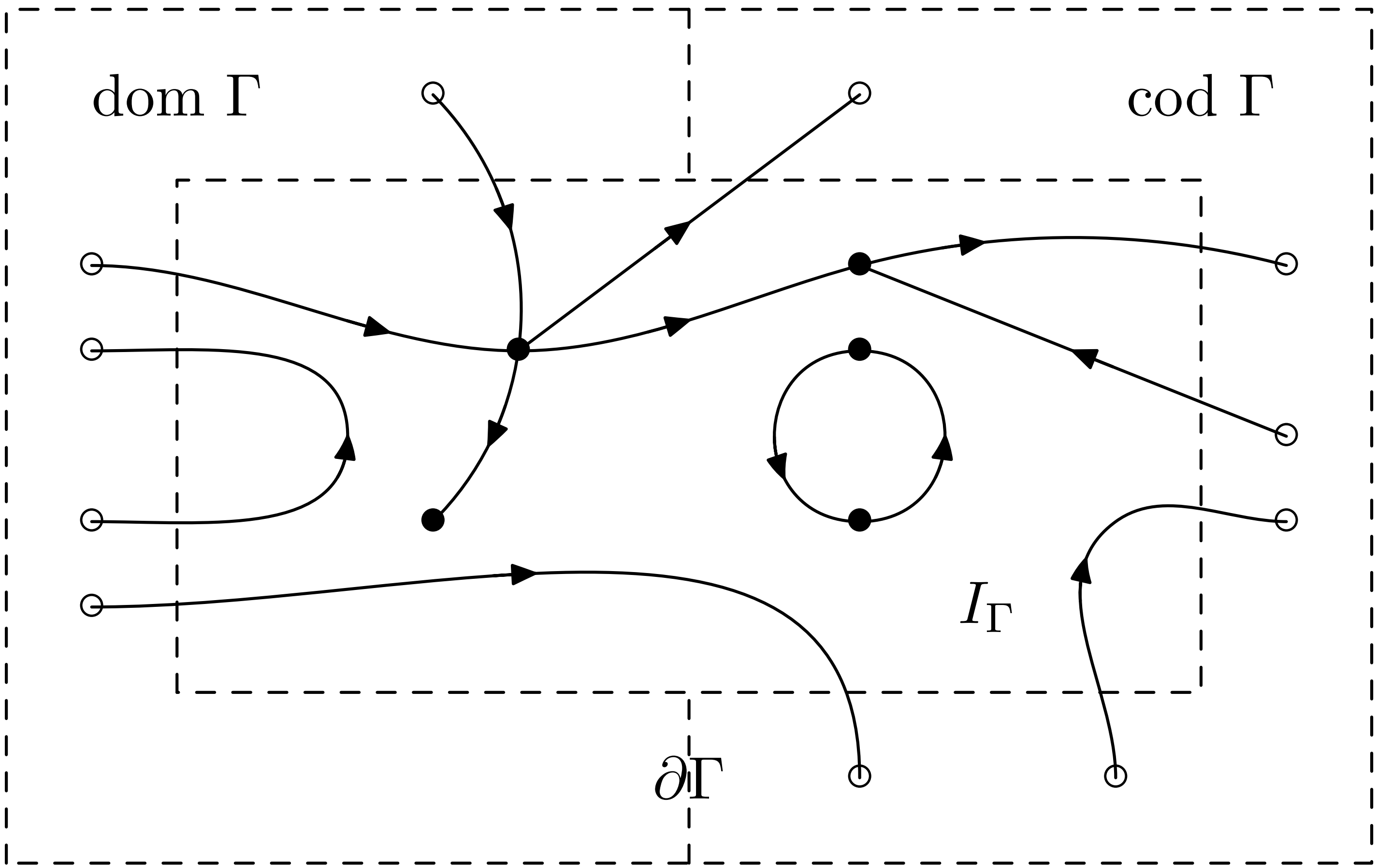
Circuits

A *graph with boundary* is a pair $(G, \partial G)$ of an underlying directed graph $G = (V, E)$ and a distinguished subset of the degree one vertices ∂G

We permit loops and parallel edges, and, in addition to the usual graph structure we permit *circles*: closed edges without any vertex..

A *circuit* is triple $\Gamma = (\Gamma, \operatorname{dom}\Gamma, \operatorname{cod}\Gamma)$ where $(\Gamma, \partial\Gamma)$ is a finite directed graph with boundary with $\partial\Gamma$ partitioned into two totally ordered subsets $\operatorname{dom}\Gamma$ and $\operatorname{cod}\Gamma$. In addition, every node x carries a total ordering on its incoming and outgoing edges; the resulting sequences are written $\operatorname{in}(x)$ and $\operatorname{out}(x)$ respectively.

Anatomy of a Circuit



Circuits form a Compact Closed Category

We construct a category of abstract circuits **Circ**.

- Objects are signed ordinals: maps $\{1, \ldots, n\} \to \{+, -\}$;
- Arrow $X \to Y$ are circuits whose domain and codomain are X^* and Y;
- Composition is by "plugging together";
- Tensor defined by "laying beside";

$\mathcal{A}$-Labelling

If we have a given polycategory $\mathcal{A}$, embed it into **Circ** using a labelling on the edges and vertices of circuits.

A pair of maps $\theta = (\theta_O, \theta_A)$ is an $\mathcal{A}$-labelling for a circuit gamma when θ_O maps each edge of Γ to an object in $\mathrm{Obj}_(\mathrm{A})$ and θ_A maps each internal node of Γ to $\mathrm{Arr}_{\mathcal{A}}$ such that for each node f, $\mathrm{in}(f) = \langle a_1, \ldots, a_n \rangle$ and $\mathrm{out}(f) = \langle b_1, \ldots, b_m \rangle$ imply

$$\mathrm{dom}(\theta f) = \theta a_1, \ldots, \theta a_n$$

$$\mathrm{cod}(\theta f) = \theta b_1, \ldots, \theta b_m.$$

$\mathbf{Circ}(\mathcal{A})$

If θ is a labelling for Γ then (Γ, θ) is an $\mathcal{A}$-*labelled circuit*.

The $\mathcal{A}$-labelled circuits form a category called $\mathbf{Circ}(\mathcal{A})$.

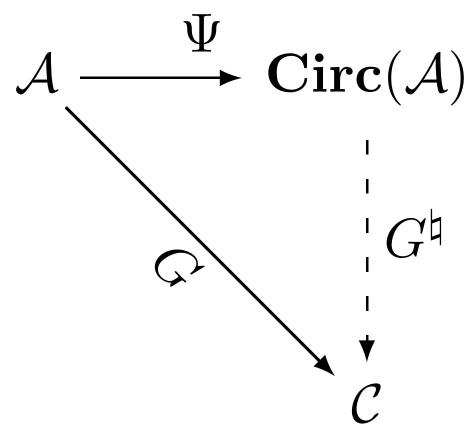
- Objects : signed vectors of objects from $\mathcal{A}$.
- Arrows : $\mathcal{A}$-labelled circuits.

There is a forgetful functor

$$\mathbf{Circ}(\mathcal{A}) \xrightarrow{U} \mathbf{Circ}$$

$\mathbf{Circ}(\mathcal{A})$ inherits compact closure from $\mathbf{Circ}$.

$\mathbf{Circ}(\mathcal{A})$ is the Free Compact closed Category on $\mathcal{A}$



Theorem. *Given any compact closed category $\mathcal{C}$, every compact closed functor $G : \mathcal{A} \to \mathcal{C}$ factors uniquely through Ψ.*

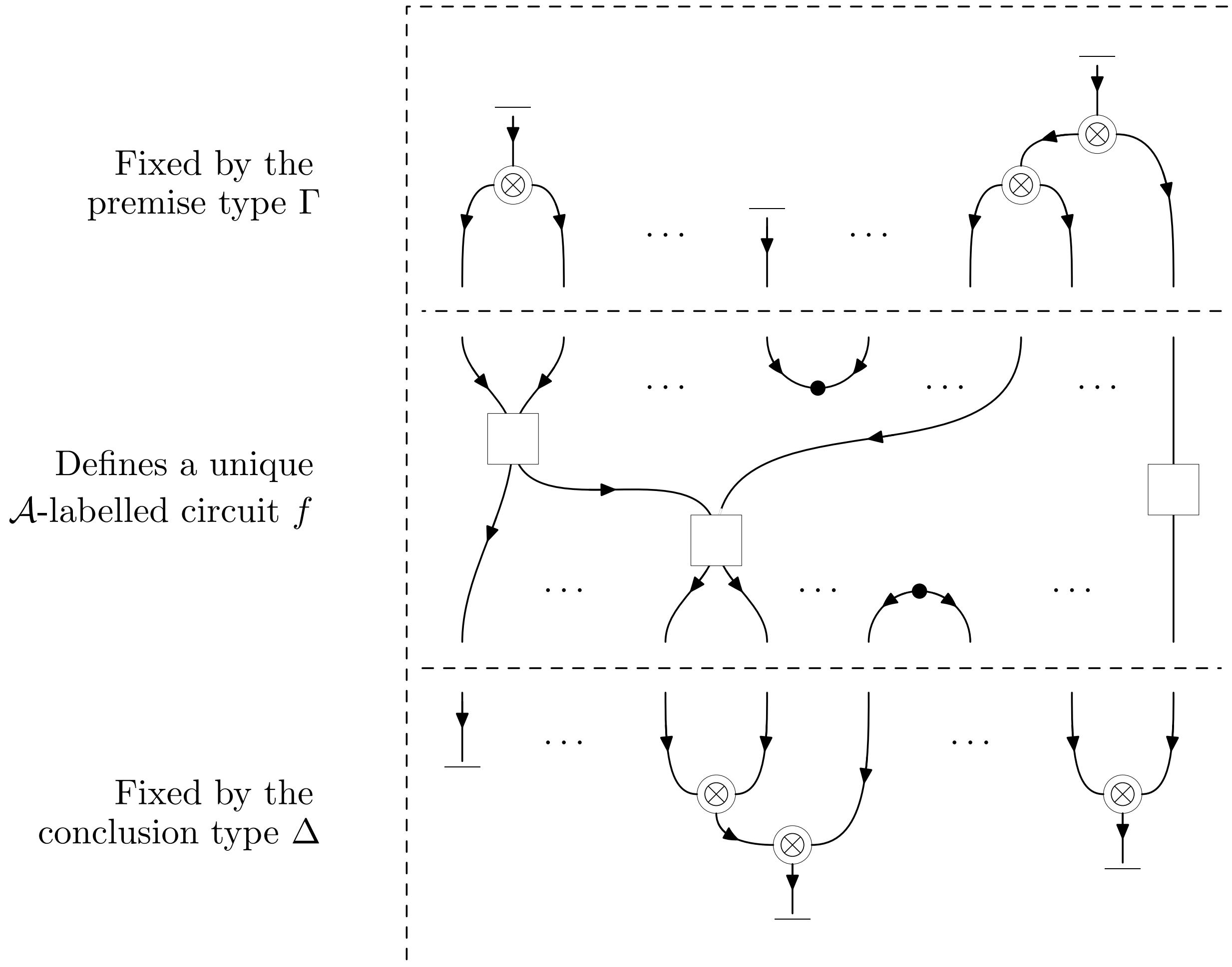
An Aside : Proofnets

Given a compact polycategory $\mathcal{A}$ we can construct a category of two-sided proof-nets $PN(\mathcal{A})$.

$PN(\mathcal{A})$ has a strongly normalising cut-elimination procedure.

$$PN(\mathcal{A}) \cong \mathbf{Circ}(\mathcal{A})$$

The normal forms of $PN(\mathcal{A})$ are the circuits of $\mathbf{Circ}(\mathcal{A})$ with some type formers attached to their domain and codomain.



There is an equivalence of categories between $\mathbf{Circ}(\mathcal{A})$ and $PN(\mathcal{A})$.

Some Questions

- We can define a maximally entangled state to be the name of a unitary (this is a theorem in **FDHilb**) — but can we give an abstract measure of entanglement in other states?
- If $\mathcal{A}$ is simply a category, the double gluing on $\mathbf{Circ}(\mathcal{A})$ gives a $*$-autonomous category where the linear logic connectives characterise the separable and entangled states. Will something similar work here?
- What about non-symmetric cases?
- Can the topology tell us anything interesting? Complexity?

2. The Measurement Calculus in Abstract Form

Measurement Calculus

Introduced by Danos, Kashefi and Pananagden for the 1-way model

1. A set S of qubits, numbered $1, \ldots n$;
2. Subsets $I \subseteq S$, $O \subseteq S$ of inputs and outputs;
3. All $q \notin I$ initialised to $|+\rangle$;
4. All $q \notin O$ must eventually be measured and not reused.

Compute using *patterns* comprised of

$$\begin{aligned} E_{ij} &= \text{Control-}Z \\ X_i, Z_j &= \text{Pauli X,Z corrections} \\ M_i^{\alpha} &= \text{1 qubit measurement in basis } |0\rangle \pm e^{i\alpha}|1\rangle \end{aligned}$$

where i, j index over qubits.

Measurement Calculus (cont.)

Theorem. *Measurement patterns are universal with respect to unitaries.*

A slight variation with only X-Y measurements is approximately universal.

Theorem. *Every measurement pattern is equivalent to a pattern where all E_{ij} precede all M_i^α which precede all X_i, Y_j.*

Further there is an effective rewriting procedure to put any pattern into this (EMC)-normal form.

Polycategorising the Measurement Calculus

We define a polycategory $\mathcal{M}$ suitable for measurement patterns,

$$\begin{aligned} \mathrm{Obj}_{\mathcal{M}} &= \{Q\} \\ \mathrm{Arr}_{\mathcal{M}} &= \{|+\rangle, \langle+|, T_\alpha, H, X, Z, E\} \end{aligned}$$

Give $\mathcal{M}$ an involution $(\cdot)^\dagger$ by

$$E^\dagger = E \qquad H^\dagger = H \qquad X^\dagger = X \qquad Z^\dagger = Z$$

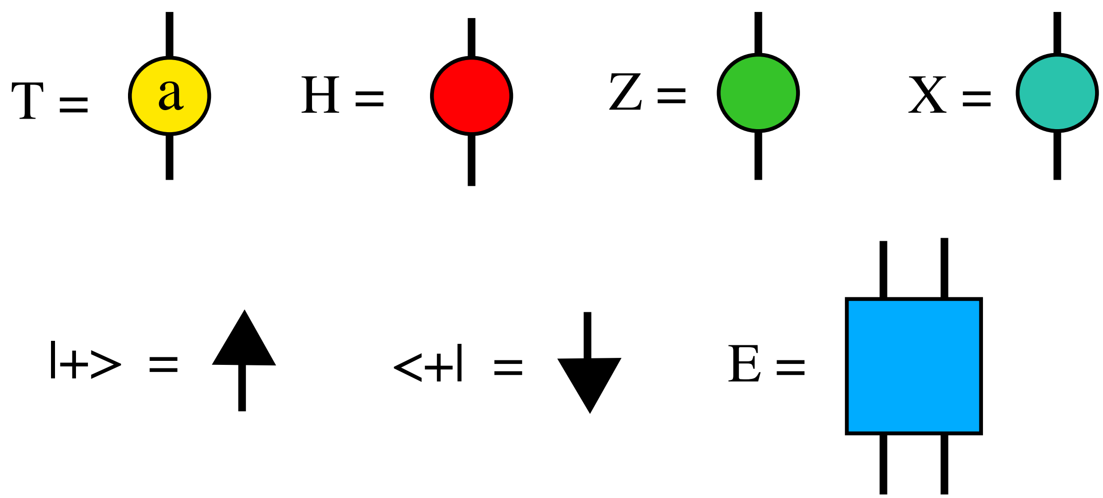
$$T_\alpha^\dagger = T_{-\alpha} \qquad\qquad |+\rangle^\dagger = \langle+|$$

Now we interpret the measurement calculus in $\mathbf{Circ}(\mathcal{M})$ by mapping each pattern to a circuit.

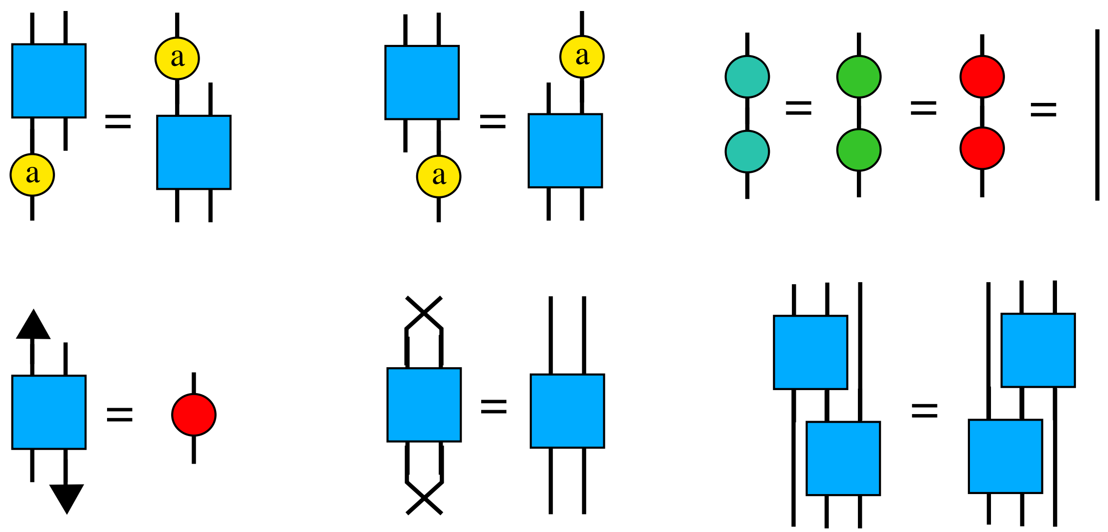
$$\begin{array}{rclrcl} E_{ij} & \mapsto & E & Z_i & \mapsto & Z \\ X_j & \mapsto & X & M^\alpha & \mapsto & \langle+| T_\alpha \end{array}$$

Graphical Notation for $\mathcal{M}$

We use the following graphical notation for the $\mathcal{M}$-labelled circuits.



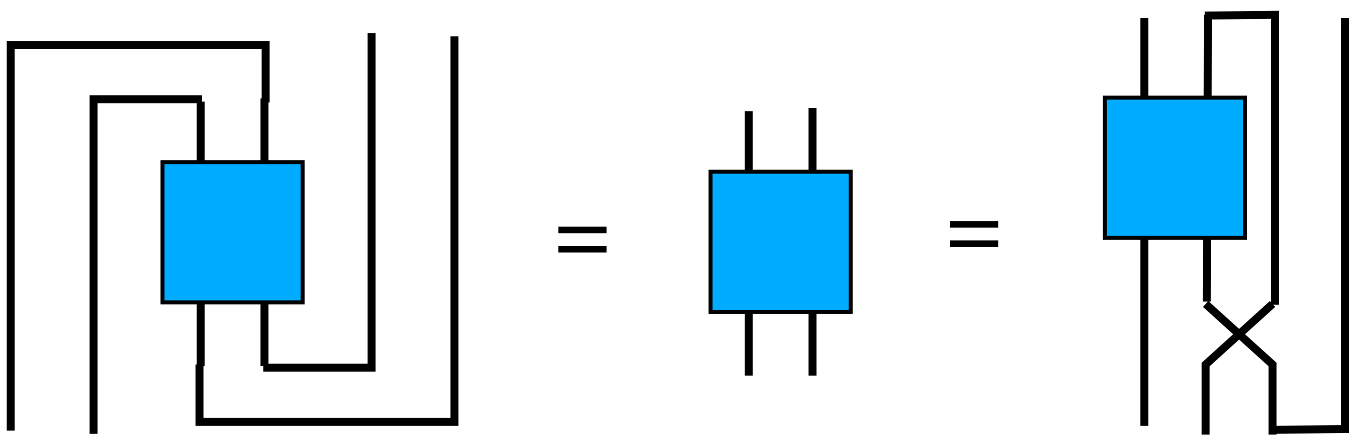
Equations in $\mathcal{M}$



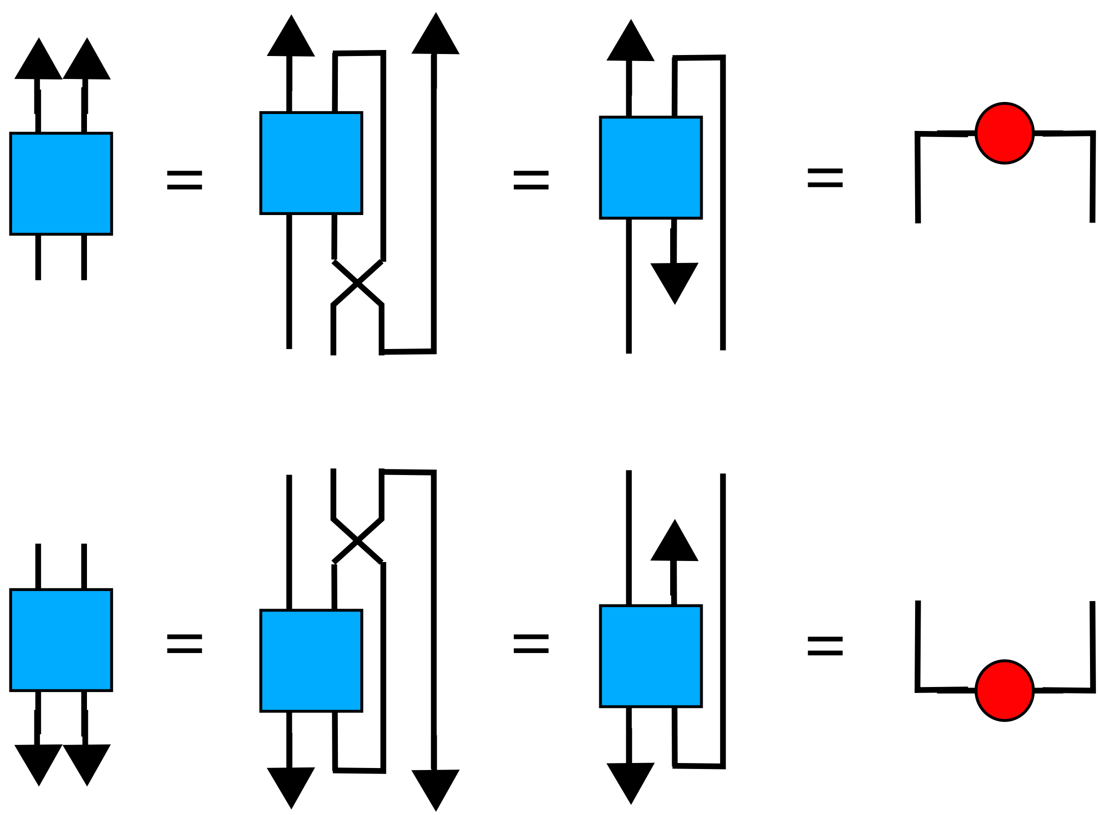
There are more: for example commutation relations.

Symmetry

E is invariant under transpose and partial transpose.



$$E\,|{+}{+}\rangle = \ulcorner H \urcorner$$

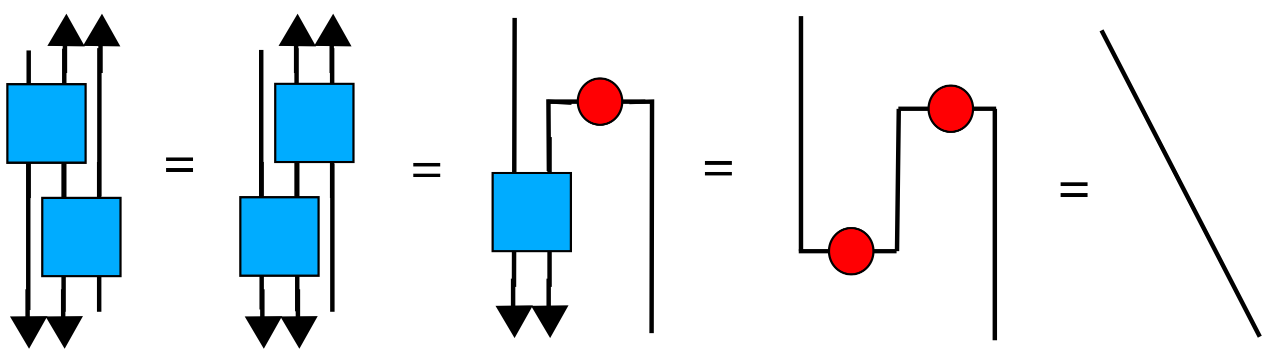


Example : Teleportation

From DKP, ignoring corrections the teleportation protocol is computed by

$$M_2^0 M_1^0 E_{23} E_{12}$$

with input 1 and output 3.

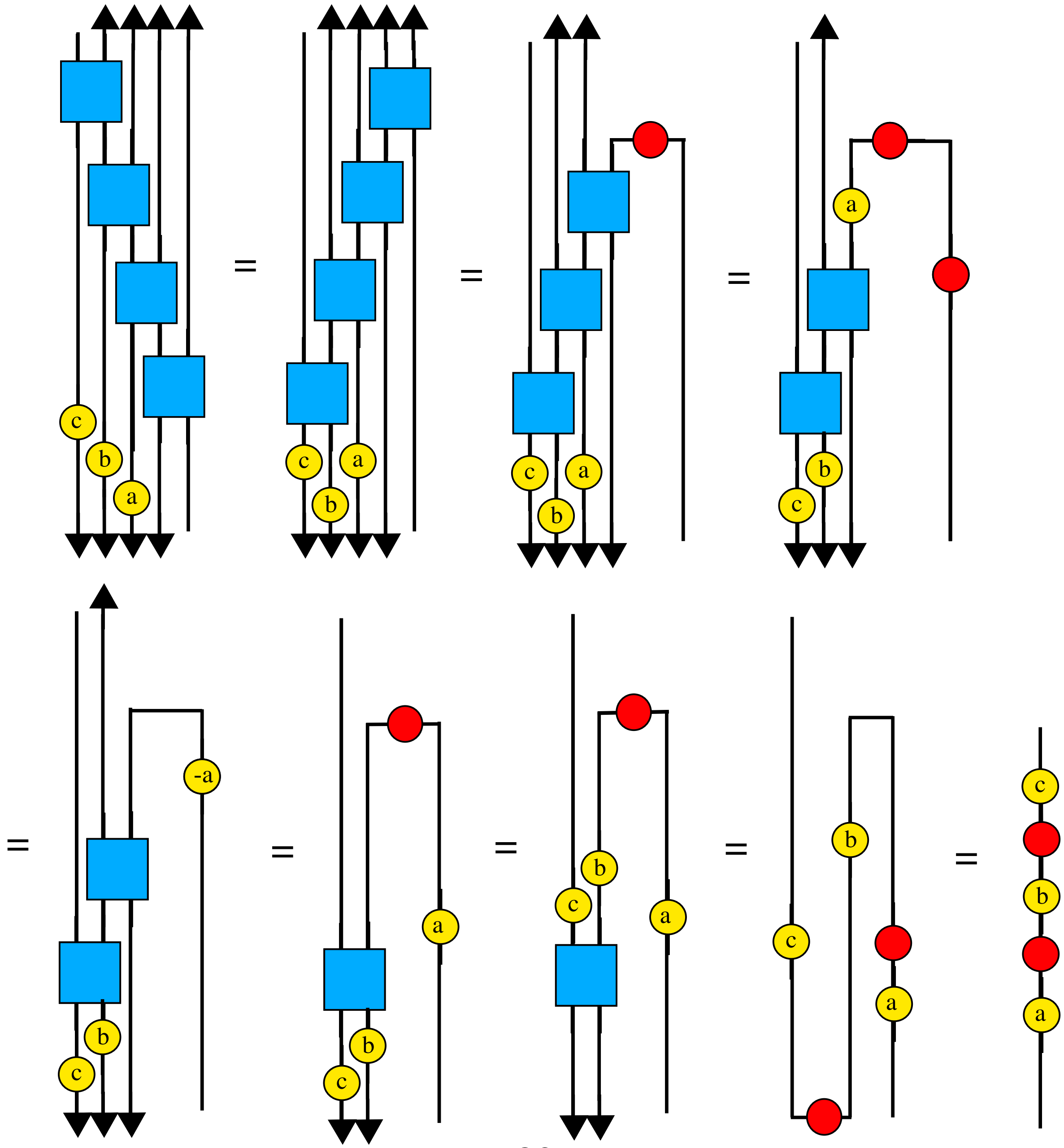


Example : General Rotation

From DKP, a one qubit rotation, given by its Euler decomposition $R_x(\gamma)R_z(\beta)R_x(\alpha)$ is computed by the pattern

$$M_4^0 M_3^\alpha M_2^\beta M_1^\gamma E_{12345}$$

with input 1 and output 5.


c
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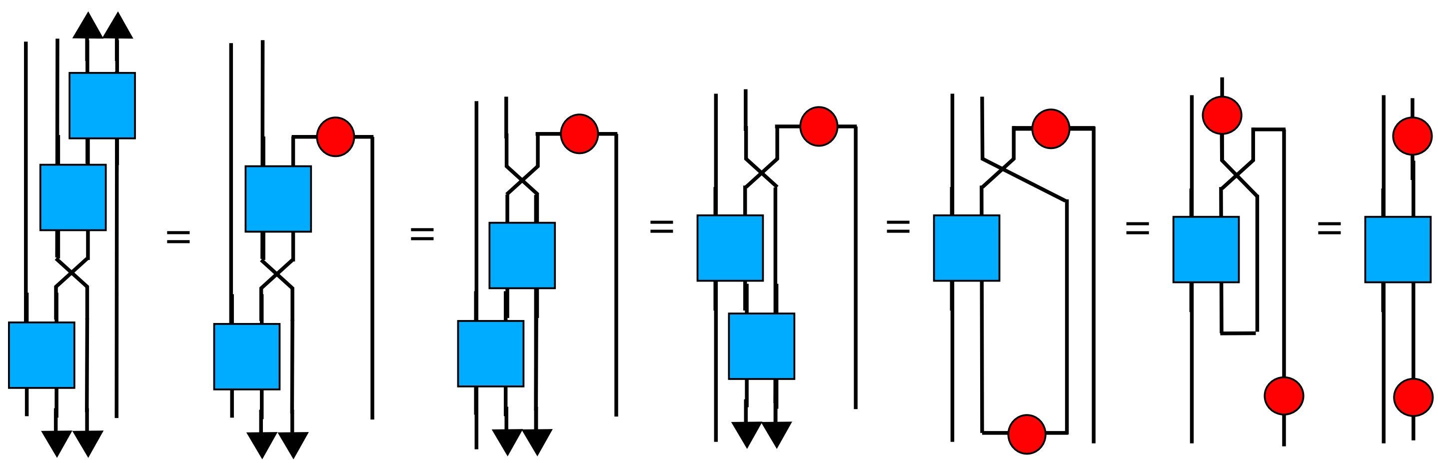
b
c
a
=
c
b
a

Example : CNOT

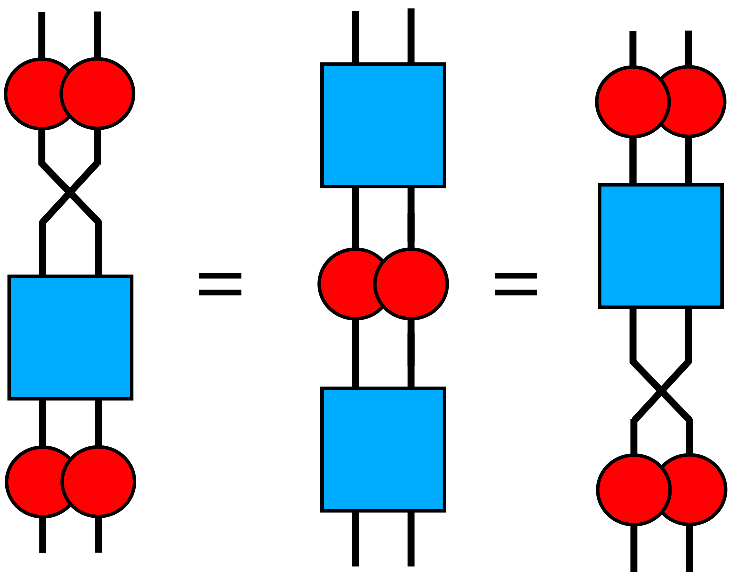
CNOT is computed by the pattern

$$M_3^0 M_2^0 E_{13} E_{23} E_{34}$$

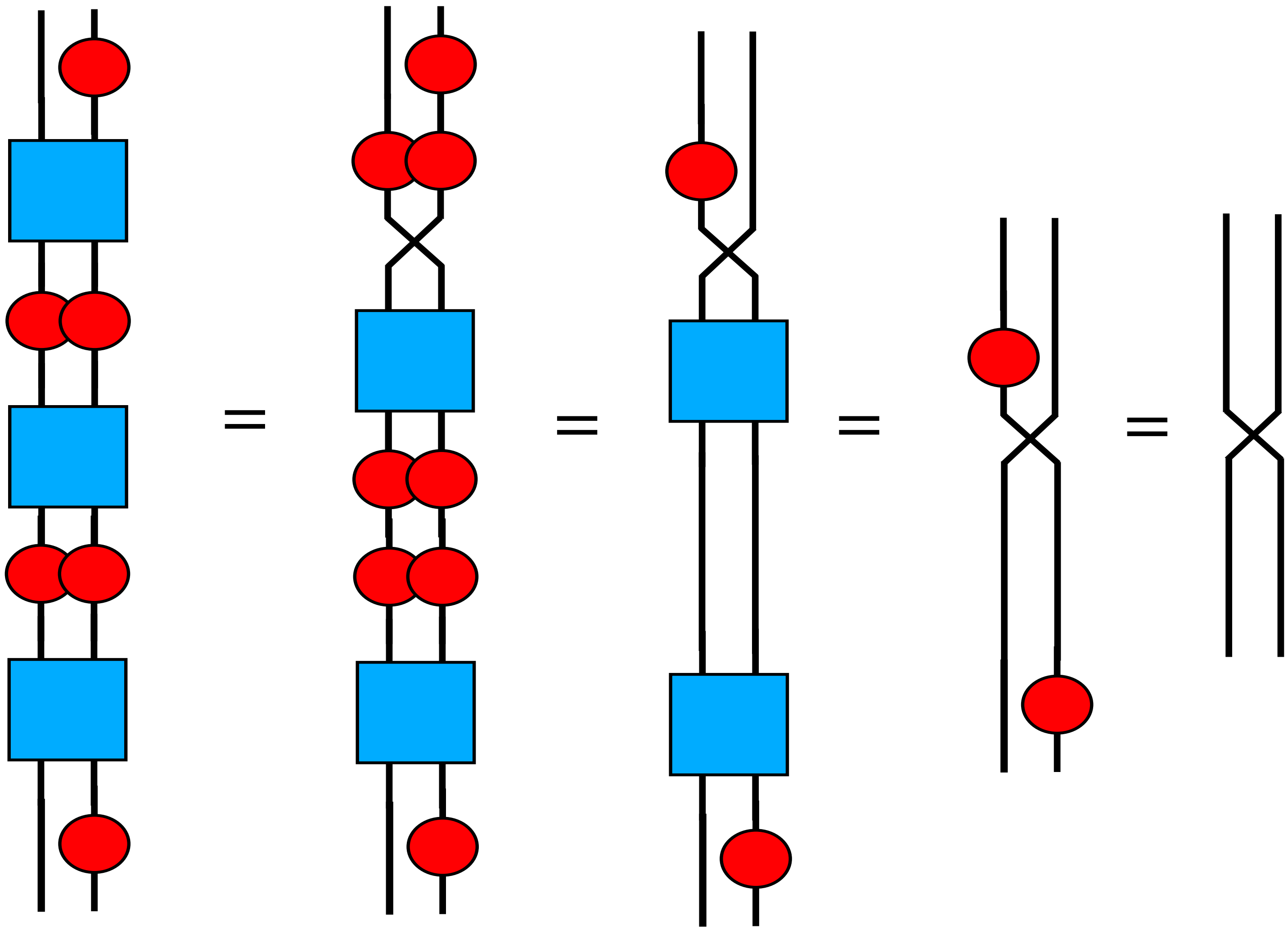
with inputs 1,2 and outputs 1,4.



Commuting Relations



The Swap Gate



Questions

- Is this set of equations complete?
- Is there a normalisation theorem?
- Is there a connection with the converse to the Flow theorem?