

# Exercise Sheet 1 for Categories, Proofs and Games: HT 2004

Samson Abramsky  
Oxford University Computing Laboratory

## Question 1

Consider the following properties of an arrow  $f$  in a category  $\mathcal{C}$ .

- $f$  is *monic* (*i.e.* a monomorphism) if for all arrows  $g$  and  $h$  (with domains and codomains such that the following equations make sense)

$$f \circ g = f \circ h \implies g = h.$$

- $f$  is *epic* if

$$g \circ f = h \circ f \implies g = h.$$

- $f$  is *iso* if for some  $g$ , both  $f \circ g$  and  $g \circ f$  are identity arrows.
- $f$  is *split monic* if for some  $g$ ,  $g \circ f$  is an identity arrow.
- $f$  is *split epic* if for some  $g$ ,  $f \circ g$  is an identity arrow.

- Prove that if  $f$  and  $g$  are arrows such that  $g \circ f$  is monic, then  $f$  is monic.
- Prove that, if  $f$  is split epic, then it is epic.
- Prove that, if  $f$  and  $g \circ f$  are iso, then  $g$  is iso.
- Prove that, if  $f$  is monic and split epic, then it is iso.
- Show that, in the category **Set**, an arrow is epic if and only if it is surjective.

## Question 2

Show that  $i : A \longrightarrow B$  has at most one inverse  $j : B \longrightarrow A$  making it an isomorphism. This justifies writing  $i^{-1}$  for *the* inverse of an isomorphism  $i$ .

### Question 3

Identify initial and terminal objects in the categories **Set**, **Mon**, **Vect<sub>k</sub>**, **Pos**. What does it mean to be an initial/terminal object in a preorder?

### Question 4

Repeat the previous exercise with respect to the notion of product.

### Question 5

Show that an  $A, B$ -pairing  $(A \times B, \pi_1, \pi_2)$  is a product if and only if for every  $A, B$ -pairing  $(C, f, g)$  there is a morphism

$$\langle f, g \rangle : C \longrightarrow A \times B$$

such that

$$\pi_1 \circ \langle f, g \rangle = f, \quad \pi_2 \circ \langle f, g \rangle = g$$

and for all  $h : C \longrightarrow A \times B$ :

$$\langle \pi_1 \circ h, \pi_2 \circ h \rangle = h.$$

(The main point is to show that the last equation is equivalent to asking for uniqueness explicitly).

### Question 6

Explicitly dualize the notion of product (the dual notion is called *coproduct*). Repeat exercise 3 with respect to coproducts.