Exercise Sheet 3 for Categories, Proofs and Games

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1. The notion of *universal arrow* is dual to that of couniversal arrow. We state it explicitly. Let $G : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor, and A an object of $\mathcal{C}$. A *universal arrow from A to G* is an object D of $\mathcal{D}$ and a morphism $f : A \longrightarrow GD$ such that, for every object D' of $\mathcal{D}$, and morphism $g : A \longrightarrow GD'$, there exists a unique morphism $h : D \longrightarrow D'$ such that $g = Gh \circ f$.

   - Show carefully that a coproduct of objects A, B of $\mathcal{C}$ is a universal arrow to the diagonal functor
   $$\Delta : \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}.$$
   - Let $U : \mathbf{Mon} \longrightarrow \mathbf{Set}$ be the 'forgetful' functor which sends a monoid to its set of elements. Show that for each set X, there is a universal arrow $X \longrightarrow UX^*$, where X^* is the monoid of finite sequences of elements of X, with concatenation as the binary operation.
   - (If you have not encountered rings, you should skip this part). Let **Ring** be the category with commutative rings with unit as objects, and ring homomorphisms as morphisms. Let $\mathbf{Ring}_*$ be the category where objects are pairs (R, a) where R is a ring, and $a \in R$; the morphisms from (R, a) to (S, b) are the ring homomorphisms $h : R \longrightarrow S$ such that $h(a) = b$. The functor $G : \mathbf{Ring}_* \longrightarrow \mathbf{Ring}$ simply sends (R, a) to R, 'forgetting' the specified element a. Show that for each ring R, there is a universal arrow from R to G. (Hint: polynomials!).

2. Assume we have a category $\mathcal{C}$ with a terminal object and binary products. Show that exponentials can be axiomatized in a purely equational fashion, as follows. For each pair of objects A, B, there is an object $A \Rightarrow B$ and a morphism
$$\mathsf{Ap}_{A,B} : (A \Rightarrow B) \times A \longrightarrow B$$
and for each morphism $f : C \times A \longrightarrow B$ there is a morphism $\Lambda(f) : C \longrightarrow (A \Rightarrow B)$ such that
$$f = \mathsf{Ap}_{A,B} \circ (\Lambda(f) \times \mathsf{id}_A).$$
Moreover, for each morphism $g : C \longrightarrow (A \Rightarrow B)$:
$$\Lambda(\mathsf{Ap}_{A,B} \circ (g \times \mathsf{id}_A)) = g.$$
Show that this is equivalent to the definition in terms of couniversal arrows given in the Notes.

3. Let X be a non-empty set. Consider the set H of all those sets of subsets Θ of X ($\Theta \subseteq P(P(X))$) with the following property:

$$U \in \Theta \land T \subseteq U \implies T \in \Theta.$$

This set H, ordered by inclusion, is a poset, and hence a category. Show that H is not closed under complements (and in fact is not a Boolean algebra), but that H is Cartesian closed.