Themes

- Abstract scalars.
- Free strongly compact closed categories.
- $\sim$ The logic of strongly compact closed categories, proof nets.

Scalars in monoidal categories

Monoidal category $(\mathcal{C}, \otimes, \mathrm{I})$. A scalar is a morphism $s : \mathrm{I} \to \mathrm{I}$.

Examples: $(\mathbf{FdVec}_{\mathbb{K}}, \otimes)$, $(\mathbf{Rel}, \times)$.

(1) $\mathcal{C}(\mathrm{I}, \mathrm{I})$ is a commutative monoid

$$\begin{tikzcd}
\mathrm{I} \arrow[r, equal] & \mathrm{I} \otimes \mathrm{I} \arrow[r, equal] & \mathrm{I} \otimes \mathrm{I} \arrow[r, equal] \arrow[d, "1 \otimes t"] & \mathrm{I} \arrow[d, "t"] \\
\mathrm{I} \arrow[u, "s"] \arrow[d, "t"'] \arrow[r, equal] & \mathrm{I} \otimes \mathrm{I} \arrow[u, "s \otimes 1"] \arrow[d, "1 \otimes t"'] \arrow[r, "s \otimes t"] & \mathrm{I} \otimes \mathrm{I} \arrow[r, equal] & \mathrm{I} \\
\mathrm{I} \arrow[r, equal] & \mathrm{I} \otimes \mathrm{I} \arrow[r, equal] & \mathrm{I} \otimes \mathrm{I} \arrow[u, "s \otimes 1"'] \arrow[r, equal] & \mathrm{I} \arrow[u, "s"']
\end{tikzcd}$$

(2) Each scalar $s : \mathrm{I} \to \mathrm{I}$ induces a natural transformation

$$s_A : A \xrightarrow{\simeq} \mathrm{I} \otimes A \xrightarrow{s \otimes 1_A} \mathrm{I} \otimes A \xrightarrow{\simeq} A\,.$$

$$\begin{tikzcd}
A \arrow[r, "s_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\
B \arrow[r, "s_B"'] & B
\end{tikzcd}$$

We write $s \bullet f$ for $f \circ s_A = s_B \circ f$. Note that

$$\begin{array}{rcl}
s \bullet (t \bullet f) & = & (s \circ t) \bullet f \\
(s \bullet g) \circ (r \bullet f) & = & (s \circ r) \bullet (g \circ f) \\
(s \bullet f) \otimes (t \bullet g) & = & (s \circ t) \bullet (f \otimes g)
\end{array}$$

Compact closed categories

A *compact closed category* is a symmetric monoidal category with, for each object A:

- a *dual* A^*,
- a *unit* $\eta_A : \mathrm{I} \longrightarrow A^* \otimes A$
- and a *counit* $\epsilon_A : A \otimes A^* \longrightarrow \mathrm{I}$.

Triangular identities:

$$\begin{tikzcd} A \arrow[r, "1_A \otimes \eta_A"] & A \otimes A^* \otimes A \arrow[r, "\epsilon_A \otimes 1_A"] & A & = & 1_A \end{tikzcd}$$

$$\begin{tikzcd} A^* \arrow[r, "\eta_A \otimes 1_A"] & A^* \otimes A \otimes A^* \arrow[r, "1_A \otimes \epsilon_A"] & A^* & = & 1_{A^*} \end{tikzcd}$$

"Every object (1-cell) has an adjoint"

But also: $*$-autonomous with $\otimes = \bindnasrepma$, $\perp = \mathrm{I}$.

Examples

- Sets, relations and cartesian product $(\mathbf{Rel}, \times)$. Here $\eta_X \subseteq \{*\} \times (X \times X)$ and we have

$$\eta_X = \epsilon_X^c = \{(*, (x, x)) \mid x \in X\}\,.$$

- Vector spaces over a field $\mathbb{K}$, linear maps and tensor product $(\mathbf{FdVec}_{\mathbb{K}}, \otimes)$. The unit and counit in $(\mathbf{FdVec}_{\mathbb{C}}, \otimes)$ are

$$\eta_V : \mathbb{C} \to V^* \otimes V :: 1 \mapsto \sum_{i=1}^{i=n} \bar{e}_i \otimes e_i$$

$$\epsilon_V : V \otimes V^* \to \mathbb{C} :: e_j \otimes \bar{e}_i \mapsto \langle \bar{e}_i \mid e_j \rangle$$

Duality, Names and Conames

For each morphism $f : A \longrightarrow B$ in a compact closed category we can construct a *dual* $f^* : A^* \longrightarrow B^*$:

$$f^* \quad = \quad B^* \xrightarrow{\eta_A \otimes 1} A^* \otimes A \otimes B^* \xrightarrow{1 \otimes f \otimes 1} A^* \otimes B \otimes B^* \xrightarrow{1 \otimes \epsilon_B} A^*$$

a *name*

$$\ulcorner f \urcorner : \mathrm{I} \to A^* \otimes B \quad = \quad \mathrm{I} \xrightarrow{\eta} A^* \otimes A \xrightarrow{1 \otimes f} A^* \otimes B$$

and a *coname*

$$\llcorner f \lrcorner : A \otimes B^* \to \mathrm{I} \quad = \quad A \otimes B^* \xrightarrow{f \otimes 1} B \otimes B^* \xrightarrow{\epsilon} \mathrm{I}$$

The assignment $f \mapsto f^*$ extends $A \mapsto A^*$ into a contravariant endofunctor with $A \simeq A^{**}$. In any compact closed category, we have

$$\mathcal{C}(A \otimes B^*, \mathrm{I}) \simeq \mathcal{C}(A, B) \simeq \mathcal{C}(\mathrm{I}, A^* \otimes B).$$

Why compact closure does not suffice

In inner-product spaces we have the adjoint:

$$\frac{A \xrightarrow{f} B}{A \xleftarrow{f^\dagger} B} \qquad\qquad \langle f\phi \mid \psi\rangle_B = \langle \phi \mid f^\dagger \psi\rangle_A$$

N.B. not the same as the dual.

In "degenerate" CCC's in which $A^* = A$, e.g. **Rel**, real inner-product spaces, we have $f^* = f^\dagger$.

In complex inner-product spaces, Hilbert spaces, the inner product is sesquilinear

$$\langle \psi \mid \phi \rangle = \overline{\langle \phi \mid \psi \rangle}$$

and the isomorphism $A \simeq A^*$ is not linear, but conjugate linear:

$$\langle \lambda \bullet \phi \mid - \rangle \quad = \quad \bar{\lambda} \bullet \langle \phi \mid - \rangle$$

and hence does not live in the category **Hilb** at all!

Solution: Strong Compact Closure

The conjugate space of a Hilbert space $\mathcal{H}$: same additive group of vectors, scalar multiplication and inner product “twisted” by complex conjugation:

$$\alpha \bullet_{\bar{\mathcal{H}}} \phi := \bar{\alpha} \bullet_{\mathcal{H}} \phi \qquad \langle \phi \mid \psi \rangle_{\bar{\mathcal{H}}} := \langle \psi \mid \phi \rangle_{\mathcal{H}}$$

We can define $\mathcal{H}^* = \bar{\mathcal{H}}$, since $\mathcal{H}$, $\bar{\mathcal{H}}$ have the same orthornormal bases, and we can define the counit by

$$\epsilon_{\mathcal{H}} : \mathcal{H} \otimes \bar{\mathcal{H}} \to \mathbb{C} :: \phi \otimes \psi \mapsto \langle \psi \mid \phi \rangle_{\mathcal{H}}$$

which is (bi)linear!

The crucial observation is this: $()^*$ has a covariant functorial extension $f \mapsto f_*$, which is essentially identity on morphisms; and then we can define

$$f^\dagger = (f^*)_* = (f_*)^*.$$

Axiomatization of Strong Compact Closure

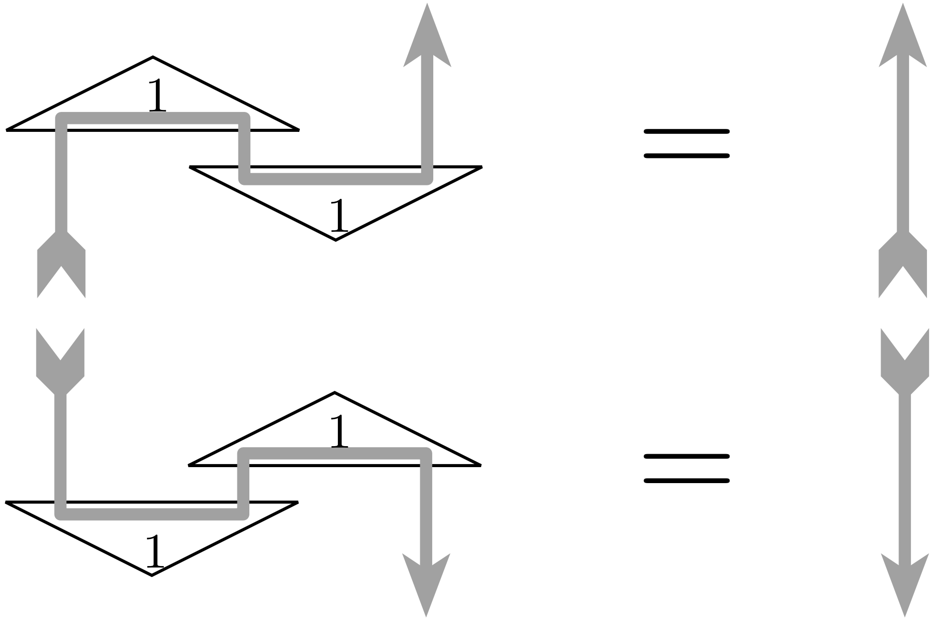
It suffices to require the following structure on a symmetric monoidal category $(\mathcal{C}, \otimes, \mathrm{I}, \sigma)$:

- A monoidal involutive assignment $()^*$ on objects.
- An identity on objects, contravariant, involutive, strictly monoidal functor $()^\dagger$ (so we take the adjoint as primitive).
- An assignment of units $\eta_A : \mathrm{I} \longrightarrow A^* \otimes A$ such that $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$.

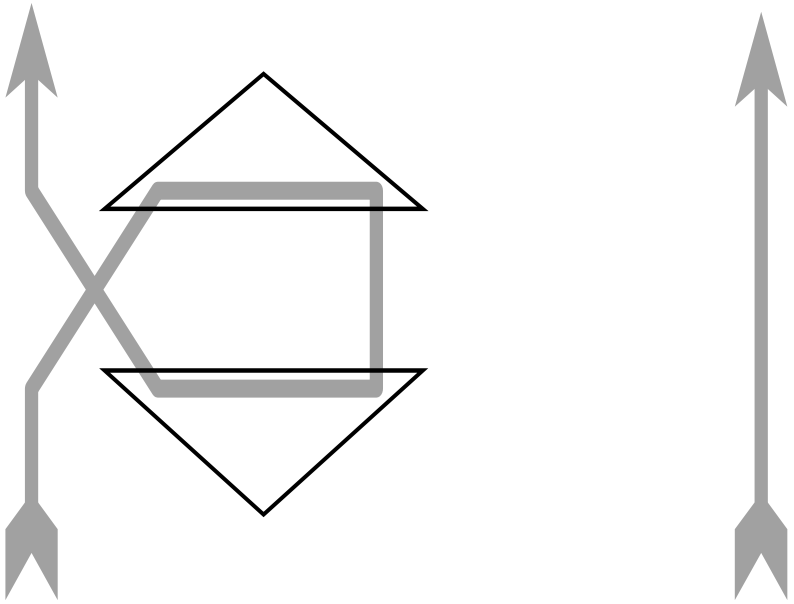
We can then define $\epsilon_A = \eta_A^\dagger \circ \sigma_{A,A^*}$, and we need only one triangular identity, which can be given in the form of a Yanking axiom:

$$A \xrightarrow{\eta_A \otimes 1_A} A^* \otimes A \otimes A \xrightarrow{1 \otimes \sigma_{A,A}} A^* \otimes A \otimes A \xrightarrow{\eta_A^\dagger \otimes 1} A \quad = \quad 1_A$$

Standard triangular identities diagrammatically



Yanking diagrammatically



Free Constructions

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, shift left=2, "F_S"] \arrow[r, phantom, "\perp"] & \mathbf{S-Cat} \arrow[l, shift left=2, "U_S"]
\end{tikzcd}$$

M	monoidal	lists
SM	symmetric monoidal	permutations
Tr	traced symmetric monoidal	loops
CC	compact closed	polarities
SCC	strong compact closed	reversals

Take $F_S(\mathbf{1})$ for 'pure' picture (one generator, no relations).

Monoidal Categories

Objects of $F_M(\mathcal{C})$: lists of objects of $\mathcal{C}$. Monoidal structure given by concatenation; the tensor unit I is the empty sequence.

Arrows:

$$\begin{tikzcd}
A_1 \arrow[d, "f_1"'] & A_2 \arrow[d, "f_2"'] & \cdots & A_n \arrow[d, "f_n"] \\
B_1 & B_2 & \cdots & B_n
\end{tikzcd}$$

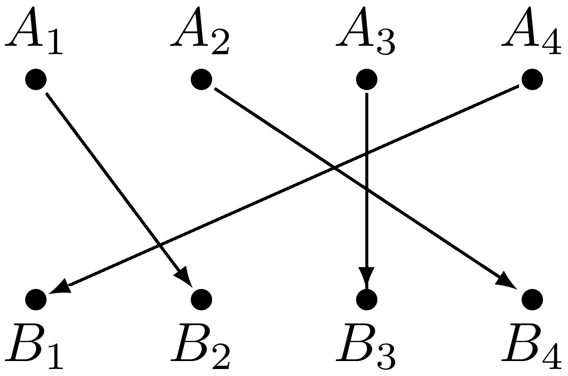
$$f_i : A_i \to B_i$$

$F_M(\mathbf{1}) = (\mathbb{N}, =, +, 0)$.

Symmetric Monoidal Categories

Same objects as in monoidal case.

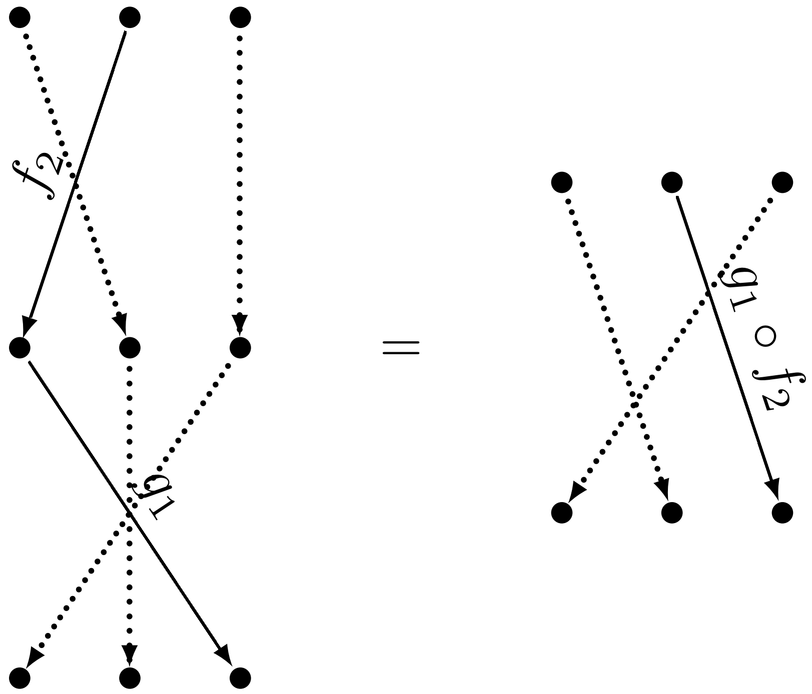
An arrow $[A_1, \ldots, A_n] \longrightarrow [B_1, \ldots, B_n]$ is given by (π, λ), where $\pi \in S(n)$, and $\lambda(i) : A_i \rightarrow B_{\pi(i)}$, $1 \leq i \leq n$.



$F_{SM}(\mathbf{1}) = \coprod_n S(n)$.

Composition in $F_{SM}(\mathcal{C})$

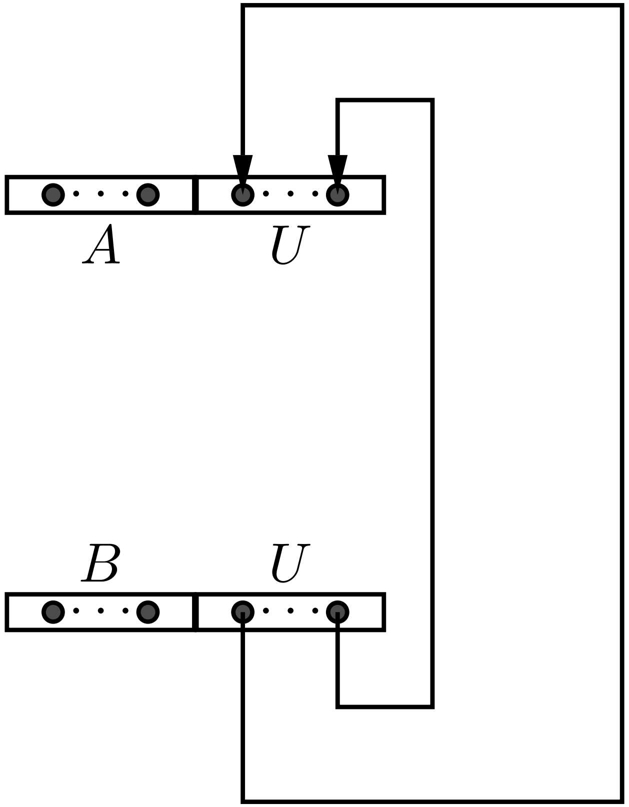
Form paths of length 2, compose the arrows from $\mathcal{C}$ labelling these paths.



Traced Symmetric Monoidal Categories

Feedback operation (or “tracing out” part of a morphism, *cf.* contraction of tensors):

$$\frac{A \otimes U \xrightarrow{\quad f \quad} B \otimes U}{A \xrightarrow{\quad \mathsf{Tr}^{U}_{A,B}(f) \quad} B}$$



Is $F_{SM}(\mathcal{C})$ traced? Yes!

$$|A| + |U| = |B| + |U| \;\;\Rightarrow\;\; |A| = |B|$$

Any path starting in A will pass some number (maybe 0) of times through U, but can never revisit any node in U, hence must eventually land in B. (Note that there may very well be cycles starting in U!) Moreover, any orbits starting from distinct nodes in A must be disjoint, hence must end in different places in B. So by following paths, we end up with a well-defined bijection between A and B. Composing the sequences of arrows labelling each path, we get a morphism in $F_{SM}(\mathcal{C})$ from A to B, as required.

However ...

$F_{SM}(\mathcal{C})$ is not the free traced monoidal category over $\mathcal{C}$. Note that $F_{SM}(\mathcal{C})(\mathrm{I},\mathrm{I}) = 1_{\mathrm{I}}$: this category only has one scalar!

So given $f : A \rightarrow A$, we are forced to assign $\mathsf{Tr}^{A}_{\mathrm{I},\mathrm{I}}(1 \otimes f) = 1_{\mathrm{I}}$: all loops are collapsed to have the same value.

In any traced monoidal category, given

$$f : A \otimes V \rightarrow B \otimes V, \qquad g : W \rightarrow W$$

we have

$$\mathsf{Tr}^{V \otimes W}_{A,B}(f \otimes g) = s \bullet \mathsf{Tr}^{V}_{A,B}(f),$$

where $s = \mathsf{Tr}^{W}_{\mathrm{I},\mathrm{I}}(g)$. So in the free case, our previous construction is what the trace must be, up to evaluation of loops as scalars.

Loops

The loops of a category $\mathcal{C}$, written $\mathcal{L}[\mathcal{C}]$, are the endomorphisms of $\mathcal{C}$ quotiented by the following equivalence relation: a composite

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] & A_2 \arrow[r, "f_2"] & \cdots & A_k \arrow[r, "f_k"] & A_1 \end{tikzcd}$$

is equated with all its cyclic permutations. A trace function on $\mathcal{C}$ is a map on the endomorphisms of $\mathcal{C}$ which respects this equivalence.

Description of $F_{\mathsf{Tr}}(\mathcal{C})$

Objects as for $F_{SM}(\mathcal{C})$. A morphism is now (π, λ, μ), where (π, λ) are as in $F_{SM}(\mathcal{C})$, and μ is a multiset of loops in $\mathcal{L}[\mathcal{C}]$, *i.e.* an element of $\mathcal{M}(\mathcal{L}[\mathcal{C}])$, the free commutative monoid generated by $\mathcal{L}[\mathcal{C}]$.

Composition of $(\pi_1, \lambda_1, \mu_1)$ with $(\pi_2, \lambda_2, \mu_2)$ extends the definition for $F_{SM}(\mathcal{C})$ by taking the multiset union of μ_1 and μ_2.

The trace is defined as in our first attempt, but in general new loops will be formed, and must be added to the multiset.

Note that $F_{\mathsf{Tr}}(\mathcal{C})(\mathrm{I}, \mathrm{I}) = \mathcal{M}(\mathcal{L}[\mathcal{C}])$.

Compact Closed Categories

We know these are constructed freely from traced categories by the $\mathcal{G}$ or (Int) construction of Joyal-Street-Verity.

Adjoints compose, so $F_{CC}(\mathcal{C}) = \mathcal{G} \circ F_{\mathsf{Tr}}(\mathcal{C})$. We can easily relate this to Kelly-Laplaza's description of $F_{CC}(\mathcal{C})$.

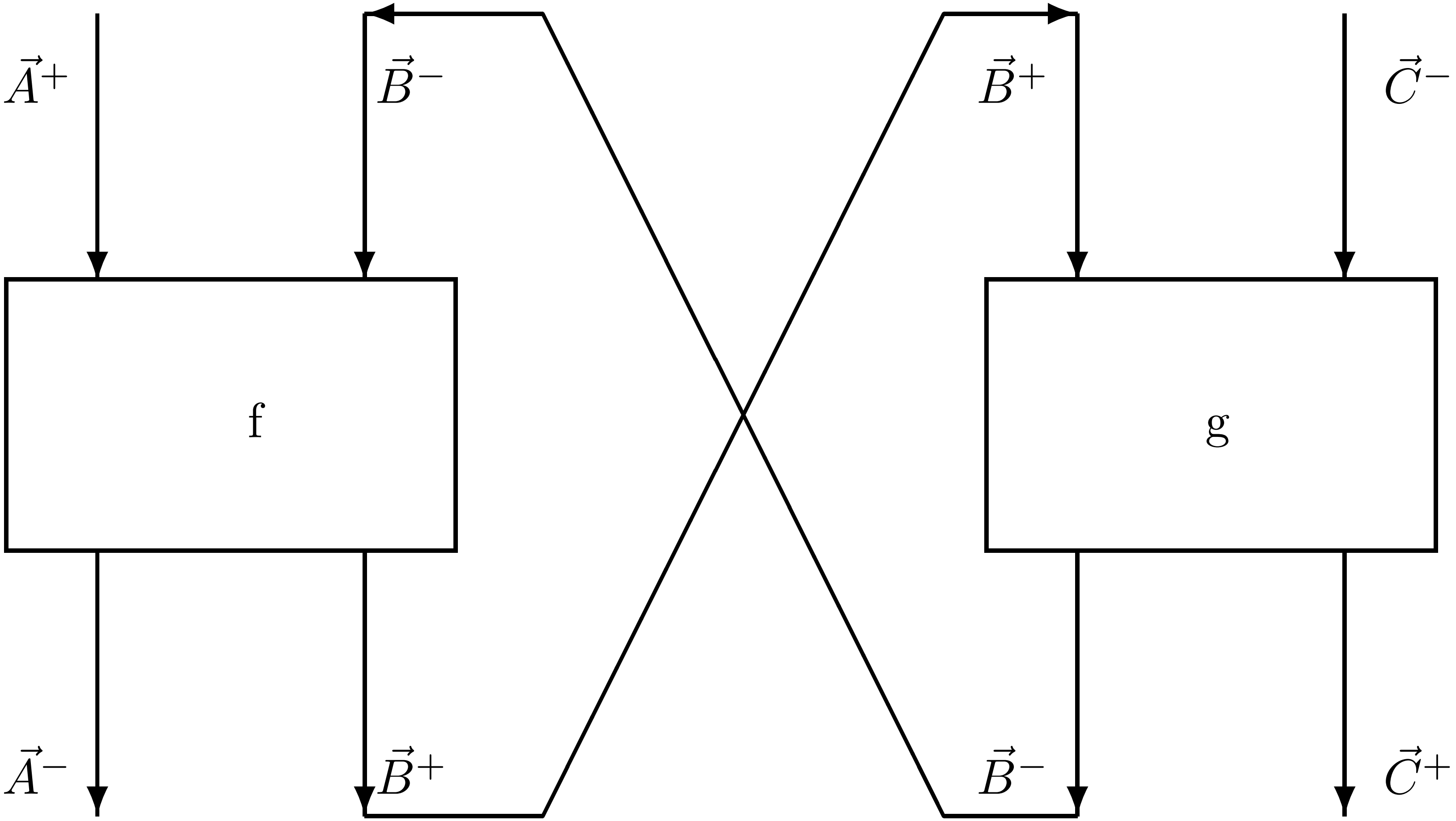
Objects are now polarized; each dot is labelled with $+$ or $-$, as well as an object from $\mathcal{C}$. $()^*$ flips polarities. Thus we can write an object as $(\vec{A}^+, \vec{A}^-)$, where we partition the elements into those labelled $+$ or $-$.

A morphism $(\vec{A}^+, \vec{A}^-) \longrightarrow (\vec{B}^+, \vec{B}^-)$ has the form (β, λ, μ), where:

- $\beta : \vec{A}^+\vec{B}^- \xrightarrow{\cong} \vec{A}^-\vec{B}^+$ is a signed bijection
- $\lambda(i) : C_i \longrightarrow D_{\beta(i)}$ is a $\mathcal{C}$-arrow, where C_i is the i'th $-$ve object, and D_j is the j'th $+$ve object.
- μ is a multiset of loops, as in $F_{\mathsf{Tr}}(\mathcal{C})$.

Compact Closed Categories Ctd.

Composition is given by the 'execution formula' (already in Kelly-Laplaza, but not easy to spot!): *i.e.* , chase paths, and compose (in $\mathcal{C}$) the morphisms labelling the paths to get the labels. In general, loops will be formed, and must be added to the multiset.


$\vec{A}^+$
$\vec{B}^-$
$\vec{B}^+$
$\vec{C}^-$
f
g
$\vec{A}^-$
$\vec{B}^+$
$\vec{B}^-$
$\vec{C}^+$

Note that identities, units and counits are really all the same(!), except that the polarities allow variables to be transposed freely between the domain and codomain.

Identity:

$$\bullet^{+} \xrightarrow[\vdash]{1} \bullet^{+}$$

Unit:

$$\vdash \qquad \bullet^{-} \xrightarrow{1} \bullet^{+}$$

Counit:

$$\bullet^{+} \xrightarrow{1} \bullet^{-} \qquad \vdash$$

Strongly Compact Closed Categories

We describe an adjunction

$$\begin{tikzcd}\mathbf{InvCat} \arrow[r, shift left=2, "F_S CC"] \arrow[r, phantom, "\perp"] & \mathbf{SCC-Cat} \arrow[l, shift left=2, "U_S CC"]\end{tikzcd}$$

InvCat: categories with a specified involution, *i.e.* an identity on objects, contravariant, involutive functor.

Our previous construction of F_{CC} lifts directly to this setting. The main point is to define $()^\dagger$ on $F_{CC}(\mathcal{C})$, under the assumption that we are given a primitive $()^\dagger$ on the generating category $\mathcal{C}$.

Given

$$(\beta, \lambda, \mu) : (\vec{A}^+, \vec{A}^-) \rightarrow (\vec{B}^+, \vec{B}^-),$$

we can define

$$(\beta, \lambda, \mu)^\dagger = (\beta^{-1}, ()^\dagger \circ \lambda, \mu^\dagger)$$

Here

$$\beta^{-1} : \vec{B}^{+}\vec{A}^{-} \xrightarrow{\cong} \vec{B}^{-}\vec{A}^{+},$$

and if

$$\begin{tikzcd} C_i \arrow[r, "f"] & D_{\beta(i)=j} \end{tikzcd}$$

then

$$\begin{tikzcd} D_j \arrow[r, "f^\dagger"] & C_{\beta^{-1}(j)=i}. \end{tikzcd}$$

Parameterizing on the monoid

There is a forgetful functor $U_{\mathcal{V}} : \mathbf{SCC{-}Cat} \longrightarrow \mathcal{V}$ where $\mathcal{V}$ has objects $(\mathcal{C}, M, \tau)$:

- $\mathcal{C}$ is a category with involution
- M is a commutative monoid with an involution
- $\tau : \mathcal{L}[\mathcal{C}] \to M$ is a trace function respecting the involution.

We can construct an adjunction

$$\begin{tikzcd} \mathcal{V} \arrow[r, shift left=2] \arrow[r, phantom, "\perp"] & \mathbf{SCC{-}Cat} \arrow[l, shift left=2, "U_{\mathcal{V}}"] \end{tikzcd}$$

which builds the free SCC on a category with prescribed scalars. (For example, we can force the scalars to be the complex numbers). This essentially acts by composition with the trace function τ on the free construction F_{SCC} given previously.