Relating coalgebraic notions of bisimulation
with applications to name-passing process calculi
(extended abstract)

Sam Staton

Computer Laboratory, University of Cambridge

Abstract. A labelled transition system can be understood as a coalgebra for a particular endofunctor on the category of sets. Generalizing, we are led to consider coalgebras for arbitrary endofunctors on arbitrary categories.

Bisimulation is a crucial notion in the theory of labelled transition systems. We identify four definitions of bisimulation on general coalgebras. The definitions all specialize to the same notion for the special case of labelled transition systems. We investigate general conditions under which the four notions coincide.

As an extended example, we consider the semantics of name-passing process calculi (such as the pi-calculus), and present a new coalgebraic model for name-passing calculi.

1 Introduction

Notions of bisimulation play a central role in the theory of transition systems. As different kinds of system are encountered, different notions of bisimulation arise, but the same questions are posed: Is there a fixed-point characterization for the maximal bisimulation, bisimilarity? Is there a minimal system, where bisimilar states are equated? And is there a procedure for constructing a minimal system, or for verifying bisimilarity?

The theory of coalgebras provides a setting in which different notions of transition system can be understood at a general level. In this paper we investigate notions of bisimulation at this general level, and determine how and when these questions can be answered.

To explain the generalization from transition systems to coalgebras, we begin with the traditional model of a labelled transition system,

$$(X,\ (\rightarrow_X) \subseteq X \times L \times X)$$

(for some set L of labels). A labelled transition system can be considered coalgebraically as a set X of states equipped with a 'next-state' function $X \to \mathcal{P}(L \times X)$. (Here, $\mathcal{P}$ is the powerset operator.) Generalizing, we are led to consider an arbitrary category $\mathcal{C}$ and an endofunctor B on it; then a coalgebra is an object X in $\mathcal{C}$ of 'states', and a 'next-state' morphism $X \to \mathrm{B}(X)$.

Coalgebras in different categories. The generalization to different endofunctors on different categories has proved appropriate in various settings: concepts from modal logic have been studied in terms of coalgebras over Stone spaces [2, 6, 30]; basic process calculi with recursion can be described using coalgebras over categories of domains [29, 37]; stochastic transition systems have been studied in terms of coalgebras over metric and measurable spaces [8, 12, 44, 45].

In this paper we revisit work [15, 17, 21, 42] on models of name-passing calculi, such as the π-calculus, where it is appropriate to work in a sheaf topos. Endofunctors that describe transition-system-like behaviour often decompose as $\mathrm{B} = P \circ \mathrm{B}'$, where B' is an endofunctor of a particularly simple form, and P is a powerset functor for a class of small maps, in the sense of algebraic set theory. A contribution of the present work is the introduction of a powerset that is appropriate for name-passing. It arises by combining the theory of semilattices with a theory of name-equality testing.

Notions of bisimulation. Once coalgebras are understood as generalized transition systems, we can consider bisimulation relations for these systems. Recall that, for labelled transition systems $(X, \rightarrow_X)$ and $(Y, \rightarrow_Y)$, a relation $R \subseteq X \times Y$ is a (strong) *bisimulation* if, whenever $x\,R\,y$, then for all $l \in L$:

- For $x \in X$, if $x \xrightarrow{l}_X x'$ then there is $y' \in Y$ such that $y \xrightarrow{l}_Y y'$ and $x'\,R\,y'$;
- For $y \in Y$, if $y \xrightarrow{l}_Y y'$ then there is $x' \in X$ such that $x \xrightarrow{l}_X x'$ and $x'\,R\,y'$.

In this article, we identify four notions of bisimulation that have been proposed in the context of coalgebras for endofunctors on arbitrary categories. Aczel and Mendler [3] introduced two definitions; we also consider a definition due to Hermida and Jacobs [23]; and lastly a definition that gives an immediate connection with 'final coalgebra' semantics.

The four notions coincide for the particular case of labelled transition systems. We investigate conditions under which the notions are related in the more general setting of coalgebras.

Relationship with the terminal sequence. Various authors have constructed terminal coalgebras as a limit of a transfinite sequence, beginning

$$1 \xleftarrow{!} B(1) \xleftarrow{B(!)} B(B(1)) \xleftarrow{B(B(!))} B(B(B(1))) \leftarrow \cdots \leftarrow \cdots \quad .$$

On the other hand, bisimulations can often be characterized as post-fixed points of a monotone operator Φ on a lattice of relations, so that a maximal bisimulation ('bisimilarity') arises as a limit of a transfinite sequence, beginning

$$X \times Y \supseteq \Phi(X \times Y) \supseteq \Phi(\Phi(X \times Y)) \supseteq \Phi(\Phi(\Phi(X \times Y))) \supseteq \cdots \supseteq \cdots \quad .$$

These sequences suggest algorithms for minimizing and computing bisimilarity for arbitrary coalgebras (see e.g. [14]). We investigate conditions under which the steps of the terminal coalgebra sequence are precisely related with the steps of this relation refinement sequence.

Other approaches not considered. In this article we are concerned with (internal) relations between the carrier objects of two fixed coalgebras: a relation is an object of the base category. Some authors (e.g. [12]) instead work with an equivalence relation on the class of all coalgebras, setting two coalgebras as 'bisimilar' if there is a span of surjective homomorphisms between them. Others work with relations as bimodules [38, 8]. We will not discuss these approaches here.

Outline. This paper is structured as follows. In Section 2 we recall some examples of coalgebras for endofunctors. We recall the four notions of bisimulation in Section 3. In Section 4 we investigate how the different notions of bisimulation are related. In Section 5 we investigate the connection between the terminal sequence and the relation refinement sequence. Finally, in Section 6, we provide a novel analysis of models of name-passing calculi.

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Many of the results in Sections 4 and 5 are well-known where $\mathcal{C} = \mathbf{Set}$. In other cases, some results are probably folklore; I have tried to ascribe credit where it is due.

2 Coalgebras: definition and examples

Recall the definition of a coalgebra for an endofunctor:

Definition 1. *Consider an endofunctor* B *on a category* $\mathcal{C}$*. A* B-coalgebra *is given by an object* X *of* $\mathcal{C}$ *together with morphism* $X \to \mathrm{B}(X)$ *in* $\mathcal{C}$*. A* homomorphism *of* B*-coalgebras, from* (X, h) *to* (Y, k)*, is a morphism* $f : X \to Y$ *that respects the coalgebra structure, i.e. such that* $\mathrm{B}f \circ h = k \circ f$*.*

We collect some examples of structures that arise as coalgebras for endofunctors. For further motivation, see [4, 24, 39].

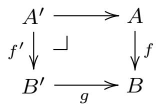
Streams and coinductive datatypes. Let A be a set, and consider the endofunctor $(A \times (-) + 1)$ on the category **Set** of sets. Coalgebras for this endofunctor are a form of stream, of elements of A. Other coinductive datatypes arise from other 'polynomial' endofunctors, built out of constants, sums and products (see e.g. [39, Sec. 10]).

Labelled transition systems. In the introduction, we discussed how labelled transition systems correspond to coalgebras for the endofunctor $\mathcal{P}(L \times (-))$. Here, $\mathcal{P}$ is the powerset functor, that acts by direct image. For finite non-determinism, and image-finite transition systems, one can instead consider the endofunctor $\mathcal{P}_{\mathrm{f}}(L \times (-))$ where $\mathcal{P}_{\mathrm{f}}$ is the finite powerset functor.

Axioms for a class $\mathcal{S}$ of small maps in a regular category:

(A1) $\mathcal{S}$ is closed under composition, and all identity morphisms are in $\mathcal{S}$.

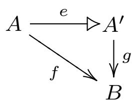
(A2) $\mathcal{S}$ is stable under pullback: in dgm. 1, if $f \in \mathcal{S}$, then $f' \in \mathcal{S}$.



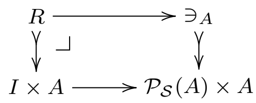
(1)

(A3) In dgm. 1, if $f' \in \mathcal{S}$ and g is a cover (aka strong epi), then $f \in \mathcal{S}$.

(A6) In the following triangle, if $f \in \mathcal{S}$ and e is a cover, then $g \in \mathcal{S}$.



(P1) For every object A of $\mathcal{C}$ there is an $\mathcal{S}$-relation $(\mathcal{P}_\mathcal{S}(A) \leftarrow \ni_B \rightarrow A)$ such that for every $\mathcal{S}$-relation $(I \leftarrow R \rightarrow A)$ there is a unique morphism $I \rightarrow P(A)$ making the following diagram a pullback.



Additional axioms for an extensive regular category:

(A4) Maps $0 \rightarrow 1$ and $1+1 \rightarrow 1$ are in $\mathcal{S}$.

(A5) If $A \rightarrow A'$ and $B \rightarrow B'$ are in $\mathcal{S}$, then so is $(A + B) \rightarrow (A' + B')$.

Additional axioms (optional):

(A7) (*Collection*, assuming (P1)) The endofunctor $\mathcal{P}_\mathcal{S}$ preserves covers.

(M) All monos in $\mathcal{C}$ are in $\mathcal{S}$.

Fig. 1. Axioms for small maps (see [27]; we do not need representability here.) Recall that a category is regular if it has finite limits and stable cover-image factorizations (e.g. [7, Vol. 2, Ch. 2], [26, A1.3]), and that it is extensive if it has finite sums that are stable under pullback (e.g. [10]). An extensive regular category is sometimes called positive (e.g. [26, A1.4]).

Powersets and small maps. A general treatment of powersets is suggested by algebraic set theory [27]. A model of algebraic set theory is a category $\mathcal{C}$ together with a class of 'small' maps $\mathcal{S}$ in $\mathcal{C}$, all subject to certain conditions — see Figure 1. In such a situation, an *$\mathcal{S}$-relation* is a jointly-monic span $(X \leftarrow R \rightarrow Y)$ of which the projection $R \rightarrow X$ is in $\mathcal{S}$. Axiom (P1) entails that there is an endofunctor $\mathcal{P}_\mathcal{S}$ on $\mathcal{C}$ that classifies $\mathcal{S}$-relations.

For instance, in the category of sets, we can say that a function $f : X \rightarrow Y$ is small if for every $y \in Y$ the set $\{x \in X \mid f(x) = y\}$ is finite. An $\mathcal{S}$-relation is precisely an image-finite one, and the $\mathcal{S}$-powerset is the finite powerset, $\mathcal{P}_\mathrm{f}$.

For more complex examples, see Section 6 and [5].

Continuous-state systems. For systems with continuous state spaces it is appropriate to work in a category of topological spaces. For a first example, consider a Stone space X, and let $K(X)$ be the space of compact subsets of X, with the finite (aka Vietoris) topology. The endofunctor K on the category of Stone spaces has attracted interest because of a duality with modal logics (e.g.[2, 30]).

Continuous phenomena also arise in other application areas. For recursively defined systems, it is reasonable to investigate coalgebras for powerdomain constructions on a category of domains (see e.g. [29, 37, 1]). For continuous stochastic systems, researchers have investigated coalgebras for probability distribution functors on categories of metric or measurable spaces (see e.g. [8, 12, 44, 45]).

3 Four definitions of bisimulation

We now recall four coalgebraic notions of bisimulation from the literature.

Context. Throughout this section, we consider an endofunctor B on a category $\mathcal{C}$. We assume that $\mathcal{C}$ has finite limits and images (i.e. (strong-epi/mono) factorizations; see e.g. [7, Vol. 2, Ch. 2], [26, A1.3]). We fix two B-coalgebras, $h : X \to \mathrm{B}(X)$ and $k : Y \to \mathrm{B}(Y)$. We write $\mathrm{Rel}(X, Y)$ for the preorder of relations, viz. jointly-monic spans $(X \leftarrow R \to Y)$.

3.1 The lifting-span bisimulation of Aczel and Mendler

Definition 2 (following [3]). *A relation R in* $\mathrm{Rel}(X, Y)$ *is an* AM-bisimulation *if there is a* B*-coalgebra structure* $R \to \mathrm{B}(R)$ *making the following diagram commute.*

$$\begin{array}{ccccc} X & \longleftarrow & R & \longrightarrow & Y \\ {\scriptstyle h}\downarrow & & \vdots & & \downarrow{\scriptstyle k} \\ \mathrm{B}(X) & \longleftarrow & \mathrm{B}(R) & \longrightarrow & \mathrm{B}(Y) \end{array}$$

3.2 The relation-lifting bisimulation of Hermida and Jacobs

For any relation R in $\mathrm{Rel}(X, Y)$, the 'relation lifting' $\bar{B}(R)$ in $\mathrm{Rel}(\mathrm{B}(X), \mathrm{B}(Y))$ is the image of the composite morphism $\mathrm{B}(R) \to \mathrm{B}(X \times Y) \to \mathrm{B}(X) \times \mathrm{B}(Y)$.

Definition 3 (following [23]). *A relation R in* $\mathrm{Rel}(X, Y)$ *is an HJ-bisimulation if there is a morphism* $R \to \bar{B}(R)$ *making the left-hand diagram below commute.*

$$\begin{array}{ccccc} X & \longleftarrow & R & \longrightarrow & Y \\ {\scriptstyle h}\downarrow & & \vdots & & \downarrow{\scriptstyle k} \\ \mathrm{B}(X) & \longleftarrow & \bar{B}(R) & \longrightarrow & \mathrm{B}(Y) \end{array} \qquad \begin{array}{ccc} \Phi^{\mathrm{HJ}}(R) & \longrightarrow & \bar{B}(R) \\ \downarrow & \lrcorner & \downarrow \\ X \times Y & \xrightarrow[h \times k]{} & \mathrm{B}(X) \times \mathrm{B}(Y) \end{array}$$

Equivalently: let the right-hand diagram be a pullback; R is an HJ-bisimulation iff $R \leq \Phi^{\mathrm{HJ}}(R)$ in $\mathrm{Rel}(X, Y)$.

Proposition 4. *The operator* Φ^{HJ} *on* $\mathrm{Rel}(X, Y)$ *is monotone.*

For an example, return to the situation where $\mathcal{C} = \mathbf{Set}$ and $\mathrm{B} = \mathcal{P}(L \times -)$. For any relation R in $\mathrm{Rel}(X, Y)$, the refined relation $\Phi^{\mathrm{HJ}}(R)$ is the set of all pairs $(x, y) \in X \times Y$ for which

$$\begin{array}{l} \text{(i) } \forall (l, x') \in h(x).\ \exists y' \in Y.\ (l, y') \in k(y) \text{ and } (x', y') \in R\ ; \\ \text{(ii) } \forall (l, y') \in k(y).\ \exists x' \in X.\ (l, x') \in h(x) \text{ and } (x', y') \in R \quad . \end{array}$$

Thus Φ^{HJ} is the construction $\mathcal{F}$ considered by Milner in [35, Sec. 4].

3.3 The congruences of Aczel and Mendler

Definition 5 (following [3]). *A relation R in* $\mathrm{Rel}(X,Y)$ *is an* AM-precongruence *if for every cospan* $(X \xrightarrow{i} Z \xleftarrow{j} Y)$,

$$\text{if} \quad \begin{array}{ccccc} & & X & & \\ & \nearrow & & \searrow^{i} & \\ R & & & & Z \\ & \searrow & & \nearrow_{j} & \\ & & Y & & \end{array} \quad \textit{commutes then so does} \quad \begin{array}{ccccc} & & X \xrightarrow{h} \mathrm{B}(X) & & \\ & \nearrow & & \searrow^{\mathrm{B}(i)} & \\ R & & & & \mathrm{B}(Z) \\ & \searrow & & \nearrow_{\mathrm{B}(j)} & \\ & & Y \xrightarrow[k]{} \mathrm{B}(Y) & & \end{array}$$

When $\mathcal{C}$ has pushouts, then we let $\Phi^{\mathrm{AM}}(R)$ be the pullback of the cospan $(X \xrightarrow{h} \mathrm{B}X \xrightarrow{i} \mathrm{B}(X +_R Y) \xleftarrow{j} \mathrm{B}Y \xleftarrow{k} Y)$. By definition, a relation R is an AM-precongruence iff $R \leq \Phi^{\mathrm{AM}}(R)$.

This definition differs from that of [3] in that we consider relations between different coalgebras. If $(X,h) = (Y,k)$, then an equivalence relation is an AM-precongruence exactly when it is a congruence in the sense of [3].

Proposition 6. *The operator Φ^{AM} on* $\mathrm{Rel}(X,Y)$ *is monotone.*

(N.B. Φ^{AM} is different from Φ^{HJ}, even when B is the identity functor on **Set**.)

3.4 Terminal coalgebras and kernel-bisimulations

Suppose for a moment that there is a terminal B-coalgebra, $(Z, z\colon Z \to \mathrm{B}(Z))$. This induces a relation in $\mathrm{Rel}(X,Y)$ as the pullback of the unique terminal morphisms $(X \to Z \leftarrow Y)$. Many authors have argued that this relation is the right notion of bisimilarity. We can formulate a related notion of bisimulation without assuming terminal coalgebras, as follows.

Definition 7. *A relation R is a* kernel-bisimulation *if there is a cospan of* B*-coalgebras,* $(X,h) \to (Z,z) \leftarrow (Y,k)$, *and R is the pullback of* $(X \to Z \leftarrow Y)$.

(The term 'cocongruence' is sometimes used to refer directly to the cospan involved, e.g. [31].)

4 Relating the notions of bisimulation

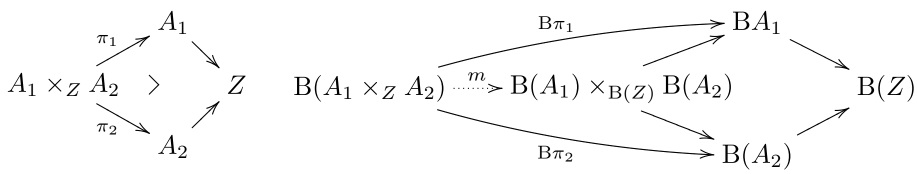
In this section we establish when the notions of bisimulation, introduced in the previous section, are related. As is well known, the four definitions coincide for the case of labelled transition systems.

4.1 Conditions on endofunctors

We begin by recalling five conditions that might be assumed of our endofunctor B.

1. We say that B *preserves relations*, if a jointly-monic span is mapped to a jointly-monic span.

To introduce the remaining conditions, we consider a cospan $(A_1 \to Z \leftarrow A_2)$ in $\mathcal{C}$, and in particular the mediating morphism m from the image of the pullback to the pullback of the image:



Here are four conditions on B, listed in order of decreasing strength:

2. B *preserves pullbacks* if m is always an isomorphism;
3. B *preserves weak pullbacks*, if m is always split epi (see e.g. [22]);
4. B *covers pullbacks* if m is always a cover (=strong epi);
5. B *preserves pullbacks along monos* if m is an iso when $A_1 \to Z$ is monic.

Tying up with item (1), note that B preserves pullbacks if and only if it preserves relations and covers pullbacks (e.g. [9, Sec. 4.3]).

4.2 Relevance of the conditions on endofunctors

We now discuss which of the conditions (1–5) are relevant for the endofunctors considered in Section 2. First, note that polynomial endofunctors on extensive categories preserve all pullbacks.

Regarding powerset functors, we have the following general result.

Proposition 8. *Let $\mathcal{C}$ be a regular category with a class $\mathcal{S}$ of small maps. Let $\mathcal{P}_\mathcal{S}$ be an $\mathcal{S}$-powerset.*

1. *The functor $\mathcal{P}_\mathcal{S}$ preserves pullbacks along monomorphisms.*
2. *If $\mathcal{S}$ contains all monomorphisms (M), then $\mathcal{P}_\mathcal{S}$ preserves weak pullbacks.*
3. *Let $\mathcal{C}$ also be extensive, and let $\mathcal{S}$ satisfy Axioms (A1–7), but not necessarily (M). The functor $\mathcal{P}_\mathcal{S}$ preserves covers and covers pullbacks.*

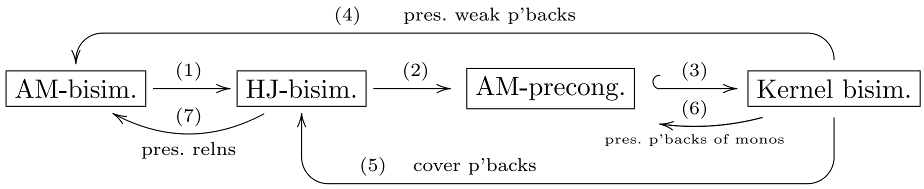
Moving to the setting of continuous state spaces, the compact-subspace endofunctor K does not preserve weak pullbacks, but it does cover pullbacks on the category of Stone spaces [6]; this seems to be an instance of Prop. 8(3). More sophisticated continuous settings are problematic. Plotkin [37] discusses problems with coalgebraic bisimulation in categories of domains. The convex powerdomain does not even preserve monomorphisms. Counterexamples to the weak-pullback-preservation of probability distributions on measurable spaces are discussed in [44].

4.3 Relating notions of bisimulation

Theorem 9. *Let* B *be an endofunctor on a category* $\mathcal{C}$ *with finite limits and images.*

1. *Every AM-bisimulation is an HJ-bisimulation.*
2. *Every HJ-bisimulation is an AM-precongruence.*
3. *Every AM-precongruence is contained in a kernel bisimulation that is an AM-precongruence, provided* $\mathcal{C}$ *has pushouts.*
4. *Every kernel bisimulation is an AM-bisimulation, provided* B *preserves weak pullbacks.*
5. *Every kernel bisimulation is an HJ-bisimulation, provided* B *covers pullbacks.*
6. *Every kernel bisimulation is an AM-precongruence, provided* B *preserves pullbacks along monos.*
7. *Every HJ-bisimulation is an AM-bisimulation, provided either* B *preserves relations, or every epi in* $\mathcal{C}$ *is split.*

In summary:



The second part of Theorem 9(7) accounts for the following well-known fact: when $\mathcal{C} = \mathbf{Set}$, assuming the axiom of choice, HJ-bisimulation is the same thing as AM-bisimulation. We can achieve close connections in more constructive settings:

Theorem 10. *Let* $\mathcal{C}$ *be a regular category with a class* $\mathcal{S}$ *of small maps, with an* $\mathcal{S}$*-powerset,* $\mathcal{P}_{\mathcal{S}}$*. Suppose that* $\mathrm{B}(-) \cong \mathcal{P}_{\mathcal{S}}(\mathrm{B}'(-))$*, for some endofunctor* B' *that preserves relations. If all monomorphisms are in* $\mathcal{S}$ *(axiom (M)), then every HJ-bisimulation is an AM-bisimulation.*

5 Bisimilarity through transfinite constructions

In this section we consider a procedure for constructing the maximal bisimulation. We relate it with the terminal sequence, which is used for finding final coalgebras.

Context. In this section we assume that the ambient category $\mathcal{C}$ is complete with images and pushouts. We fix an endofunctor B on $\mathcal{C}$, and fix two B-coalgebras, $h : X \to \mathrm{B}(X)$ and $k : Y \to \mathrm{B}(Y)$.

Relation refinement sequences. HJ-bisimulations and AM-precongruences can be understood as postfixed points of operators Φ^{HJ} and Φ^{AM} respectively. When $\mathcal{C}$ is well-powered, greatest fixed points can be obtained as limits. For the case of HJ-bisimulations, we define an ordinal-indexed cochain $(r_{\beta,\alpha} : R^{\mathrm{HJ}}_{\beta} \rightarrowtail R^{\mathrm{HJ}}_{\alpha})_{\alpha\leq\beta}$ in $\mathrm{Rel}(X, Y)$, in the usual way (see e.g. [11]):

- *Limiting case*: If λ is limiting, $R^{\mathrm{HJ}}_{\lambda} = \bigcap_{\alpha<\lambda} R^{\mathrm{HJ}}_{\alpha}$; e.g. $R^{\mathrm{HJ}}_{0} = X \times Y$.
- *Inductive case*: $R^{\mathrm{HJ}}_{\alpha+1} = \Phi^{\mathrm{HJ}}(R^{\mathrm{HJ}}_{\alpha})$.

If this sequence is eventually stationary then it achieves the maximal post-fixed point of Φ^{HJ}, the greatest HJ-bisimulation.

Similarly, we consider a cochain $(R^{\mathrm{AM}}_{\beta} \rightarrowtail R^{\mathrm{AM}}_{\alpha})_{\alpha\leq\beta}$ for the operator Φ^{AM}.

For the case of the endofunctor $\mathcal{P}(L \times (-))$ on **Set**, the relation refinement sequences $(R^{\mathrm{HJ}}_{\alpha})_{\alpha}$ and $(R^{\mathrm{AM}}_{\alpha})_{\alpha}$ coincide, giving a transfinite extension of Milner's sequence $\sim_0 \supseteq \sim_1 \supseteq \cdots \supseteq \sim$ (c.f. [34, Sec 5.7]).

The terminal sequence. The terminal sequence is an ordinal-indexed cochain $(z_{\beta,\alpha} : Z_{\beta} \to Z_{\alpha})_{\alpha\leq\beta}$ that can be used to construct a final coalgebra for an endofunctor (see e.g. [46]). The cochain commutes and satisfies the following conditions:

- *Limiting case:* If λ is limiting, $Z_{\lambda} = \lim\{z_{\beta,\alpha} : Z_{\beta} \to Z_{\alpha} \mid \alpha \leq \beta < \lambda\}$, and the cone $\{z_{\lambda,\alpha} : Z_{\lambda} \to Z_{\alpha} \mid \alpha < \lambda\}$ is the limiting one; e.g. $Z_0 = 1$.
- *Inductive case:* $Z_{\alpha+1} = \mathrm{B}(Z_{\alpha})$; and $z_{\beta+1,\alpha+1} = \mathrm{B}(z_{\beta,\alpha}) : Z_{\beta+1} \to Z_{\alpha+1}$.

5.1 Relating the relation and terminal sequences

The coalgebra (X, h), (Y, k) determine two cones over the terminal sequence: $(x_{\alpha} : X \to Z_{\alpha})_{\alpha}$ and $(y_{\alpha} : X \to Z_{\alpha})_{\alpha}$. The cone $(x_{\alpha})_{\alpha}$ is given as follows.

- *Limiting case:* If λ is a limit ordinal then the morphisms $x_{\alpha} : X \to Z_{\alpha}$ for $\alpha < \lambda$ form a cocone over the cochain $(z_{\beta,\alpha} : Z_{\beta} \to Z_{\alpha})_{\alpha\leq\beta<\lambda}$, with apex X. We let $x_{\lambda} : X \to Z_{\lambda}$ be the unique mediating morphism. For instance, when $\lambda = 0$, then $x_{\lambda} : X \to Z_{\lambda}$ is the terminal map $X \to 1$.
- *Inductive case:* Let $x_{\alpha+1}$ be the composite $X \xrightarrow{h} \mathrm{B}(X) \xrightarrow{\mathrm{B}x_{\alpha}} \mathrm{B}(Z_{\alpha}) = Z_{\alpha+1}$.

The other cone, $(y_{\alpha} : X \to Z_{\alpha})_{\alpha}$, is defined similarly.

Proposition 11. *Consider an ordinal α, and consider the pullback $X \times_{Z_{\alpha}} Y$ of the cospan $(X \xrightarrow{x_{\alpha}} Z_{\alpha} \xleftarrow{y_{\alpha}} Y)$.*

1. *If* B *preserves pullbacks along monos then $X \times_{Z_{\alpha}} Y = R^{\mathrm{AM}}_{\alpha}$.*
2. *If* B *covers pullbacks, then $X \times_{Z_{\alpha}} Y = R^{\mathrm{HJ}}_{\alpha} \quad (= R^{\mathrm{AM}}_{\alpha})$.*

The relation refinement sequence may converge before the terminal sequence:

Corollary 12. *Suppose that* B *covers pullbacks. If, for some ordinal α, the morphism $z_{\alpha+1,\alpha} : Z_{\alpha+1} \to Z_{\alpha}$ is monic, then the relation refinement sequence converges at α.*

If $\mathcal{C}$ is **Set** and B preserves filtered colimits, then the terminal sequence does not converge until $(\omega + \omega)$, but it becomes monic at ω [46].

6 Models of name-passing process calculi

As a case study, we now investigate models for name-passing process calculi. A fragment of the π-calculus is given in Figure 2. To simplify the presentation, we omit restriction (name-generation) for now; we return to this issue in Section 6.4.

It is unreasonable to model name-passing in the category of sets. One alternative is the category of nominal substitutions [16, 18], as we now investigate. We describe an endofunctor, for which coalgebras capture the model-theoretic properties of the π-calculus, and for which bisimulation is open bisimulation [40].

Throughout this section we fix an infinite set $\mathbb{A}$ of channel names.

6.1 Labelled transition systems in nominal substitutions

Definition 13. *A nominal substitution is a set X together with a function $\mathsf{sub} : \mathbb{A} \times \mathbb{A} \times X \to X$, written $\{^b/_a\}x = \mathsf{sub}(b, a, x)$, such that*

1. $\{^a/_a\}x = x$
2. $\{^c/_b\}\{^b/_a\}x = \{^c/_b\}\{^c/_a\}x$
3. *If* $c \neq b \neq a \neq d$ *then* $\{^d/_b\}\{^c/_a\}x = \{^c/_a\}\{^d/_b\}x$.
4. *If* $a \neq b$ *then* $\{^c/_a\}\{^b/_a\}x = \{^b/_a\}x$.
5. *For each* $x \in X$*, the set* $\mathrm{supp}(x) = \{a \in \mathbb{A} \mid \exists b \in \mathbb{A}.\, \{^b/_a\}x \neq x\}$ *is finite.*[1]

A homomorphism of nominal substitutions, $f\colon (X, \mathsf{sub}) \to (Y, \mathsf{sub})$, is a function that preserves the structure: $f\{^b/_a\}x = \{^b/_a\}(f(x))$. We write **NomSub** *for the resulting category.*

The set $\mathbb{A}$ is itself a nominal substitution. Other important examples are set T_π of π-calculus terms up-to α-equivalence, and the set L_π of labels, both with the evident (capture-avoiding) substitution structure.

The transition relation for the π-calculus, as given in Figure 2, is a subset $(\longrightarrow) \subseteq T_\pi \times L_\pi \times T_\pi$. Moreover, the relation is a relation in **NomSub**; that is to say, $(\longrightarrow)$ has a nominal substitution structure, and $(\longrightarrow) \rightarrowtail T_\pi \times L_\pi \times T_\pi$ preserves the structure. That is, if $t \xrightarrow{l} u$, then $\{^b/_a\}t \xrightarrow{\{^b/_a\}l} \{^b/_a\}u$. (see e.g. [43]).

[1] I am grateful to Andy Pitts for pointing out this simplified condition.

Terms (T_π): $p \ ::= \ a(b).p \mid \bar{a}b.p \mid p|p \mid \mathbf{0}$
Labels (L_π): $l \ ::= \ ab \mid \bar{a}b \mid \tau$

Structural operational semantics:

$$\frac{}{a(b).p \xrightarrow{ac} \{^c/_b\}p} \qquad \frac{}{\bar{a}b.p \xrightarrow{\bar{a}b} p} \qquad \frac{p \xrightarrow{l} p'}{p|q \xrightarrow{l} p'|q} \qquad \frac{p \xrightarrow{\bar{a}b} p' \quad q \xrightarrow{ab} q'}{p|q \xrightarrow{\tau} p'|q'}$$

Fig. 2. A fragment of the π-calculus [36]. Symmetric versions of rules are elided.

6.2 Coalgebras and notions of non-determinism

We now investigate how the transition relation for the π-calculus can be viewed in a coalgebraic way. As a starting point, we note that **NomSub** is a topos, and hence to give an substitution-closed transition relation is to give a coalgebra for $\mathcal{P}(L_\pi \times (-))$, i.e. a homomorphism $X \to \mathcal{P}(L_\pi \times X)$ (writing $\mathcal{P}$ for the power object of **NomSub**). This approach is not without drawbacks, however: (i) an explicit description of $\mathcal{P}$ is cumbersome; (ii) the functor $\mathcal{P}$ is not finitary, and there is no final coalgebra for cardinality reasons. To remedy this, we now consider an explicit description of a finitary subfunctor of $\mathcal{P}$.

A theory of equality testing and non-determinism. In Figure 3, we present a theory of 'conditional-semilattices'. A conditional-semilattice has enough structure to describe both nondeterminism and name-equality testing. Conditional-semilattices form a category $\mathcal{CSL}$, with the evident morphisms, and the forgetful functor $\mathcal{CSL} \to$ **NomSub** is monadic. The free conditional-semilattice, $\mathcal{P}_{\mathrm{csl}}(X)$, on a nominal substitution X, can be constructed by considering the set of all well-formed terms built from $+$, $\mathbf{0}$, if, and elements of X, quotiented by the equations in Figure 3. Alternatively, Figure 3 can be understood as a presentation of an enriched algebraic theory in **NomSub**, following [28], and a finitary monad $\mathcal{P}_{\mathrm{csl}}$ arises from the general results there.

Proposition 14. *The monad $\mathcal{P}_{\mathrm{csl}}$ classifies homomorphisms $f\colon Y \to X$ in* **NomSub** *for which $\{y \in Y \mid f(y) = x\}$ is finite, for each $x \in X$. That is: the following data are equivalent, for all nominal substitutions X and Y.*

1. *A homomorphism $X \to \mathcal{P}_{\mathrm{csl}}(Y)$.*
2. *A subset $R \subseteq X \times Y$ such that*
   (a) If $x\ R\ y$ then $\{^b/_a\}x\ R\ \{^b/_a\}y$ ("R is substitution closed"); and
   (b) For all $x \in X$, the image $\{y \in Y \mid x\ R\ y\}$ is finite.

We write $\mathcal{S}_{\mathrm{csl}}$ for the class of morphisms that $\mathcal{P}_{\mathrm{csl}}$ classifies. This class satisfies all the axioms of Figure 1.

For the curious topos theorist, we record that $\mathcal{P}_{\mathrm{csl}}$ is the free semilattice generated by the partial map classifier (see also [13, 32]).

A *conditional-semilattice* is a nominal substitution X together with three homomorphisms

$$+\colon X \times X \to X \qquad \mathbf{0}\colon 1 \to X \qquad \mathsf{if}\colon \mathbb{A} \times \mathbb{A} \times X \to X$$

that satisfy the following equations. (We abbreviate $\mathsf{if}(a, b, x)$ by $[a = b]\, x$.)

$$x + x = x$$
$$x + y = y + x$$
$$x + (y + z) = (x + y) + z$$
$$x + \mathbf{0} = x$$

$$[a = b]\, \mathbf{0} = \mathbf{0}$$
$$[a = b]\, (x + y) = ([a = b]\, x) + ([a = b]\, y)$$
$$[a = b]\, x + x = x$$

$$[a = b]\, [a = b]\, x = [a = b]\, x$$
$$[a = b]\, [c = d]\, x = [c = d]\, [a = b]\, x$$
$$[a = b]\, [b = c]\, x = [a = b]\, [a = c]\, x$$
$$[a = a]\, x = x$$
$$[a = b]\, x = [b = a]\, x$$

$$[a = b]\, x = [a = b]\, (\{^a/_b\}x)$$

Fig. 3. Theory of conditional-semilattices (c.f. [40, Sec. 4.1])

Coalgebraic models of name-passing calculi. Coalgebras for the endofunctor $\mathcal{P}_{\mathrm{csl}}(L_\pi \times (-))$ are the same thing as substitution closed, image-finite relations. Unfortunately, the transition system for the π-calculus is not image-finite — the process $a(b).\mathbf{0}$ can input an infinite number of different names. The π-calculus behaviour is essentially finite, though, in a sense that we now make precise.

For each nominal substitution X, we define a set $X^{\mathbb{A}} = (\mathbb{A} \times X)/_{=_\alpha}$, where $=_\alpha$ is the equivalence relation generated by setting $(a, x) =_\alpha (b, \{^b/_a\}x)$, for any name b not in $\mathrm{supp}(x)$. This construction extends to an endofunctor on nominal substitutions. We thus have an appropriate endofunctor for name-passing behaviour: let $\mathrm{B}_\pi = \mathcal{P}_{\mathrm{csl}}(\mathbb{A} \times (-)^{\mathbb{A}} + \mathbb{A} \times \mathbb{A} \times (-) + (-))$.

Theorem 15. *The following data are equivalent.*

1. *A* B_π*-coalgebra.*
2. *A substitution-closed relation* $(\longrightarrow) \subseteq X \times L_\pi \times X$ *for which*
   - *(a) for every* $x \in X$*, the sets* $\{(a, b, x') \mid x \xrightarrow{\bar{a}b} x'\}$*,* $\{x' \mid x \xrightarrow{\tau} x'\}$*, are both finite;*
   - *(b) for all* $x \in X$ *there is a name* c *and a finite subset* $\{(a_1, y_1), \ldots, (a_n, y_n)\}$ *of* $\mathbb{A} \times X$ *such that if* $x \xrightarrow{ab} x'$ *then, for some* $i \le n$*, we have* $a = a_i$ *and* $x' = \{^b/_c\}y_i$*;*
   - *(c) if* $x \xrightarrow{ab} x'$ *then there is a name* $c \notin \mathrm{supp}(x)$ *and* $x'' \in X$ *such that* $x \xrightarrow{ac} x''$ *and* $\{^b/_c\}x'' = x'$*.*

The transition relation for the π-calculus, given in Figure 2, satisfies the three conditions of Theorem 15(2), and indeed these conditions essentially exhaust the model-theoretic properties of this fragment of the π-calculus (see [41, Sec. 1.4.2]).

For instance, in the 'next state' coalgebra $T_\pi \to \mathrm{B}_\pi(T_\pi)$ for the π-calculus, the process $(a(c).x \mid \bar{b}d.y)$ maps to the equivalence class in $\mathrm{B}_\pi(T_\pi)$ that represents the following element. We specify the input and output transitions, and, although the process cannot perform a τ action, we must record that it could, if a and b were equal.

$$\mathsf{inj}_1(a, [c, (x \mid \bar{b}d.y)]_{=_\alpha}) \; + \; \mathsf{inj}_2(b, d, (a(c).x \mid y)) \; + \; \mathsf{if}(a, b, \mathsf{inj}_3((\{^d/_c\}x) \mid y)). \quad (2)$$

Free semilattices aren't enough. In (2), we see that the operator 'if' in the theory of conditional-semilattices is essential for modelling name-passing. To emphasise this, consider the free semilattice monad $\mathcal{P}_{\mathrm{sl}}$ on **NomSub**: $\mathcal{P}_{\mathrm{sl}}(X)$ is the set of finite subsets S of X, with the pointwise substitution structure. To understand why this is inadequate for name-passing, consider the following result:

Proposition 16. *Consider nominal substitutions* X *and* Y*. The following data are equivalent.*

1. *A homomorphism* $X \to \mathcal{P}_{\mathrm{sl}}(Y)$*.*
2. *A substitution-closed subset* $R \subseteq X \times Y$ *such that for all* $x \in X$*, the image* $\{y \in Y \mid x \; R \; y\}$ *is finite, and such that if* $\{^b/_a\}x \; R \; y$ *then there is* $y' \in Y$ *such that* $\{^b/_a\}y' = y$ *and* $x \; R \; y'$*.*

6.3 Bisimulation for name-passing calculi

HJ-bisimulations for B_π are essentially open bisimulations [40, Sec. 3.3]:

Proposition 17. *Let (X,h) and (Y,k) be B_π-coalgebras, corresponding to transition systems $(X,\to_X)$ and $(Y,\to_Y)$. A substitution-closed relation $R\subseteq X\times Y$ is an HJ-bisimulation if and only if it is a bisimulation between the transition systems in the classical sense.*

The class $\mathcal{S}_{\mathrm{csl}}$ contains all monomorphisms, so, by the results in Section 4, an HJ-bisimulation is the same thing as an AM-bisimulation, and the greatest such bisimulation is the greatest AM-precongruence and the kernel of the final map.

6.4 Remarks on functor categories and other models of name-passing

The category of nominal substitutions is equivalent to the category $[\mathbf{F}_{\mathrm{ne}},\mathbf{Set}]$ of set-valued functors, where $\mathbf{F}_{\mathrm{ne}}$ is the category of non-empty finite subsets of $\mathbb{A}$ and all functions between them[2]. Via Prop. 14 and the Yoneda lemma, we have an explicit description of $\mathcal{P}_{\mathrm{csl}}$.

In some circumstances it is appropriate to work in the category $[\mathbf{I},\mathbf{Set}]$, where $\mathbf{I}$ is the category of all finite subsets of $\mathbb{A}$ and injections between them. The free semilattice monad $\mathcal{P}_{\mathrm{sl}}$ on $[\mathbf{I},\mathbf{Set}]$ has a simple description: for F in $[\mathbf{I},\mathbf{Set}]$ and $A\subseteq_{\mathrm{f}}\mathbb{A}$, the set $(\mathcal{P}_{\mathrm{sl}}F)(A)$ is the set of finite subsets of $F(A)$. Unlike the situation in **NomSub**, the monad $\mathcal{P}_{\mathrm{sl}}$ on $[\mathbf{I},\mathbf{Set}]$ is an appropriate monad for non-determinism in the semantics of name-passing languages [15, 17].

In any topos, the free semilattice monad $\mathcal{P}_{\mathrm{sl}}$ classifies the morphisms that are Kuratowski-finite. By Prop. 8, $\mathcal{P}_{\mathrm{sl}}$ covers pullbacks, and if the topos is Boolean, then it also preserves weak pullbacks: note that monos are Kuratowski-finite if and only if they are complemented (see also [25]). Indeed, the example of [25] can be adapted to show that $\mathcal{P}_{\mathrm{sl}}$ does *not* preserve weak pullbacks in $[\mathbf{I},\mathbf{Set}]$ (correcting oversights in [17, 21, 42]). In spite of this, HJ-bisimulations can still be related with AM-bisimulations:

Proposition 18 (c.f. [42, Sec. 3.3.3]). *Let B' be an endofunctor on $[\mathbf{I},\mathbf{Set}]$ that preserves complemented relations, and consider the composite endofunctor $\mathcal{P}_{\mathrm{sl}}\circ\mathrm{B}'$. The $\neg\neg$-closure of an HJ-bisimulation is an AM-bisimulation.*

The situation simplifies if we restrict attention to the category of $\neg\neg$-sheaves. The sheaves can be understood as 'nominal sets' [19, 42] or as 'named-sets with symmetry', giving a connection with 'history dependent automata' [14, 15, 20].

Modelling restriction. To simplify the presentation, we have so-far ignored the restriction operator in the π-calculus, which introduces the facility of name generation. To properly handle name generation, the category **NomSub** is inadequate. Instead, following [21, 33], one can work in the category $[\mathbf{D},\mathbf{Set}]$ of set-valued functors, where $\mathbf{D}$ is the category of finite irreflexive undirected graphs

[2] I am grateful to Alexander Kurz for pointing this out.

and homomorphisms between them. The idea is to keep track of which names are known to be different, described by an edge in a graph.

The free semilattice monad on $[\mathbf{D}, \mathbf{Set}]$ is not a sufficient notion of non-determinism, for the reasons discussed above (correcting an oversight in [21]). Instead, one can work with the theory in Figure 3, with an additional axiom saying that when names a and b are distinct then $[a = b]x = \mathbf{0}$ (c.f. [40, Sec. 6, Ax. P]). The resulting monad corresponds to a class of small maps, but not all monos are small. It would be interesting to find a generalization of Prop. 18 that applies to this situation.

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