# Positive structural operational semantics and monotone distributive laws

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**Abstract**

We describe a correspondence between (i) rule-based inductive definitions in the Positive GSOS format, i.e. without negative premises, and (ii) distributive laws in the spirit of Turi and Plotkin (LICS'97), that are suitably monotone. This result can be understood as an isomorphism of lattices, in which the prime elements are the individual rules.

## Context

We fix a signature, i.e. a set $S$ of operators, each $\mathrm{op} \in S$ having an arity, $\mathrm{ar}(\mathrm{op}) \in \mathbb{N}$. We also fix a set $L$ of labels. We are concerned with labelled transition relations over the terms of the signature.

For an example, we recall a fragment of Milner's CCS. Fix a set $A$ of actions. There is a binary operation for parallel composition (written $x\,|\,y$), a constant for deadlock, and two unary operations for every action $a \in A$, for prefix and co-action prefix.

The set of labels contains actions, co-actions, and silent steps: $L = A + A + 1$. A labelled transition relation over CCS terms is defined by a transition system specification (TSS). This is a set of rules, including the one in the box on the right.

$$\frac{x \xrightarrow{a} x' \qquad y \xrightarrow{\bar{a}} y'}{x\,|\,y \xrightarrow{\tau} x'\,|\,y'}$$

## Background on the Positive GSOS rule format

A transition system specification (TSS) over the signature is said to be in the GSOS format [2] if it comprises a set of rules of the form shown in the box on the right, where the variables are all distinct, and the variables in the term $t$ already appear in the premises or on the left hand side of the conclusion. We will allow the sets $J_i$ and $K_i$ of premises to be infinite.

$$\frac{\{x_i \xrightarrow{l_{ij}} y_{ij} \mid i \leq \mathrm{ar}(\mathrm{op}), j \in J_i\} \quad \{x_i \stackrel{l_{ik}}{\not\longrightarrow} \mid i \leq \mathrm{ar}(\mathrm{op}),\ k \in K_i\}}{\mathrm{op}(x_1, \ldots, x_{\mathrm{ar}(\mathrm{op})}) \xrightarrow{l} t}$$

The TSS is said to be in the Positive GSOS format if there are no negative premises: the sets $K_i$ are all empty. The specification of CCS is in the Positive GSOS format. Aceto et al. [1] give an overview.

## Background on distributive laws and mathematical operational semantics

We now recall some aspects of the 'mathematical operational semantics' of Turi and Plotkin [6]. The signature $S$ induces an endofunctor $\Sigma_S$ and a monad $T_S$ on **Set**: for a set $X$, let $\Sigma_S(X) = \biguplus_{\mathrm{op} \in S} X^{\mathrm{ar}(\mathrm{op})}$, and let $T_S(X)$ be the set of terms built from the signature and the elements of $X$. We will also consider an endofunctor $B_L$ on **Set**: for a set $X$, let $B_L(X) = \mathscr{P}(L \times X)$. (Here, $\mathscr{P}$ is the covariant powerset functor.) Coalgebras for $B_L$ are in bijective correspondence with labelled transition systems.

We define a *distributive law* to be a family of functions $\rho_X \colon \Sigma_S(X \times B_L(X)) \to B_L(T_S(X))$, natural in $X \in \mathbf{Set}$. (Technically, this data defines a distributive law of a monad over a copointed endofunctor — see [4].) Every GSOS TSS gives rise to a distributive law, as follows. (For a set $X$, we define an *X-instance* of a rule to be a formal substitution of elements of $X$ for variables in the rule.)

$$\rho_X\Big(\mathrm{op}\big((x_1,\beta_1),\ldots,(x_{\mathrm{ar}(\mathrm{op})},\beta_{\mathrm{ar}(\mathrm{op})})\big)\Big) = \left\{ (l,t) \;\middle|\; \begin{array}{l} \textit{there is an } X\textit{-instance of a rule in the TSS for which} \\ \textit{the conclusion is } \mathrm{op}(x_1,\ldots,x_{\mathrm{ar}(\mathrm{op})}) \xrightarrow{l} t, \textit{ and for each} \\ \textit{premise } x_i \xrightarrow{l_{ij}} y_{ij}, \textit{ the pair } (l_{ij},y_{ij}) \textit{ is in } \beta_i. \end{array} \right\} \quad (\dagger)$$

**Proposition 1** (Turi and Plotkin)**.** *Every distributive law arises from a GSOS TSS.*

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## Positivity and monotonicity

We now provide an analogue of Proposition 1 for the Positive GSOS format. For any set $X$, the set $\Sigma_S(X \times B_L(X))$ has a natural partial order, induced by the subset ordering of the powerset. If $\beta_1 \subseteq \beta'_1, \ldots,$ and $\beta_{\mathrm{ar(op)}} \subseteq \beta'_{\mathrm{ar(op)}}$, then we let $\mathrm{op}((x_1, \beta_1), \ldots, (x_{\mathrm{ar(op)}}, \beta_{\mathrm{ar(op)}}))) \ \leq \ \mathrm{op}((x_1, \beta'_1), \ldots, (x_{\mathrm{ar(op)}}, \beta'_{\mathrm{ar(op)}})))$. The set $B_L(T_S(X))$ can also be partially ordered by subset inclusion. We say that a distributive law $\{\rho_X : \Sigma_S(X \times B_L(X)) \to B_L(T_S(X))\}_{X \in \mathbf{Set}}$ is monotone if each function $\rho_X$ is monotone with respect to the above orderings.

**Theorem 2.** *Every Positive GSOS TSS induces (via (†)) a monotone distributive law. Every monotone distributive law is induced by a Positive GSOS TSS.*

## A lattice isomorphism

By a *model* of a Positive GSOS TSS, we understand a model of the first-order theory whose function symbols are the operators of the algebraic signature, together with a binary relation symbol $\xrightarrow{l}$ for each label $l \in L$, and a Horn clause for every rule in the TSS.

We say that two TSSs in the Positive GSOS format are equivalent if they have the same models. This accounts for a change in variables in the rules of a TSS, and for redundant premises. For instance, if we add the rule in the box on the right to the TSS for CCS, then the resulting TSS is equivalent.

$$\frac{v \xrightarrow{a} v' \quad w \xrightarrow{\bar{a}} w' \quad w \xrightarrow{\tau} w''}{v \,|\, w \xrightarrow{\tau} v' \,|\, w'}$$

The collection of equivalence classes of Positive GSOS TSSs (over our signature $S$) forms a set. We can equip this set with the following complete lattice structure. The join $\bigsqcup \mathscr{C}$ of a collection $\mathscr{C}$ of TSSs is the TSS that contains all the rules in all the TSSs in $\mathscr{C}$. This lattice is prime algebraic, and the TSSs that only contain one rule are the primes (viz. the TSSs that are inaccessible by joins).

The union structure of the powerset induces a pointwise join-semilattice structure on the collection of monotone distributive laws.

**Theorem 3.** *The construction in* (†) *defines an isomorphism between the complete lattice of Positive GSOS TSSs, and the complete lattice of monotone distributive laws.*

## Mathematical operational semantics

This development suggests two abstract ideas in the context of mathematical operational semantics. Firstly, $B_L$ can be replaced by an arbitrary functor $\mathbf{Set} \to \mathbf{Poset}$ (see [3]), and monotone distributive laws are nothing but natural transformations between functors $\mathbf{Set} \to \mathbf{Poset}$. We can thus provide a categorical account of the following fact: *for a Positive GSOS TSS, similarity is a precongruence* (see also [5, §4]). Secondly, when $B_L$ is a functor $\mathbf{Set} \to \mathbf{PrAlgLat}$ into prime algebraic lattices and join preserving maps, the primes in the lattice of monotone distributive laws are an abstract form of solitary rule.

## References

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