# Freyd categories are enriched Lawvere theories

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**Abstract**

Lawvere theories provide a categorical formulation of the algebraic theories from universal algebra. Freyd categories are categorical models of first-order effectful programming languages.

The notion of *sound limit doctrine* has been used to classify accessible categories. We provide a definition of Lawvere theory that is enriched in a closed category that is locally presentable with respect to a sound limit doctrine.

For the doctrine of finite limits, we recover Power's enriched Lawvere theories. For the empty limit doctrine, our Lawvere theories are Freyd categories, and for the doctrine of finite products, our Lawvere theories are distributive Freyd categories. In this sense, computational effects are algebraic.

*Keywords:* Freyd categories, Lawvere theories, monads and notions of computation.

## 1 Introduction

Strong monads have helped to organize the semantics of impure programming languages from at least two perspectives: firstly by examining the crucial properties of concrete models of programming languages; secondly by axiomatizing the equations between programs that must hold in all models [24,25].

However, more refined perspectives have since emerged.

- Firstly, the monads involved in many concrete models of impure programming languages actually arise as free algebras for equational theories, in the setting of enriched category theory (e.g. [28,29]).
- Secondly, when we separate first-order effectful computation from higher-order types, we arrive at the notion of Freyd categories as an axiomatization of first-order effectful computation. (Moggi's monad-models can be recovered as closed Freyd categories, see e.g. [19].)

In this paper I explain that the second development can be seen as an instance of the former.

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**Informal overview**

The generalization from traditional equational theories to enriched ones proceeds as follows. Recall that in a traditional algebraic signature there is a set of $n$-ary operations for each natural number $n$, and so a structure for the signature comprises a function $X^n \to X$ for each $n$-ary operation.

This is enriched by replacing the category of sets by a category $\mathcal{V}$ (perhaps the simplest interesting example of $\mathcal{V}$ to have in mind is the category of posets and monotone functions). The $n$-ary operations are no longer required to form a set, but rather an object $O_n$ of $\mathcal{V}$; the arities $n$ are no longer natural numbers, but rather 'finitary' objects of $\mathcal{V}$; and a structure for such a signature in a given $\mathcal{V}$-enriched category $\mathcal{A}$ comprises a morphism $O_n \to \mathcal{A}(X^n, X)$ in $\mathcal{V}$, where $X^n$ is a power. (This line of thought goes back to Kelly's work [14,15]; our starting point is Power's development [30]; see also [34] for an overview.)

A traditional equational theory determines a Lawvere theory, which is a category where the objects are natural numbers, and a morphism $m \to 1$ is a term in $m$ variables modulo the equations, and in general a morphism $m \to n$ is a family of $n$ terms-mod-equations in $m$ variables. The categories arising in this way can be characterized as categories $\mathbb{L}$ with finite products equipped with a functor $J : \mathbb{N}^{\mathrm{op}} \to \mathbb{L}$, where $\mathbb{N}$ is the category of natural numbers and functions between them; the functor $J$ is required to be identity-on-objects (i.e. $\mathbb{N}^{\mathrm{op}}$ and $\mathbb{L}$ have the same objects) and to preserve products.

Similarly a $\mathcal{V}$-enriched Lawvere theory [30] is defined to be a $\mathcal{V}$-enriched category $\mathbb{L}$ with 'finitary' powers and an identity-on-objects finitary-power-preserving $\mathcal{V}$-functor $\mathbb{F}^{\mathrm{op}} \to \mathbb{L}$, where $\mathbb{F}$ is the category of finitary objects of $\mathcal{V}$.

On the other hand, the notion of Freyd category arose in the work of Levy, Power and Thielecke [19,33] as a categorical framework for first order effectful programs. Recall the basic ideas of the categorical interpretation of type theory: that types are denoted by objects of a category, that a context is denoted by the product of its constituent types, and that a judgement $\Gamma \vdash t : \tau$ is interpreted as a morphism $\Gamma \to \tau$ in the category. A Freyd category comprises two categories with the same objects: one $\mathbb{V}$, whose morphisms denote pure, value judgements, and one $\mathbb{C}$, whose morphisms denote judgements of computations, together with an identity-on-objects functor $J : \mathbb{V} \to \mathbb{C}$. For example, $\mathbb{C}$ might be the Kleisli category for a strong monad on $\mathbb{V}$. Since the order of effectful computation matters, $\mathbb{C}$ typically does not have products, but it does have a product-like structure, and the functor $J$ is required to preserve it. This was initially described in terms of premonoidal categories [32]. Subsequently, Levy used a formulation based on actions of monoidal categories [18, App. B] (see also [23]) and that is what we use in this paper.

Coming back to the definition of enriched Lawvere theory, notice that, naively put, there is some choice in what is meant by 'finitary' when it comes to the arities. When $\mathcal{V} = \mathbf{Set}$, 'finitary' means finite. Power takes $\mathcal{V}$ to be a locally finitely presentable category, and 'finitary' means finitely presentable. If $\mathbb{V}$ is a category with finite products, and $\mathcal{V}$ is the functor category $[\mathbb{V}^{\mathrm{op}}, \mathbf{Set}]$, then we can take 'finitary' to mean representable. In this case, an enriched Lawvere theory is the same thing as a Freyd category (Theorem 3.3).

Several authors have found profit in analyzing the 'arities' of monads and Lawvere theories, including in the study of computational effects [4,7,16,20,21]. The line of work best suited for us is the classification of accessible categories by Adámek et al. [1]. This is based on a notion of sound limit doctrine $\mathfrak{D}$, and includes concepts of $\mathfrak{D}$-presentable object, which form our arities, and locally $\mathfrak{D}$-presentable categories. A locally $\mathfrak{D}$-presentable category is tightly connected with its subcategory of $\mathfrak{D}$-presentable objects, since each is determined by the other. We consider enriched Lawvere theories in this setting, following Lack and Rosický [16]

By considering different sound limit doctrines we recover familiar concepts:

- For the sound doctrine of finite limits, enriched Lawvere theories are the concept defined by Power [30].
- For the sound doctrine of finite products, enriched Lawvere theories are the same as distributive Freyd categories (Theorem 3.5).
- For the empty sound doctrine, enriched Lawvere theories are the same as Freyd categories (Theorem 3.3).

## 2 Preliminaries

### *2.1 Sound limit doctrines*

A sound limit doctrine is a class of limits that admits a well-behaved refinement of the theory of accessible and locally presentable categories [1].

**Definition 2.1** [[1]] A *doctrine* is a set $\mathfrak{D}$ of small categories. A $\mathfrak{D}$-limit is a limiting cone whose diagram is indexed by a category in $\mathfrak{D}$. Dually a $\mathfrak{D}$-*colimit* is a colimiting cone whose diagram is indexed by a category in $\mathfrak{D}$. We write $\mathfrak{D}^{\mathrm{op}}$ for the doctrine $\{\mathbb{D}^{\mathrm{op}} \mid \mathbb{D} \in \mathfrak{D}\}$.

A set of small categories $\mathfrak{D}$ is a *sound limit doctrine* if for any functor $F : \mathcal{A} \to \mathbf{Set}$ the left Kan extension $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$ of $F$ along the Yoneda embedding $\mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ preserves $\mathfrak{D}$-limits if and only if it preserves $\mathfrak{D}$-limits of representables.

The condition of soundness ensures that the theory of accessible and locally presentable categories, which is traditionally based on $\lambda$-small limits, makes sense for $\mathfrak{D}$-limits. Examples of sound limit doctrines include:

- $\mathfrak{FinLim}$: the doctrine of finite limits;
- $\mathfrak{FinProd}$: the doctrine of finite products;
- $\emptyset$: the empty doctrine.

Those are the three doctrines that we study in this paper.

**Definition 2.2** [[1]] Let $\mathfrak{D}$ be a sound limit doctrine. A small category $\mathbb{C}$ is $\mathfrak{D}$-*filtered* if $\mathbb{C}$-indexed colimits commute in $\mathbf{Set}$ with $\mathfrak{D}$-limits (i.e. the functor *colim* : $[\mathbb{C}, \mathbf{Set}] \to \mathbf{Set}$ preserves $\mathfrak{D}$-limits). A $\mathfrak{D}$-*filtered colimit* is a colimiting cone whose diagram is indexed by a $\mathfrak{D}$-filtered category.

For example:

- A $\mathfrak{FinLim}$-filtered category is normally just called a filtered category.

- A $\mathfrak{FinProd}$-filtered category is sometimes called a sifted category [3]. Roughly speaking, sifted colimits are built from filtered colimits and reflexive coequalizers.
- All categories are trivially $\emptyset$-filtered, so a $\mathfrak{D}$-filtered colimit is the same thing as a colimit.

### *2.2 Locally presentable categories*

**Definition 2.3** Let $\mathfrak{D}$ be a sound limit doctrine. Let $\mathcal{A}$ be a category with all small colimits. An object $a$ of $\mathcal{A}$ is $\mathfrak{D}$*-presentable* if the representable functor $\mathcal{A}(a,-) : \mathcal{A} \to \mathbf{Set}$ preserves $\mathfrak{D}$-filtered colimits. The cocomplete category $\mathcal{A}$ is *locally* $\mathfrak{D}$*-presentable* if there is a set $\mathbb{F}$ of $\mathfrak{D}$-presentable objects such that every object of $\mathcal{A}$ is a $\mathfrak{D}$-filtered colimit of objects from $\mathbb{F}$.

For example:

- A locally $\emptyset$-presentable category is a presheaf category $[\mathbb{F}^{\mathrm{op}}, \mathbf{Set}]$ and the $\emptyset$-presentable objects are retracts of representables.
- Any locally $\emptyset$-presentable category is also locally $\mathfrak{FinProd}$-presentable. More generally, the category of models for a multi-sorted algebraic theory is always a locally $\mathfrak{FinProd}$-presentable category, and all locally $\mathfrak{FinProd}$-presentable categories arise in this way (e.g. [3]). In particular, the category of sets is locally $\mathfrak{FinProd}$-presentable, and the $\mathfrak{FinProd}$-presentable objects are the finite sets.
- A locally $\mathfrak{FinLim}$-presentable category is normally called a locally finitely presentable category (e.g. [2]). Any locally $\mathfrak{FinProd}$-presentable category is also locally $\mathfrak{FinLim}$-presentable. More generally, the category of models for an 'essentially algebraic' theory is always a locally finitely presentable category, and all locally finitely presentable categories arise in this way.

### *2.3 Locally presentable symmetric monoidal closed categories*

The following definition is a mild generalization of the standard concept of a locally finitely presentable closed category [15].

**Definition 2.4** Let $\mathfrak{D}$ be a sound limit doctrine. A symmetric monoidal closed category $(\mathcal{V}, \otimes, I)$ is *locally* $\mathfrak{D}$*-presentable as a symmetric monoidal closed category* if it is locally $\mathfrak{D}$-presentable, if $I$ is $\mathfrak{D}$-presentable, and $a \otimes b$ is $\mathfrak{D}$-presentable when $a$ and $b$ are.

If $(\mathcal{V}, \otimes, I)$ is locally $\mathfrak{D}$-presentable as a closed category, then we define a *basis* for $\mathcal{V}$ to be a small full subcategory $\mathbb{F}$ of $\mathcal{V}$ whose objects are $\mathfrak{D}$-presentable, which is closed under $\mathfrak{D}^{\mathrm{op}}$-colimits and $\otimes$ and $I$, and which is such that every object of $\mathcal{V}$ is a $\mathfrak{D}$-filtered colimit of objects from $\mathbb{F}$.

Note that an object of a locally $\mathfrak{D}$-presentable closed category is $\mathfrak{D}$-presentable if and only if it is a retract of an object in the basis. If $\mathfrak{D}$ contains the category with one object and one idempotent non-identity morphism, e.g. if $\mathfrak{D} = \mathfrak{FinLim}$, then the basis is closed under retracts and so all bases are equivalent.

Recall (e.g. [2, Prop. 1.45]) that for any category $\mathbb{F}$ with $\mathfrak{D}^{\mathrm{op}}$-colimits, the category $[\mathbb{F}^{\mathrm{op}}, \mathbf{Set}]_{\mathfrak{D}}$ of $\mathfrak{D}$-limit-preserving set-valued functors and natural transformations has a universal property as a cocompletion. It has small colimits; the Yoneda embedding restricts to a functor $\mathbb{F} \hookrightarrow [\mathbb{F}^{\mathrm{op}}, \mathbf{Set}]_{\mathfrak{D}}$ that preserves $\mathfrak{D}^{\mathrm{op}}$-colimits; and every $\mathfrak{D}^{\mathrm{op}}$-colimit-preserving functor $G : \mathbb{F} \to \mathcal{A}$ with $\mathcal{A}$ cocomplete extends to a colimit-preserving-functor $G^* : [\mathbb{F}^{\mathrm{op}}, \mathbf{Set}]_{\mathfrak{D}} \to \mathcal{A}$, unique up-to unique isomorphism. The extension $G^*$ has a right adjoint, $G_* : \mathcal{A} \to [\mathbb{F}^{\mathrm{op}}, \mathbf{Set}]_{\mathfrak{D}}$, with $G_*(x) = \mathcal{A}(G(-), x)$.

Moreover, if $(\mathbb{F}, \otimes, i)$ is a symmetric monoidal category with $\mathfrak{D}^{\mathrm{op}}$-colimits and $(a \otimes -) : \mathbb{F} \to \mathbb{F}$ preserves $\mathfrak{D}^{\mathrm{op}}$-colimits for all $a$, then $[\mathbb{F}^{\mathrm{op}}, \mathbf{Set}]_{\mathfrak{D}}$ is a symmetric monoidal closed category and the embedding $\mathbb{F} \to [\mathbb{F}^{\mathrm{op}}, \mathbf{Set}]_{\mathfrak{D}}$ preserves the symmetric monoidal structure [8,11].

**Proposition 2.5** *Let $\mathfrak{D}$ be a sound limit doctrine.*

- *Let $\mathbb{F}$ be a small symmetric monoidal category $\mathbb{F}$ with $\mathfrak{D}^{\mathrm{op}}$-colimits such that $a \otimes -$ preserves $\mathfrak{D}^{\mathrm{op}}$-colimits for all $a$ in $\mathbb{F}$. Then $[\mathbb{F}^{\mathrm{op}}, \mathbf{Set}]_{\mathfrak{D}}$ is locally $\mathfrak{D}$-presentable as a closed category, with basis $\mathbb{F}$.*
- *Let $\mathcal{V}$ be locally $\mathfrak{D}$-presentable as a closed category, with basis $\mathbb{F}$. It is equivalent to $[\mathbb{F}^{\mathrm{op}}, \mathbf{Set}]_{\mathfrak{D}}$.*

### *2.4 Actions and powers*

The relationship between monoidal actions and enrichment is widely understood (e.g. [13]) and has proved useful in studying algebraic theories and notions of computation (e.g. [9], [10, Ch. 6], [18], [23]). Proposition 2.7 is the main step towards our two main theorems.

**Definition 2.6** Let $(\mathbb{C}, \otimes, i)$ be a monoidal category and let $\mathcal{A}$ be an ordinary category. An *action* of $\mathbb{C}$ on $\mathcal{A}$ is a functor $M : \mathbb{C} \times \mathcal{A} \to \mathcal{A}$ together with natural isomorphisms

$$M(i, x) \cong x \qquad\qquad M(c \otimes d, x) \cong M(c, M(d, x))$$

satisfying the evident coherence conditions.

Note that any monoidal category acts on itself in the obvious way.

Recall [14] that, for a symmetric monoidal category $(\mathcal{V}, \otimes, I)$, a $\mathcal{V}$-*enriched category* $\underline{\mathcal{C}}$ is like an ordinary category except that between a pair of objects $x$, $y$ in $\underline{\mathcal{C}}$, we have an object $\underline{\mathcal{C}}(x, y)$ of $\mathcal{V}$ instead of a set of morphisms. Any enriched category $\underline{\mathcal{C}}$ has an underlying ordinary category $\mathcal{C}$ with the same objects and with hom-sets $\mathcal{C}(x, y) = \mathcal{V}(I, \underline{\mathcal{C}}(x, y))$. An *enrichment* of an ordinary category $\mathcal{C}$ in $\mathcal{V}$ is defined to be a $\mathcal{V}$-*enriched category* $\underline{\mathcal{C}}$ whose underlying ordinary category is $\mathcal{C}$. For example, if $\mathcal{V}$ is a symmetric monoidal closed category, then the closed structure provides an enrichment of $\mathcal{V}$ in itself.

Finally, recall the definition of powers (aka cotensors) in a category $\underline{\mathcal{C}}$ enriched in a symmetric monoidal closed category $(\mathcal{V}, \otimes, I)$. For $x \in \underline{\mathcal{C}}$, $a \in \mathcal{V}$, a *power* is an object $x^a$ together with an isomorphism $\mathcal{V}(a, \underline{\mathcal{C}}(y, x)) \cong \underline{\mathcal{C}}(y, x^a)$ that is natural in $y$.

**Proposition 2.7** *Let $\mathfrak{D}$ be a sound limit doctrine. Let $(\mathcal{V}, \otimes, I)$ be locally $\mathfrak{D}$-presentable as a closed category with a basis $\mathbb{F}$. Let $\mathcal{C}$ be an ordinary category. The following data are equivalent.*

(i) *An action $M$ of the monoidal category $\mathbb{F}^{\mathrm{op}}$ on $\mathcal{C}$ such that for each $x$ in $\mathcal{C}$ the functor $M(-, x) : \mathbb{F}^{\mathrm{op}} \to \mathcal{C}$ preserves $\mathfrak{D}$-limits.*

(ii) *An enrichment of $\mathcal{C}$ in $\mathcal{V}$ with powers by objects in $\mathbb{F}$.*

*Proof notes.* From (ii) to (i): let $M(a, x)$ be the power $x^a$. From (i) to (ii): we define the hom-object $\underline{\mathcal{C}}(x, y)$ in $\mathcal{V}$ by working up to the equivalence $\mathcal{V} \simeq [\mathbb{F}^{\mathrm{op}}, \mathbf{Set}]_{\mathfrak{D}}$: let $\underline{\mathcal{C}}(x, y)(a) = \mathcal{C}(x, y^a)$. □

Proposition 2.7 is probably known quite widely. An instance ($\mathfrak{D} = \emptyset$) of Proposition 2.7 is implicit in Levy's work on call-by-push-value [18] and more recently explicit in Melliès work ([22, Prop. 11], [21, Lecture 6]).

## 3 Enriched Lawvere theories and Freyd categories

We now consider a definition of Lawvere theory enriched in a locally $\mathfrak{D}$-presentable closed category. We recall the definitions of Freyd category and distributive Freyd category, and show that they are instances of the concept of Lawvere theory.

### *3.1 Enriched Lawvere theories*

**Definition 3.1** Let $\mathfrak{D}$ be a sound limit doctrine, and let $(\mathcal{V}, \otimes, I)$ be locally $\mathfrak{D}$-presentable as a closed category, with a basis $\mathbb{F}$. A *$\mathcal{V}$-Lawvere theory* is given by

- a category $\mathbb{L}$ enriched in $\mathcal{V}$ with powers by objects of $\mathbb{F}$.
- an identity-on-objects $\mathcal{V}$-functor $\mathbb{F}^{\mathrm{op}} \to \mathbb{L}$ that preserves powers by objects of $\mathbb{F}$.

The choice of basis $\mathbb{F}$ is irrelevant to the following extent. Define a *change of basis* $(\rho, r, s) : \mathbb{F} \to \mathbb{F}'$ to be given by, for each $a$ in $\mathbb{F}$ a choice of a section/retraction pair, $(a \overset{s_a}{\rightarrowtail} \rho_a \overset{r_a}{\twoheadrightarrow} a) = \mathsf{id}$, with $\rho_a$ in $\mathbb{F}'$. This choice determines an assignment from Lawvere theories $\mathbb{L}'$ wrt $\mathbb{F}'$ to Lawvere theories $\mathbb{L}$ wrt $\mathbb{F}$: let $\mathbb{L}(a, b)$ be the equalizer

$$\begin{tikzcd} \mathbb{L}(a,b) \arrow[r] & \mathbb{L}'(\rho_a, \rho_b) \arrow[r, shift left, "s_b \cdot r_b \cdot -"] \arrow[r, shift right, "- \cdot s_a \cdot r_a"'] & \mathbb{L}'(\rho_a, \rho_b) \end{tikzcd}$$

(We could simplify this situation by requiring bases to be closed under retracts, but this would complicate our main theorems, 3.3 and 3.5.)

When $\mathfrak{D} = \mathfrak{FinLim}$, Definition 3.1 is the definition of Power [30]. When, moreover, $\mathcal{V}$ is the category of sets with cartesian product structure, this is the original definition of Lawvere [17]. For a broader study of notions of Lawvere theory, including this one, see the article by Lack and Rosický [16]. It follows from the results in [16, §7] that, for a locally $\mathfrak{D}$-presentable closed category $\mathcal{V}$, to give a Lawvere $\mathcal{V}$-theory is to give an enriched monad on $\mathcal{V}$ that preserves $\mathfrak{D}$-filtered colimits.

### 3.2 Freyd categories

We recall a formulation of Freyd categories proposed by Levy [18, App. B].

**Definition 3.2** A *Freyd category* is given by

- a small category $\mathbb{V}$ with finite products;
- a small category $\mathbb{C}$;
- an action of $\mathbb{V}$ on $\mathbb{C}$ (with the finite products providing a symmetric monoidal structure for $\mathbb{V}$);
- an identity on objects functor $\mathbb{V} \to \mathbb{C}$ that preserves the actions.

**Theorem 3.3** *The following data are equivalent.*

- *A Freyd category.*
- *A Lawvere theory enriched in a locally $\emptyset$-presentable cartesian closed category.*

Proof notes. A Freyd category, i.e. an identity-on-objects action-preserving-functor $\mathbb{V} \to \mathbb{C}$, can equivalently be described as an identity-on-objects action-preserving-functor $\mathbb{V}^{\mathrm{op}} \to \mathbb{C}^{\mathrm{op}}$, which (by Prop. 2.7) is the same thing as a $[\mathbb{V}^{\mathrm{op}}, \mathbf{Set}]$-enriched power-preserving functor $\mathbb{V}^{\mathrm{op}} \to \mathbb{C}^{\mathrm{op}}$, which is the same thing as a Lawvere theory enriched in a locally $\emptyset$-presentable cartesian closed category.

### 3.3 Distributive Freyd categories

Recall that a distributive category is a category with finite sums and products such that for all objects $a$ the functor $a \times (-)$ preserves sums. This is a model for simple first order type theory with sums and products. A distributive Freyd category [19,31,23], then, is a model for an effectful first order language with sums and products.

**Definition 3.4** A *distributive Freyd category* is given by

- a distributive category $\mathbb{V}$;
- a category $\mathbb{C}$ with finite coproducts;
- an action of $\mathbb{V}$ on $\mathbb{C}$ that distributes over coproducts (i.e. $M(a, -)$ preserves coproducts for all $a$ in $\mathbb{V}$);
- an identity on objects functor $\mathbb{V} \to \mathbb{C}$ that preserves the action and coproducts.

**Theorem 3.5** *The following data are equivalent.*

- *A distributive Freyd category.*
- *A Lawvere theory enriched in a locally $\mathfrak{FinProd}$-presentable cartesian closed category.*

*Remark.* In this paper we focused on three sound limit doctrines: finite limits, finite products, and the empty doctrine. I am only aware of three other kinds of sound limit doctrine: terminal objects (whose enriched Lawvere theories are like distributive Freyd categories but with an initial object instead of all finite coproducts), finite connected limits, and $\lambda$-small limits for a regular cardinal $\lambda$.

*Other work in this direction.*

Power [31] already used ordinary Lawvere theories to build examples of distributive Freyd categories. He moreover showed how to build an enriched monad on $[\mathbb{V}^{\mathrm{op}}, \mathbf{Set}]$ from a Freyd category $\mathbb{V} \to \mathbb{C}$, and how to build an enriched monad on $[\mathbb{V}^{\mathrm{op}}, \mathbf{Set}]_{\mathfrak{FinProd}}$ from a distributive Freyd category $\mathbb{V} \to \mathbb{C}$. (There, enriched monads are explained in terms of closed Freyd categories.)

Several other authors have discussed the relationships between Freyd categories, monads and the Yoneda embedding [4,5,6,12].

My own main starting point was my work with Møgelberg [23]. We considered 'effect theories', which are a programming language syntax for those distributive Freyd categories where $\mathbb{V}$ is a free finite coproduct completion of a category with finite products. In that work we used effect theories in the same way that one uses classical algebraic theories, by considering their models and comodels. Subsequently I developed 'parameterized algebraic theories' [35,36], which are an alternative syntax and deduction system for the same structures (with syntax inspired by [26,27]). One could say that the programming language syntax is for distributive Freyd categories $\mathbb{V} \to \mathbb{C}$ whereas the algebraic syntax is for the corresponding enriched Lawvere theories $\mathbb{V}^{\mathrm{op}} \to \mathbb{C}^{\mathrm{op}}$.

The purpose of this paper was to emphasise the relationship between effectful computation and universal algebra.

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