Effect Algebras, Presheaves, Non-locality and Contextuality☆

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Abstract

Non-locality and contextuality are among the most counterintuitive aspects of quantum theory. They are difficult to study using classical logic and probability theory. In this paper we start with an effect algebraic approach to the study of non-locality and contextuality. We will see how different slices over the category of set valued functors on the natural numbers induce different settings in which non-locality and contextuality can be studied. This includes the Bell, Hardy and Kochen-Specker-type paradoxes. We link this to earlier sheaf theoretic approaches by defining a fully faithful embedding of the category of effect algebras in this presheaf category over the natural numbers.

Keywords: Effect Algebras, Presheaves, Non-locality, Contextuality

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1. Introduction

This paper is about generalized theories of probability that allow us to analyze the non-locality and contextuality paradoxes from quantum theory. Informally, the paradoxes have to do with the idea that it might not be possible to explain the outcomes of measurements in a classical way.

The paper is in two parts. In the first we establish new relationships between two generalized theories of probability. In the second we analyze the paradoxes of contextuality using our theories of probability, and we use this to recover earlier formulations of them in different frameworks.

1.1. Generalized probability measures

Recall that a finite measurable space (X, Ω) comprises a finite set X and a sub-Boolean algebra Ω of the powerset $\Omega \subseteq \mathcal{P}(X)$, and recall:

Definition 1. *A probability distribution on a finite measurable space* (X, Ω) *is a function* $p : \Omega \to [0, 1]$ *such that* $p(X) = 1$ *and if* $A_1 \dots A_n$ *are disjoint sets in* Ω*, then* $\sum_{i=1}^{n} p(A_i) = p(\bigcup_{i=1}^{n} A_i)$.

☆This article is a significant extension of the extended abstract in Proc. ICALP 2015 [37].

We now analyze this definition to propose two general notions of probability measure. To this end, here we will focus on finite probability spaces, because this is sufficient for our examples. We intend to return to infinite spaces in future work.

Partial monoids. Our first generalization involves partial monoids. Notice that the conditions on the probability distribution $p : \Omega \to [0,1]$ do not involve the space $\mathcal{P}(X)$. We only used the disjoint union structure of Ω. More generally, we can define a *pointed partial commutative monoid* (PPCM) to be a structure $(E, \ovee, 0, 1)$ where $\ovee : E \times E \to E$ is a commutative, associative partial binary operation with a unit 0. Then $(\Omega, \uplus, \emptyset, X)$ and the interval $([0,1], +, 0, 1)$ are PPCMs. A probability distribution is now the same thing as a PPCM homomorphism, $(\Omega, \uplus, \emptyset, X) \to ([0,1], +, 0, 1)$. Thus PPCMs are a candidate for a generalized probability theory. (This is a long-established position; see e.g. [14].)

Functors. Our second generalization goes as follows. Every finite Boolean algebra Ω is isomorphic to one of the form $\mathcal{P}(N)$ for a finite set N. The elements of N are the atoms of Ω. Now, a probability distribution $p : \Omega \to [0,1]$ is equivalently given by a function $q : N \to [0,1]$ such that $\sum_{a \in N} q(a) = 1$. Let

$$D(N) = \{q : N \to [0,1] \mid \textstyle\sum_{a \in N} q(a) = 1\} \tag{1}$$

be the set of all distributions on a finite set N. It is well-known that D extends to a functor $D : \mathbf{FinSet} \to \mathbf{Set}$. The Yoneda lemma gives a bijection between distributions in $D(N)$ and natural transformations $\mathbf{FinSet}(N, -) \to D$. Thus we are led to say that a generalized finite measurable space is a functor $F : \mathbf{FinSet} \to \mathbf{Set}$ (aka presheaf), and a probability distribution on F is a natural transformation $F \to D$. (This appears to be a new position.)

Relationship. Our main contribution in Section 2 and 3 is an adjunction between the two kinds of generalized measurable spaces: PPCMs, and presheaves $\mathbf{FinSet} \to \mathbf{Set}$. 'Effect algebras' are a special class of PPCMs [16, 12]. We show that our adjunction restricts to a reflection from effect algebras into presheaves $\mathbf{FinSet} \to \mathbf{Set}$, which gives us a slogan that 'effect algebras are well-behaved generalized finite measurable spaces'.

1.2. Relating non-locality and contextuality arguments

In the second part of the paper we investigate three paradoxes from quantum theory, attributed to Bell, Hardy and Kochen-Specker. We justify our use of effect algebras and presheaves by establishing relationships with earlier work by Abramsky and Brandenburger [2], Hamilton, Isham and Butterfield [22] and with the test space approach [6]. For the purposes of introduction, we focus on the Bell paradox, and we focus on the mathematics. (Some physical intuitions are given in Section 4.)

The Bell paradox in terms of effect algebras and presheaves. As we show, the Bell scenario can be understood as a morphism of effect algebras $E \xrightarrow{t} [0,1]$, i.e., a generalized probability distribution. The paradox is that although this has a quantum realization, in that it factors through $\mathit{Proj}(\mathcal{H})$, the projections on a Hilbert space $\mathcal{H}$, it has no explanation in classical probability theory, in that there it does not factor through a given Boolean algebra Ω. Informally:

$$
\begin{tikzcd}
E \arrow[rr, "t"] \arrow[dr] & & {[0,1]} \\
& \mathit{Proj}(\mathcal{H}) \arrow[ur] &
\end{tikzcd}
\quad \text{but} \quad
\begin{tikzcd}
E \arrow[rr, "t"] \arrow[dr] & & {[0,1]} \\
& \Omega \arrow[ur, "|" marking] &
\end{tikzcd}
\tag{2}
$$

Relationship with earlier sheaf-theoretic work on the Bell paradox. In [2], Abramsky and Brandenburger have studied Bell-type scenarios in terms of presheaves. We recover their results from our analysis in terms of generalized probability theory. Our first step is to notice that effect algebras essentially fully embed in the functor category $[\mathbf{FinSet} \to \mathbf{Set}]$. We step even closer by recalling the slice category construction. This is a standard technique of categorical logic for working relative to a particular object. As we explain in Section 4, the slice category $[\mathbf{FinSet} \to \mathbf{Set}]/\Omega$ is again a presheaf category. It is more-or-less the category used in [2]. Moreover, our non-factorization (2) transports to the slice category: Ω becomes terminal, and E is a subterminal object. Thus the non-factorization in diagram (2) can be phrased in the sheaf-theoretic language of Abramsky and Brandenburger: 'the family t has no global section'.

Other paradoxes. Alongside the Bell paradox we study two other paradoxes:

- The Hardy paradox is similar to the Bell paradox, except that it uses possibility rather than probability. We analyze this by replacing the unit interval $([0,1],+,0,1)$ by the PPCM $(\{0,1\},\vee,0,1)$ where $\vee$ is bitwise-or. Although this monoid is not an effect algebra, everything still works and we are able to recover the analysis of the Hardy paradox by Abramsky and Brandenburger.

- The Kochen-Specker paradox can be understood as saying that there is no PPCM morphism

$$\mathit{Proj}(\mathcal{H}) \to (\{0,1\},\ovee,0,1) \tag{3}$$

  with $\dim \mathcal{H} \geq 3$ and where $\ovee$ is like bitwise-or, except that $1 \ovee 1$ is undefined. Now, the slice category $[\mathbf{FinSet} \to \mathbf{Set}]/\mathit{Proj}(\mathcal{H})$ is again a presheaf category, and it is more-or-less the presheaf category used by Hamilton, Isham and Butterfield. The non-existence of a homomorphism (3) transports to this slice category: $\mathit{Proj}(\mathcal{H})$ becomes the terminal object, and $(\{0,1\},\ovee,0,1)$ becomes the so-called 'spectral presheaf'. We are thus able to rephrase the non-existence of a homomorphism (3) in the same way as Hamilton, Isham and Butterfield [22]: 'the spectral presheaf does not have a global section'.

2. Pointed Partial Commutative Monoids and Effect Algebras

A probability distribution on a finite measurable space is a function satisfying certain conditions. We can understand this function as a homomorphism of partial algebraic structures, as follows.

Definition 2 (e.g. [16]). *A* pointed partial commutative monoid *(or* PPCM*)* $(E, 0, 1, \varovee)$ *consists of a set* E *with chosen elements* 0 *and* $1 \in E$ *and a partial function* $\varovee : E \times E \to E$*, such that for all* $x, y, z \in E$ *we have:*

1. *If* $x \varovee y$ *is defined, then* $y \varovee x$ *is also defined and* $x \varovee y = y \varovee x$*.*
2. $x \varovee 0$ *is always defined and* $x = x \varovee 0$*.*
3. *If* $x \varovee y$ *and* $(x \varovee y) \varovee z$ *are defined, then* $y \varovee z$ *and* $x \varovee (y \varovee z)$ *are defined and* $(x \varovee y) \varovee z = x \varovee (y \varovee z)$*.*

We write $x \perp y$ *(say* x *is perpendicular to* y*), if* $x \varovee y$ *is defined. When we write* $x \varovee y$*, we tacitly assume* $x \perp y$*. We refer to* $x \varovee y$ *as the* sum *of* x *and* y*.*

A morphism $f : E \to F$ *of PPCMs is a map such that* $f(0) = 0$*,* $f(1) = 1$ *and* $f(a \varovee b) = f(a) \varovee f(b)$ *whenever* $a \perp b$*. This entails the category* **PPCM***.*

Example 3. *Any Boolean algebra* $(B, \vee, \wedge, 0, 1)$ *can be understood as a PPCM* $(B, 0, \varovee, 1)$ *where* $x \perp y$ *iff* $x \wedge y = 0$*, and then* $x \varovee y \overset{def}{=} x \vee y$*. For example, for any set* X*, the powerset forms a PPCM, with disjoint union:* $(\mathcal{P}(X), \emptyset, \uplus, X)$*.*

The unit interval $[0, 1]$ *also forms a PPCM* $([0, 1], 0, +, 1)$*, where* $x \perp y$ *if and only if* $x + y \leq 1$*.*

Let (X, Ω) *be a finite measurable space. Since* Ω *is Boolean, we can view it as a PPCM. Then a function* $\Omega \to [0, 1]$ *is a probability distribution if and only if it is a PPCM morphism.*

PPCMs seem to be the broadest class of structures that allow us to understand the literal definition of 'probability distribution' (Def. 1) as a morphism of algebraic structures. However, it is natural to ask whether there is a class of well-behaved PPCMs that still contains both the Boolean algebras and the interval $[0, 1]$. Consensus has emerged around effect algebras [16, 12], which include Boolean algebras and the unit interval. Effect algebras are well-behaved in many ways. One example is our Corollary 10; another example is construction of universal vector spaces [e.g. 16, §10], although we won't use that here.

Definition 4 ([16]). *An* effect algebra $(E, 0, \varovee, 1)$ *is a PPCM* $(E, 0, \varovee, 1)$ *such that*

1. *For every* $x \in E$ *there exists a unique* $x^\perp$ *such that* $x \perp x^\perp$ *and* $x \varovee x^\perp = 1$*.*
2. $x \perp 1$ *implies* $x = 0$*.*

We call $x^\perp$ the 'orthocomplement of x'. PPCM morphisms between effect algebras always preserve orthocomplements. We denote by **EA** the full subcategory of **PPCM** whose objects are effect algebras.

Definition 5. *An* atom *in a PPCM E is a non-zero element $a \in E$, such that there are no non-zero elements $b, c \in E$ with $a = b \otimes c$.*

Example 6.

- *The PPCM associated to a Boolean algebra is an effect algebra; $x^\perp$ is the complement in the Boolean algebra. In fact, Boolean algebras form a full subcategory of PPCMs, that is, a function between Boolean algebras is a Boolean algebra homomorphism if and only if it is a morphism of PPCMs.*
- *The unit interval $[0, 1]$ is an effect algebra; $x^\perp$ is $(1 - x)$.*
- *We will consider the set $2 = \{0, 1\}$ as a PPCM in two ways.*
  - *The initial PPCM $(2, \otimes, 0, 1)$ has $0 \otimes 0 = 0$ and $1 \otimes 0 = 0 \otimes 1 = 1$. The term $1 \otimes 1$ is undefined; this comes from the initial Boolean algebra and so it is an effect algebra.*
  - *The monoid $(2, \vee, 0, 1)$ with $0 \vee 0 = 0$ and $1 \vee 0 = 0 \vee 1 = 1 \vee 1 = 1$; this is not an effect algebra.*
- *The projections on a Hilbert space form an effect algebra $(Proj(\mathcal{H}), 0, +, 1)$ where $p \perp q$ if their ranges are orthogonal.*

As noted in the introduction, any finite Boolean algebra Ω is isomorphic to a powerset of a finite set, $\mathcal{P}(N)$. Therefore, the PPCM morphisms $\Omega \to [0, 1]$ are in bijective correspondence with probability distributions on this underlying set N. This motivates our next section.

3. Presheaves and tests

In this section we consider a different generalization of probability spaces. Recall that for any finite set N we have a set $D(N)$ of distributions (Equation (1)). This construction is functorial in N. Consider the category $\mathbb{N}$, the skeleton of **FinSet**, whose objects are natural numbers considered as sets, $N = \{1, \ldots, n\}$, and whose morphisms are functions. Then $D : \mathbb{N} \to \mathbf{Set}$ is functorial with $((D\,f)(q))(i) = \sum_{j \in f^{-1}(i)} q(j)$.

This leads us to a notion of generalized probability space via the Yoneda lemma. Write $\mathbf{Set}^{\mathbb{N}}$ for the category of functors $\mathbb{N} \to \mathbf{Set}$ (aka 'covariant presheaves' and natural transformations. The Yoneda lemma says $D(N) \cong \mathbf{Set}^{\mathbb{N}}(\mathbb{N}(N, -), D)$.

More generally we can thus understand natural transformations $F \to D$ as 'distributions' on a functor $F \in \mathbf{Set}^{\mathbb{N}}$. Informally, $F(N)$ is the set of partitions of F into N disjoint components. We can use a similar motivation to build a presheaf of tests from any PPCM.

Definition 7. *Let E be a PPCM. An n-*test *in E is an n-tuple $(e_1, \ldots, e_n)$ of elements in E such that $e_1 \otimes \ldots \otimes e_n = 1$.*

The tests of a PPCM E form a presheaf $T(E) \in \mathbf{Set}^{\mathbb{N}}$, where $T(E)(N)$ is the set of n-tests in E, and if $f : N \to M$ is a function then

$$T(E)(f)(e_1, \ldots, e_n) = (\bigovoid_{i \in f^{-1}(j)} e_i)_{j=1,\ldots,m}$$

This extends to a functor $T : \mathbf{PPCM} \to \mathbf{Set}^{\mathbb{N}}$. If $\psi : E \to A$ is a PPCM morphism, then we obtain the natural transformation $T(\psi)$ with components $T(\psi)_N(e_1, \ldots, e_n) = (\psi(e_1), \ldots, \psi(e_n))$. (See also [25, Def. 6.3].)

Example 8.

- *$T(2, \ovee, 0, 1) \in \mathbf{Set}^{\mathbb{N}}$ is the inclusion: $(T(2, \ovee, 0, 1))(N) = N$.*
- *$T(2, \vee, 0, 1) \in \mathbf{Set}^{\mathbb{N}}$ is the non-empty powerset functor: $(T(2, \vee, 0, 1))(N) = \{S \subseteq N \mid S \neq \emptyset\}$.*
- *Any finite Boolean algebra $(B, \vee, \wedge, 0, 1)$ is of the form $\mathcal{P}(N)$ for a finite set N; we have $T(B, \ovee, 0, 1) = \mathbb{N}(N, -)$, the representable functor.*
- *For the unit interval, $T([0, 1], +, 0, 1) = D$, the distribution functor.*

Our main result in this section is that the test functor essentially exhibits effect algebras as a full subcategory of $\mathbf{Set}^{\mathbb{N}}$.

Theorem 9. *Let A and B be PPCMs. If A is an effect algebra, the induced functor $T_{A,B} : \mathbf{PPCM}(A, B) \to \mathbf{Set}^{\mathbb{N}}(TA, TB)$ is a bijection.*

Proof. We denote functions from an n-element set to an m-element set as lines from n nodes above to m nodes below. For example, $(\;) : \{1, 2, 3\} \to \{1, 2\}$ is the map $1 \mapsto 1$ and $2, 3 \mapsto 2$. We use this notation as well for the image under a functor. Now if A is an effect algebra, every element $a \in A$ is part of a 2-test $(a, a^{\perp})$. It is then clear that $T_{A,B}$ is injective. To show surjectivity, suppose we have some natural transformation $\mu : T(A) \to T(B)$. We need to find a PPCM morphism $\psi_\mu : A \to B$ such that $T(\psi_\mu) = \mu$. For $a \in A$, consider the 2-test $(a, a^{\perp})$ and let $\psi_\mu(a) = x$, where $(x, x') = \mu_2(a, a^{\perp})$. Note that x' is any complement of x, not necessarily unique. We show ψ_μ is indeed a PPCM morphism. So let $a \perp b$ in A and let $c = (a \ovee b)^{\perp}$. Then (a, b, c) is a 3-test. Let $\mu_3(a, b, c) = (x, y, z)$, then we immediately see $x \perp y$ in B. Furthermore, by naturality of μ we have

$$\begin{aligned} \mu_2(a, a^{\perp}) &= \mu_2 (\;) (a, b, c) \\ &= (\;) \mu_3(a, b, c) \\ &= (x, y \ovee z). \end{aligned}$$

Therefore $\psi_\mu(a) = x$ and by a similar argument $\psi_\mu(b) = y$.

We then calculate

$$\begin{aligned} \mu_2(a \ovee b, c) &= \mu_2 (\;) (a, b, c) \\ &= (\;) \mu_3(a, b, c) \\ &= (x \ovee y, z). \end{aligned}$$

Showing that indeed $\psi_\mu(a \ovee b) = x \ovee y$.

To show ψ_μ preserves 1 we calculate

$$\begin{aligned}
\mu_2(1_A, 0_A) &= \mu_2\,(\;)\,(1_A) \\
&= (\;)\,\mu_1(1_A) \\
&= (\;)\,1_B \\
&= (1_B, 0_B).
\end{aligned}$$

Similarly ψ_μ preserves 0. By construction we have $T_{A,B}(\psi_\mu) = \mu$, so $T_{A,B}$ is indeed a bijection.

Corollary 10. *The restriction to effect algebras, $T : \mathbf{EA} \to \mathbf{Set}^{\mathbb{N}}$, is full and faithful.*

To summarize Sections 2 and 3, we have the following generalizations of Boolean algebras. Each arrow is an embedding, i.e., a full and faithful functor.

$$\begin{tikzcd}
 & & \mathbf{PPCM} \\
\mathbf{BA} \arrow[r] & \mathbf{EA} \arrow[ur] \arrow[dr] & \\
 & & {[\mathbb{N}, \mathbf{Set}]}
\end{tikzcd}$$

Remark on nerves and realizations. We remark that a more abstract way to view the test functor is through the framework of nerves and realizations. In general, a functor $F : \mathbb{C} \to \mathbf{D}$ induces a 'nerve' functor $\mathbf{D}(F(-), =) : \mathbf{D} \to [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$. The usual motivating example is the nerve of a topological space X, which is a simplicial set $\mathbf{Top}(F(-), X) : \Delta^{\mathrm{op}} \to \mathbf{Set}$; this is induced by considering each simplex as a space, $F : \Delta \to \mathbf{Top}$. Moreover, the nerve functor has a left adjoint $[\Delta^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Top}$, the left Kan extension of F along the Yoneda embedding, which takes a simplicial set to a topological realization. Another example is the nerve of a category X, which is a simplicial set $\mathbf{Cat}(F(-), X) : \Delta^{\mathrm{op}} \to \mathbf{Set}$; this is induced by considering each simplex as a category, $F : \Delta \to \mathbf{Cat}$. In this case, the nerve functor $\mathbf{Cat} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ is full and faithful. In other words, every category is a canonical colimit of simplices, and so we say that $F : \Delta \to \mathbf{Cat}$ is dense.

In our setting, we have finite Boolean algebras instead of simplices and effect algebras instead of topological spaces. For any natural number N, the powerset $\mathcal{P}(N)$ is a Boolean algebra and hence an effect algebra. This extends to a functor $\mathcal{P} : \mathbb{N}^{\mathrm{op}} \to \mathbf{PPCM}$, and the test functor is the corresponding 'nerve' functor, $T = \mathbf{PPCM}(=, P(-)) : \mathbf{PPCM} \to [\mathbb{N}, \mathbf{Set}]$.

Theorem 11. *The test functor $T : \mathbf{PPCM} \to [\mathbb{N}, \mathbf{Set}]$ has a left adjoint.*

Proof. See for instance Theorem 2 of [31, Ch. I.5]. The category **PPCM** is cocomplete since it is a category of models of an essentially algebraic theory [7, Theorem 3.36].

This left adjoint is the left Kan extension of $\mathcal{P}$ along the Yoneda embedding. Theorem 9 can be phrased 'the counit is an isomorphism at effect algebras'. Corollary 10 can be phrased 'finite Boolean algebras are dense in effect algebras'. This last statement uses the fact that $\mathbb{N}^{\mathrm{op}} \simeq \mathbf{FinBA}$, the category of finite Boolean algebras, so that $[\mathbb{N}, \mathbf{Set}] \cong [\mathbb{N}^{\mathrm{op}\,\mathrm{op}}, \mathbf{Set}] \simeq [\mathbf{FinBA}^{\mathrm{op}}, \mathbf{Set}]$ which is the free cocompletion of the category of finite Boolean algebras. So every effect algebra is a canonical colimit of finite Boolean algebras.

As an aside, we remark that our category $\mathbb{N}$ is equivalent to finite Hausdorff spaces, which yields a connection to a recent related result: compact Hausdorff spaces are dense in piecewise C^*-algebras [15, Thm 4.5].

There also appears to be a connection with the theory of 'contextuality by default' [13], which presents a scenario as (informally) a gluing of random variables; this relationship deserves to be explored more carefully.

4. Bell scenarios: tables and effect algebras

In probability theory, questions of contextuality arise from the problem that the joint probability distribution for all outcomes of all measurements may not exist. We suppose a simple framework where Alice and Bob each have a measurement device with two settings. For simplicity we suppose that the device will emit 0 or 1, as the outcome of a measurement. We write a_0:0 for 'Alice measured 0 with setting a_0', b_1:0 for 'Bob measured 0 with setting b_1', and so on. To model this in classical probability theory we would consider a sample space S_{A} for Alice whose elements are functions $\{\mathrm{a}_0, \mathrm{a}_1\} \to \{0, 1\}$, i.e., assignments of outcomes to measurements. Similarly we have a sample space S_{B} for Bob. We would then consider a joint probability distribution on S_{A} and S_{B}. While Alice and Bob are allowed to classically communicate with each other, in this model, we implicitly assume that Alice and Bob can not signal to each other. That is to say, for any distribution on joint measurements, we can define marginal distributions each for Alice and Bob, which do not depend on the settings chosen by the other person. However, the classical model does include an assumption: that Alice is able to record the outcome of the measurement in both settings. In reality, and in quantum physics in particular, once Alice has recorded an outcome using one measurement setting, she cannot then know what the outcome would have been using the other measurement setting. Effect algebras provide a way to describe a kind of probability distribution that takes this measure-only-once phenomenon into account.

The non-locality 'paradox' is as follows: there are probability distributions in this effect algebraic sense (without signalling), which are physically realizable, but cannot be explained in a classical probability theory without signaling. The first proof of such a system was given by John Bell [8], hence the name Bell scenario.

The main purpose of this section is not to study non-locality and contextuality in different systems, but rather to give a general framework to study them. We use this to recover earlier frameworks.

4.1. Tables

Definition 12. *A function*

$$\tau : \{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1\} \times \{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1\} \to [0, 1]$$

is called a probability table*, or just a* table*, if*

- each experiment certainly has an outcome: *for* $i, j \in \{0, 1\}$,
$$\sum_{o, o' \in \{0,1\}} \tau(\mathrm{a}_i{:}o, \mathrm{b}_j{:}o') \;=\; 1.$$
- it has marginalization, aka no signalling: *for all* $i, j, o \in \{0, 1\}$,
$$\tau(\mathrm{a}_i{:}o, \mathrm{b}_0{:}0) + \tau(\mathrm{a}_i{:}o, \mathrm{b}_0{:}1) = \tau(\mathrm{a}_i{:}o, \mathrm{b}_1{:}0) + \tau(\mathrm{a}_i{:}o, \mathrm{b}_1{:}1),$$
$$\tau(\mathrm{a}_0{:}0, \mathrm{b}_j{:}o) + \tau(\mathrm{a}_0{:}1, \mathrm{b}_j{:}o) = \tau(\mathrm{a}_1{:}0, \mathrm{b}_j{:}o) + \tau(\mathrm{a}_1{:}1, \mathrm{b}_j{:}o).$$

The standard *Bell table* $\tau : \{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1\} \times \{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1\} \to [0, 1]$ is as follows:

t	$\mathrm{a}_0{:}0$	$\mathrm{a}_0{:}1$	$\mathrm{a}_1{:}0$	$\mathrm{a}_1{:}1$
$\mathrm{b}_0{:}0$	$\frac{1}{2}$	0	$\frac{3}{8}$	$\frac{1}{8}$
$\mathrm{b}_0{:}1$	0	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{3}{8}$
$\mathrm{b}_1{:}0$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
$\mathrm{b}_1{:}1$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(4)

In this simple scenario we have two observers, each with two measurement settings, each with two outcomes, but it is straightforward to generalize to more elaborate Bell-like settings. We will come back to this in Section 4.3.

The classical way of modelling the Bell scenario involves a sample space with 16 points; to be precise, we let the sample space be the set of 4-tuples $\{0, 1\}^4$, regarding a 4-tuple $(o_{\mathrm{a}_0}, o_{\mathrm{a}_1}, o_{\mathrm{b}_0}, o_{\mathrm{b}_1})$ as describing an outcome for each measurement setting. Any probability distribution on this set,

$$p : \{0, 1\}^4 \to [0, 1]$$

induces a table by marginalizing the settings that are not tested, for instance

$$t(\mathrm{a}_0{:}o_{\mathrm{a}_0}, \mathrm{b}_0{:}o_{\mathrm{b}_0}) = \sum_{o_{\mathrm{a}_1}, o_{\mathrm{b}_1}} p(o_{\mathrm{a}_0}, o_{\mathrm{a}_1}, o_{\mathrm{b}_0}, o_{\mathrm{b}_1}). \tag{5}$$

Definition 13. *A table is* classically realizable *if it arises from a distribution on* $\{0, 1\}^4$ *according to* (5).

We now have the following:

Proposition 14. *The Bell table is not classically realizable.*

This proposition is almost folklore and there are multiple proofs. See [2] for an example. We shall give a proof in an effect algebraic setting in Corollary 25, after we have introduced tensor products.

A general notion of table. Before we continue we introduce a more general notion of table by replacing $[0,1]$ with an arbitrary PPCM. This general definition is used in two ways. First, in Section 6.1, we consider possibility rather than probability by replacing $[0,1]$ with the PPCM $(\{0,1\}, \vee, 0, 1)$. Second, we use it to classify the tables in terms of effect algebras and presheaves (Prop. 19, Prop. 27, Prop. 29).

Definition 15. *Let X be a PPCM (e.g. $X = [0,1]$). A function*

$$\tau : \{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1\} \times \{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1\} \to X$$

*is called a X-*table *if*

- each experiment certainly has an outcome: *for $i, j \in \{0,1\}$,*
$$\bigovee_{o,o' \in \{0,1\}} \tau(\mathrm{a}_i{:}o, \mathrm{b}_j{:}o') \;=\; 1.$$
- it has marginalization, aka no signalling: *for all $i, j, o \in \{0,1\}$,*
$$\tau(\mathrm{a}_i{:}o, \mathrm{b}_0{:}0) \ovee \tau(\mathrm{a}_i{:}o, \mathrm{b}_0{:}1) = \tau(\mathrm{a}_i{:}o, \mathrm{b}_1{:}0) \ovee \tau(\mathrm{a}_i{:}o, \mathrm{b}_1{:}1),$$
$$\tau(\mathrm{a}_0{:}0, \mathrm{b}_j{:}o) \ovee \tau(\mathrm{a}_0{:}1, \mathrm{b}_j{:}o) = \tau(\mathrm{a}_1{:}0, \mathrm{b}_j{:}o) \ovee \tau(\mathrm{a}_1{:}1, \mathrm{b}_j{:}o).$$

An X-table t is classically realizable *if there is a function $p : \{0,1\}^4 \to X$ such that $\bigovee_{s \in \{0,1\}^4} p(s) = 1$ and $t(\mathrm{a}_0{:}o_{\mathrm{a}_0}, \mathrm{b}_0{:}o_{\mathrm{b}_0}) = \bigovee_{o_{\mathrm{a}_1}, o_{\mathrm{b}_1}} p(o_{\mathrm{a}_0}, o_{\mathrm{a}_1}, o_{\mathrm{b}_0}, o_{\mathrm{b}_1})$.*

4.2. Bell's paradox in effect algebras

The above discussion cannot by phrased in terms of distributions on measurable spaces, but it can be phrased in terms of distributions on effect algebras. The effect algebra is built from four sample spaces: one for each of Alice's measurement settings, and one for each of Bob's measurement settings.

$$E_{\mathrm{A},0} \stackrel{\mathrm{def}}{=} \mathcal{P}(\{\mathrm{a}_0 : 0, \mathrm{a}_0 : 1\}) \qquad E_{\mathrm{A},1} \stackrel{\mathrm{def}}{=} \mathcal{P}(\{\mathrm{a}_1 : 0, \mathrm{a}_1 : 1\})$$
$$E_{\mathrm{B},0} \stackrel{\mathrm{def}}{=} \mathcal{P}(\{\mathrm{b}_0 : 0, \mathrm{b}_0 : 1\}) \qquad E_{\mathrm{B},1} \stackrel{\mathrm{def}}{=} \mathcal{P}(\{\mathrm{b}_1 : 0, \mathrm{b}_1 : 1\})$$

For example, the outcomes of an experiment performed by Alice alone, just using measurement setting 0, are defined by a distribution $E_{\mathrm{A},0} \to [0,1]$.

The outcomes of the entire Bell scenario form a distribution on an effect algebra that is built by combining these four effect algebras using two constructions: the sum ($\oplus$) and the tensor ($\otimes$) of effect algebras. Ultimately, a table (Def. 12) amounts to a distribution $(E_{\mathrm{A},0} \oplus E_{\mathrm{A},1}) \otimes (E_{\mathrm{B},0} \oplus E_{\mathrm{B},1}) \to [0,1]$ (see Prop. 19).

4.2.1. Sums of effect algebras

To define effect algebras for Alice and Bob alone we take the sum of the effect algebras for their two measurement settings. The *sum*, also called *horizontal sum* in [12], is defined as follows.

Definition 16. *Let A and B be non-degenerate PPCMs* ($0 \neq 1$). *The* sum $A \oplus B$ *is the PPCM* $(A \setminus \{0,1\}) \uplus (B \setminus \{0,1\}) \uplus \{0,1\}$. *The orthogonality is given by setting $x \perp y$ if either $x = 0$, or $y = 0$, or $x \perp y$ in A, or $x \perp y$ in B.*

The following is now a straightforward categorical computation.

Proposition 17. *The sum $A \oplus B$, with the evident injections $A \to A \oplus B \leftarrow B$, has the universal property of a coproduct in the category of PPCMs. It induces a natural bijection*

$$\mathbf{PPCM}(A \oplus B, C) \cong \mathbf{PPCM}(A, C) \times \mathbf{PPCM}(B, C)$$

Moreover $A \oplus B$ is an effect algebra if A and B are effect algebras.

We use sums to define effect algebras for Alice and Bob alone as follows:

$$E_{\mathrm{A}} \stackrel{\text{def}}{=} E_{\mathrm{A},0} \oplus E_{\mathrm{A},1} \quad E_{\mathrm{B}} \stackrel{\text{def}}{=} E_{\mathrm{B},0} \oplus E_{\mathrm{B},1}$$

In detail, the effect algebra E_{A} has $\{0, \mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1, 1\}$ as its underlying set. Sums are defined by $0 \ovee x = x$ and $\mathrm{a}_i{:}0 \ovee \mathrm{a}_i{:}1 = 1$. Similarly, the effect algebra E_{B} is $\{0, \mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1, 1\}$ with similar sums.

There are some points to note here.

1. By Proposition 17, a distribution for Alice, $E_{\mathrm{A}} \to [0,1]$, amounts to a pair of distributions $E_{\mathrm{A},0} \to [0,1]$, $E_{\mathrm{A},1} \to [0,1]$, one for each measurement setting that she may choose.
2. The effect algebra E_{A} is not a Boolean algebra. It cannot arise in classical probability theory.
3. Having said this, classical probability theory has no problem with Alice alone, in view of point 1 it is just a pair of distributions, both of which are completely classical. We just cannot make it into a single distribution. The interesting thing happens when we consider joint measurements for Alice and Bob.

4.2.2. Bimorphisms

We now introduce a notion of bimorphism, which captures the notion of a probability distribution on joint measurements. Later we will see that bimorphisms are classified by a tensor product.

Definition 18. *Let A, B and C be pointed partial commutative monoids. A* bimorphism $A, B \to C$ *is a function $f : A \times B \to C$ such that for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$ with $a_1 \perp a_2$ and $b_1 \perp b_2$ we have*

$$f(a, b_1 \ovee b_2) = f(a, b_1) \ovee f(a, b_2) \qquad f(a_1 \ovee a_2, b) = f(a_1, b) \ovee f(a_2, b)$$
$$f(a, 0) = f(0, b) = 0 \qquad f(1, 1) = 1$$

That is to say, both $f(-, 1)$ and $f(1, -)$ are effect algebra morphisms.

We now describe the Bell scenario in the introduction to this section using bimorphisms. A distribution on the joint measurements of Alice and Bob is a bimorphism $E_\mathrm{A}, E_\mathrm{B} \to [0, 1]$. We now give an elementary description of these bimorphisms.

Each bimorphism $t : E_\mathrm{A}, E_\mathrm{B} \to [0, 1]$ restricts to a function

$$\tau : \{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1\} \times \{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1\} \to [0, 1].$$

In fact, this gives a bijective correspondence between bimorphisms and tables. This is true for generalized tables and so we show the general fact.

Proposition 19. *Let X be a PPCM (e.g. $[0, 1]$). A function*

$$\{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1\} \times \{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1\} \to X$$

arises as the restriction of a bimorphism $E_\mathrm{A}, E_\mathrm{B} \to X$ if and only if it is an X-table (Def. 15).

Proof. First, we note the restriction of a bimorphism always satisfies the two conditions; this follows from the definition of bimorphism. To understand the marginalization requirement, notice that for any bimorphism $t : E_\mathrm{A}, E_\mathrm{B} \to X$ we have, for instance,

$$t(\mathrm{a}_0{:}0, \mathrm{b}_0{:}0) \ovee t(\mathrm{a}_0{:}0, \mathrm{b}_0{:}1) = t(\mathrm{a}_0{:}0, 1) = t(\mathrm{a}_0{:}0, \mathrm{b}_1{:}0) \ovee t(\mathrm{a}_0{:}0, \mathrm{b}_1{:}1)$$

since $\mathrm{b}_0{:}0 \ovee \mathrm{b}_0{:}1 = 1 = \mathrm{b}_1{:}0 \ovee \mathrm{b}_1{:}1$.

Second, suppose τ is a table satisfying the two conditions. We extend it to a bimorphism $t : E_\mathrm{A}, E_\mathrm{B} \to X$ as follows:

$$\begin{aligned} t(\mathrm{a}_i{:}o, \mathrm{b}_j{:}o') &= \tau(\mathrm{a}_i{:}o, \mathrm{b}_j{:}o') & t(\mathrm{a}_i{:}o, 1) &= \tau(\mathrm{a}_i{:}j, \mathrm{b}_0{:}0) \ovee \tau(\mathrm{a}_i{:}j, \mathrm{b}_0{:}1) \\ t(x, 0) &= 0 & t(1, \mathrm{b}_j{:}o) &= \tau(\mathrm{a}_0{:}0, \mathrm{b}_j{:}o) \ovee \tau(\mathrm{a}_0{:}1, \mathrm{b}_j{:}o) \\ t(0, y) &= 0 & t(1, 1) &= 1 \end{aligned}$$

By Proposition 19 the Bell table (4) extends to a bimorphism

$$E_\mathrm{A}, E_\mathrm{B} \to [0, 1].$$

Quantum realization. A table has a 'quantum realization' if there is a way to obtain it by performing quantum experiments. Recall that a quantum system is modelled by a Hilbert space $\mathcal{H}$, and a yes-no question such as "is the outcome of measuring a_0 equal to 1" is given by a projection on this Hilbert space. The projections form an effect algebra $\mathit{Proj}(\mathcal{H})$.

Definition 20. *A* quantum realization *for a distribution on joint measurements* $t : E, E' \to [0,1]$ *is given by finite dimensional Hilbert spaces* $\mathcal{H}, \mathcal{H}'$*, two PPCM maps* $r : E \to \mathit{Proj}(\mathcal{H})$ *and* $r' : E' \to \mathit{Proj}(\mathcal{H}')$*, and a bimorphism* $p : \mathit{Proj}(\mathcal{H}), \mathit{Proj}(\mathcal{H}') \to [0,1]$*, such that for all* $e \in E$ *and* $e' \in E'$ *we have* $p(r(e), r'(e')) = t(e, e')$*.*

The Bell table (4) has a quantum realization, with $\mathcal{H} = \mathcal{H}' = \mathbb{C}^2$ (see [4]). The relation between (mixed) states and our notion of quantum realization is as follows. Let $\mathcal{H} \otimes \mathcal{H}'$ be the tensor product of Hilbert spaces. By Gleason's theorem [27] there is a bijection between morphisms $\mathit{Proj}(\mathcal{H} \otimes \mathcal{H}') \to [0,1]$ and density matrices on $\mathcal{H} \otimes \mathcal{H}'$ if $\mathit{dim}(\mathcal{H} \otimes \mathcal{H}') > 3$, which is certainly the case for $\mathcal{H}, \mathcal{H}' = \mathbb{C}^2$. The canonical map $\mathit{Proj}(\mathcal{H}), \mathit{Proj}(\mathcal{H}') \to \mathit{Proj}(\mathcal{H} \otimes \mathcal{H}')$ given by $p, q \mapsto p \otimes q$, where $p \otimes q(h \otimes h') = p(h) \otimes q(h')$, is a bimorphism. Therefore, any density matrix gives rise to a bimorphism $\mathit{Proj}(\mathcal{H}), \mathit{Proj}(\mathcal{H}') \to [0,1]$.

Classical realization. Classically, every time Alice and Bob perform a measurement, nature determines an assignment of outcomes for all measurements, which determines the outcomes for Alice and Bob. In such a deterministic theory we can calculate a probability for things like $\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}1$, in which case if Alice chose a_0 and Bob chose b_1, they would get the outcome 0 and 1, respectively.

Definition 21. *A* classical realization *for a bimorphism* $t : E, E' \to [0,1]$ *is given by two Boolean algebras* B, B'*, two effect algebra morphisms* $r : E \to B$*,* $r' : E' \to B'$ *and a bimorphism* $p : B, B' \to [0,1]$ *such that for all* $e \in E$ *and* $e' \in E'$ *we have* $p(r(e), r'(e')) = t(e, e')$*.*

To link this definition with Definition 13, we consider the Boolean algebra B_{A}, with atoms $\{\mathrm{a}_0{:}i \wedge \mathrm{a}_1{:}j \mid i, j \in \{0,1\}\}$. Note that B_{A} is a free completion of the effect algebra E_{A} to a Boolean algebra, in that, under identification of terms like $(\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0) \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1)$ with $\mathrm{a}_0{:}0$, we have $E_{\mathrm{A}} \subseteq B_{\mathrm{A}}$ and every morphism $E_{\mathrm{A}} \to B$, with B a Boolean algebra, must factor through B_{A}. Similarly, we have the algebra B_{B} for Bob. A distribution $\{0,1\}^4 \to [0,1]$ then corresponds with a bimorphism $B_{\mathrm{A}}, B_{\mathrm{B}} \to [0,1]$. Therefore Proposition 14 can be written as:

Proposition 22. *The bimorphism corresponding to Table 4 does not factor via the canonical maps* $r_{\mathrm{A}} : E_{\mathrm{A}} \to B_{\mathrm{A}}$ *and* $r_{\mathrm{B}} : E_{\mathrm{B}} \to B_{\mathrm{B}}$*. Therefore, Table 4 has no classical realization.*

4.2.3. Tensor products

Tensor products allow us to consider bimorphisms $E_{\mathrm{A}}, E_{\mathrm{B}} \to [0,1]$ as a distribution $E_{\mathrm{A}} \otimes E_{\mathrm{B}} \to [0,1]$.

Definition 23. *The* tensor product *of two PPCMs* E*,* E' *is given by a PPCM* $E \otimes E'$ *and a bimorphism* $i : E, E' \to E \otimes E'$*, such that for every bimorphism* $f : E, E' \to F$ *there is a unique morphism* $g : E \otimes E' \to F$ *such that* $f = g \circ i$*.*

This gives a bijective correspondence between morphisms $E \otimes E' \to F$ and bimorphisms $E, E' \to F$. In fact, all tensor products of effect algebras exist (see e.g. [26]; but they can be trivial [17]).

We return to the example of Alice and Bob. We know (Prop 19) that bimorphisms $E_{\mathrm{A}}, E_{\mathrm{B}} \to X$ correspond bijectively with X-tables, and so morphisms $E_{\mathrm{A}} \otimes E_{\mathrm{B}} \to X$ correspond to X-tables too. From the perspective of category theory, the effect algebra $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$ is thus a representation of the functor $\mathrm{Table} : \mathbf{PPCM} \to \mathbf{Set}$ with $\mathrm{Table}(X)$ the set of X-tables.

We now give concrete descriptions of $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$ and $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$.

Proposition 24.

- *The tensor product of Boolean algebras, $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$, is the free Boolean algebra on the four elements $\{\mathrm{a}_0, \mathrm{a}_1, \mathrm{b}_0, \mathrm{b}_1\}$, where we identify, for example, $\mathrm{a}_1{:}1$ with a_1 and $\mathrm{a}_1{:}0$ with $\neg\mathrm{a}_1$.*
- *The tensor product of effect algebras $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$ is the effect algebra with 16 atoms of the form $\mathrm{a}_i{:}k \wedge \mathrm{b}_j{:}l$ for $i, j, k, l \in \{0, 1\}$. Its atomic tests [1] (and hence the perpendicularity relations) are given by expressions of the form*

  $$(a \wedge b, a \wedge b^{\perp}, a^{\perp} \wedge b', a^{\perp} \wedge \tilde{b}^{\perp})$$

  or

  $$(a \wedge b, a^{\perp} \wedge b, \tilde{a} \wedge b^{\perp}, \tilde{a}^{\perp} \wedge b^{\perp}),$$

  where $a, \tilde{a}$ are atoms in E_{A} and $b, \tilde{b}$ are atoms in E_{B}. Note that there are 8 such tests per expression, but 4 of these overlap, so we have 12 of these atomic tests in total.

Proof. We prove the second statement. Let E be the effect algebra as in the proposition. First we note that for any effect algebra X, any elements $a, \tilde{a} \in E_{\mathrm{A}}$, $b, \tilde{b} \in E_{\mathrm{B}}$, and bimorphism $f : E_{\mathrm{A}} \times E_{\mathrm{B}} \to X$, we have

$$\begin{aligned} 1 &= f(1,1) \\ &= f(a \ovee a^{\perp}, 1) \\ &= f(a, 1) \ovee f(a^{\perp}, 1) \\ &= f(a, b) \ovee f(a, b^{\perp}) \ovee f(a^{\perp}, \tilde{b}) \ovee f(a^{\perp}, \tilde{b}^{\perp}) \end{aligned}$$

and

$$\begin{aligned} 1 &= f(1,1) \\ &= f(1, b \ovee b^{\perp}) \\ &= f(1, b) \ovee f(1, b^{\perp}) \\ &= f(a, b) \ovee f(a^{\perp}, b) \ovee f(\tilde{a}, b^{\perp}) \ovee f(\tilde{a}^{\perp}, b^{\perp}). \end{aligned}$$

Now let $g : E_{\mathrm{A}}, E_{\mathrm{B}} \to X$ be a bimorphism. We define $\tilde{g} : E \to X$ by extension of $\tilde{g}(\mathrm{a}_i{:}k \wedge \mathrm{b}_j{:}l) = g(\mathrm{a}_i{:}k, \mathrm{b}_j{:}l)$. We need to show $g(a \wedge b) \perp g(a' \wedge b')$ whenever $a \wedge b \perp a' \wedge b'$, but this follows from our first observation.

[1] With this we mean tests in which only atoms occur.

The map $g \mapsto \tilde{g}$ is easily seen to be invertible. Indeed, given $\tilde{g}$, define $g(a,b) = \tilde{g}(a \wedge b)$. We conclude the proof by noting that we now have bijections:

$$\begin{array}{cc} E \to X & \text{effect algebra morphism} \\ \hline\hline A \times B \to X & \text{bimorphism} \\ \hline\hline A \otimes B \to X & \text{effect algebra morphism} \end{array}$$

So that by the Yoneda lemma we have $E \cong E_{\mathrm{A}} \otimes E_{\mathrm{B}}$.

The statement of Bell's paradox can now be written in terms of homomorphisms, rather than bimorphisms:

Corollary 25. *Table 4, $t : E_{\mathrm{A}} \otimes E_{\mathrm{B}} \to [0,1]$, does not factor through the embedding $E_{\mathrm{A}} \otimes E_{\mathrm{B}} \to B_{\mathrm{A}} \otimes B_{\mathrm{B}}$.*

Proof. We write the following atoms in $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$ as sums of atoms in $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$:

$$\begin{aligned}
\mathrm{a}_0{:}0 \wedge \mathrm{b}_0{:}0 =& (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \\
& \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \qquad (\star) \\
\mathrm{a}_0{:}0 \wedge \mathrm{b}_1{:}1 =& (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \\
& \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}1) \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}1) \\
\mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 =& (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \vee (\mathrm{a}_0{:}1 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \\
& \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \vee (\mathrm{a}_0{:}1 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \\
\mathrm{a}_1{:}0 \wedge \mathrm{b}_1{:}0 =& (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \vee (\mathrm{a}_0{:}1 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \\
& \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}0) \vee (\mathrm{a}_0{:}1 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}0)
\end{aligned}$$

Let $\phi : B_{\mathrm{A}} \otimes B_{\mathrm{B}} \to [0,1]$ be the supposed probability distribution on $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$. If we apply ϕ to both sides of the above equations we find in the Bell table that the sum of the right hand sides must add up to $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{8}$, $\frac{1}{8}$, respectively.

However, adding the last three equations we obtain

$$\begin{aligned}
\frac{3}{8} =& \phi(\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \\
& + \phi(\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \\
& + \phi(\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \\
& + \phi(\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \\
& + \phi(\text{other terms}),
\end{aligned}$$

but the first four terms already add up to $\frac{1}{2}$ by the first equation $(\star)$ above, and since ϕ takes values in $[0,1]$, this cannot be.

This proof reveals more. Every element $\mathrm{a}_i{:}k \wedge \mathrm{b}_j{:}l$ can be written as the sum of join of four elements in the Boolean algebra $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$. Now if we have any non-signalling probability table, that is, a probability distribution on the effect algebra $E_{\mathrm{A}}, E_{\mathrm{B}}$, we might wonder if it factors through this free Boolean algebra. So we are looking for a probability distribution on the atoms of B_4 such that, whenever some elements of B_4 sum to an element of E, we must have that the sum of the values on the atoms of $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$ equals the value of the corresponding element in E. This way we obtain 16 equations which must be satisfied simultaneously. These equations can be put into a matrix and this way we obtain the *incidence matrix* from Abramsky and Brandenburger in [2].

4.3. Generalization of the Bell 2,2,2 type

As promised, we say a few words on the generalization of the Bell 2,2,2 type systems, where we have 2 observers, each with 2 measurement settings, each with two possible outcomes, to more general systems. We first consider just one observer, Alice. She has a set of measurement settings $\mathrm{a}_1, \ldots, \mathrm{a}_k$. Each of those settings a_i has a set of outcomes O_{a_i}. The powerset $\mathcal{P}(O_{\mathrm{a}_i})$ is an effect algebra in the regular way. The coproduct of these effect algebras, $\bigoplus_i \mathcal{P}(O_{\mathrm{a}_i})$, is now the effect algebra of Alice, E_{A}. We obtain such an effect algebra E_{A_i} for every observer A_i. It is the tensor product $\bigotimes E_{\mathrm{A}_i}$ that we finally want to consider. This procedure restricts to the above case $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$ when we are in the 2,2,2 type scenario.

5. Sheaf theoretic characterization

We now phrase Bell's paradox in the presheaf category, using the language of sheaf theory: the table determines a matching family, but it has no amalgamation. We recall some standard definitions.

Definition 26. *Let $\mathbb{C}$ be a category, let $F : \mathbb{C} \to \mathbf{Set}$ be a functor, and let $(f_i : c \to d_i)_{i \in I}$ be a family of morphisms in $\mathbb{C}$ with common domain. A family of elements $x_i \in F(d_i)$ $(i \in I)$ is a* matching family *if for all $i, j \in I$, and all pairs of morphisms $g_i : d_i \to e$, $g_j : d_j \to e$ such that $g_i f_i = g_j f_j$, we have $F(g_i)(x_i) = F(g_j)(x_j)$. An* amalgamation *for a matching family $(x_i)_{i \in I}$ is an element $x \in F(c)$ such that $x_i = F(f_i)(x)$.*

(Since we consider covariant presheaves we have used a covariant formulation of the concepts from sheaf theory here and in Section 5.1.)

Consider the family of functions $\pi_{i,j} : \{0,1\}^4 \to \{0,1\}^2$ in $\mathbb{N}$, indexed by $(i,j) \in \{0,1\}^2$, given by

$$\pi_{i,j}(o_{\mathrm{a}_0}, o_{\mathrm{a}_1}, o_{\mathrm{b}_0}, o_{\mathrm{b}_1}) = (o_{\mathrm{a}_i}, o_{\mathrm{b}_j}). \tag{6}$$

Let $D : \mathbb{N} \to \mathbf{Set}$ be the distributions functor. We now show that a matching family for (6) in D corresponds to a table (Def. 12). It follows from the following general result about X-tables (Def. 15).

Proposition 27. *Let X be a PPCM (e.g. $[0,1]$) and let $T(X) : \mathbb{N} \to \mathbf{Set}$ be the presheaf of tests.*

A matching family $(\mathrm{d}_{i,j})_{i,j\in\{0,1\}}$ in $(T(X))(\{0,1\}^2)$ for $\{\pi_{i,j} \mid i,j \in \{0,1\}\}$ determines an X-table, with $\tau(\mathrm{a}_i{:}o, \mathrm{b}_j{:}o') = d_{i,j}(o,o')$. Conversely, every X-table arises from a matching family.

An X-table has classical realization if and only if the corresponding matching family has an amalgamation.

Proof. The first requirement on tables corresponds to the fact that $d_{i,j}$ is a test. The second requirement corresponds to the compatibility condition on matching families. For the second part, note that a distribution on the classical sample space $\{0,1\}^4$ is an element of $(T(X))(\{0,1\}^4)$.

The Bell table (4) thus induces a matching family for $(\pi_{i,j})_{i,j}$ in the distributions functor $D = T[0,1]$. It does not have an amalgamation.

5.1. Relating the effect-algebraic and sheaf-theoretic characterizations

The definitions of matching family and amalgamation have a different, equivalent form, which is less elementary but more categorical and which allows us to make a connection with formulations of the paradox in different categories.

Definition 28. *Let c be an object of a category $\mathbb{C}$. A* sieve *on c is a set of elements with common domain, $S \subseteq \{f \mid f : c \to d\}$ that is closed under post-composition (i.e. $f \in S \implies gf \in S$).*

Every family of elements with common domain determines a sieve, by closing under post-composition. A sieve S on c can equivalently be described as a functor $\bar{S} : \mathbb{C} \to \mathbf{Set}$ that is a subfunctor of the hom-functor $\mathbb{C}(c,-)$: let $(\bar{S})(d) = \{f : c \to d \mid f \in S\}$.

Consider a family of maps $(f_i)_{i\in I}$ with domain c, generating a sieve S. We can use the Yoneda lemma to rearrange the definitions of matching family and amalgamation, as follows. A matching family for $(f_i)_{i\in I}$ in $F : \mathbb{C} \to \mathbf{Set}$ is equivalently given by a natural transformation $\bar{S} \to F$. An amalgamation for a matching family $\bar{S} \to F$ is a factorization of $\bar{S} \to F$ through the inclusion $\bar{S} \rightarrowtail \mathbb{C}(c,-)$. We can now rewrite Bell's paradox as a non-factorization in the presheaf category $\mathbf{Set}^{\mathbb{N}}$:

$$\begin{tikzcd} S \arrow[r, "\tau"] \arrow[d] & D \\ \mathbb{N}(\{0,1\}^4,-) \arrow[ur, "/" marking] & \end{tikzcd} \tag{7}$$

where $S(N) = \{f : \{0,1\}^4 \to N \mid \exists i,j,g.\, f = g \circ (\pi_{i,j})\}$.

Another way to obtain a non-factorization statement in the presheaf category is to apply the functor $T : \mathbf{EA} \to \mathbf{Set}^{\mathbb{N}}$, which is full and faithful from effect algebras (Cor. 10), to our effect algebra formulation of the Bell scenario. The resulting diagram

$$\begin{tikzcd} T(E_{\mathrm{A}} \otimes E_{\mathrm{B}}) \arrow[r, "T\tau"] \arrow[d] & T([0,1]) \\ T(B_{\mathrm{A}} \otimes B_{\mathrm{B}}) \arrow[ur, "/" marking] & \end{tikzcd} \tag{8}$$

is isomorphic to (7): for $T(B_{\mathrm{A}} \otimes B_{\mathrm{B}}) \cong T(\mathcal{P}(\{0,1\}^4))$, and $T([0,1]) \cong D$, and also $T(E_{\mathrm{A}} \otimes E_{\mathrm{B}})$ is isomorphic to the sieve S generated by $(\pi_{i,j})_{i,j}$. Indeed, we identify $\{0,1\}^4$ with the atoms of $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$. The maps $\pi_{i,j}$ are now ways to make atoms of $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$ from the Boolean atoms. A function in $S(N)$ is a composite $g \circ \pi_{i,j}$ and hence a test in $T(E_{\mathrm{A}} \otimes E_{\mathrm{B}})$.

5.2. Relationship with the work of Abramsky and Brandenburger

Abramsky and Brandenburger [2] also phrase Bell's paradox in terms of a compatible family with no amalgamations. We now relate our statement with theirs.

Transferring the paradox to other categories. We can use adjunctions to transfer statements of non-factorization (such as Corollary 25) between different categories. Let $\mathcal{C}$ be a category and let $R : \mathbf{EA} \to \mathcal{C}$ be a functor with a left adjoint $L : \mathcal{C} \to \mathbf{EA}$. Let $j : X \to Y$ be a morphism in $\mathcal{C}$, and let $f : L(X) \to A$ be a morphism in $\mathbf{EA}$. Then f factors through $L(j)$ if and only if $f^\sharp : X \to R(A)$ factors through j, where $f^\sharp$ is the transpose of f.

$$\begin{tikzcd} L(X) \arrow[rr, "f"] \arrow[dr, "L(j)"'] & & A \\ & L(Y) \arrow[ur, "|" marking] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f^\sharp"] \arrow[dr, "j"'] & & R(A) \\ & Y \arrow[ur, "|" marking] & \end{tikzcd} \tag{9}$$

We use this technique to derive several equivalent statements of Bell's paradox. To start, the equivalence of the non factoring of the triangles (8) (§5.1) and (2) (§1.2) is immediate from the adjunction between the test functor and its left adjoint (see Theorem 11).

No global section. Recall that if X is an object of a category $\mathcal{C}$ then the objects of the *slice category* $\mathcal{C}/X$ are pairs (C, f) where $f : C \to X$. Morphisms are commuting triangles. The slice category $\mathcal{C}/X$ always has a terminal object, (X, id_X). The projection map $\Sigma_X : \mathcal{C}/X \to \mathcal{C}$, with $\Sigma_X(C, f) = C$, has a right adjoint $\Delta_X : \mathcal{C} \to \mathcal{C}/X$ with $\Delta_X(C) = (C \times X, \pi_2)$. First, notice that, using the adjunction $\Sigma_{\mathbb{N}(16,-)} \dashv \Delta_{\mathbb{N}(16,-)}$ we can rewrite diagram (8) in the slice category $(\mathbf{Set}^{\mathbb{N}})/\mathbb{N}(16,-)$ as:

$$\begin{tikzcd} (T(E_{\mathrm{A}} \otimes E_{\mathrm{B}}), Ti) \arrow[rr, "\langle Tt, Ti \rangle"] \arrow[dr] & & (D \times \mathbb{N}(16,-), \pi_2) \\ & (\mathbb{N}(16,-), \mathrm{id}) \arrow[ur, "|" marking] & \end{tikzcd} \tag{10}$$

Since $(\mathbb{N}(16,-), \mathrm{id})$ is terminal, we can phrase Bell's paradox as "*the local section* $\langle Tt, Ti \rangle : (T(E_{\mathrm{A}} \otimes E_{\mathrm{B}}), Ti) \to (D \times \mathbb{N}(16,-), \pi_2)$ *has no global section*".

Measurement covers. The analysis of Abramsky and Brandenburger is based on a 'measurement cover', which corresponds to our effect algebra $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$.

Fix a finite set X of measurements. In our Bell example, $X = \{\mathrm{a}_0, \mathrm{a}_1, \mathrm{b}_0, \mathrm{b}_1\}$. Also fix a finite set of O of outcomes. In our example, $O = \{0, 1\}$, so $O^X = 16$. Abramsky and Brandenburger work in the category of presheaves $\mathcal{P}(X)^{\mathrm{op}} \to \mathbf{Set}$ on the powerset $\mathcal{P}(X)$ (ordered by subset inclusion). They explain Bell-type paradoxes by considering the family of morphisms (i.e. inclusions) in $\mathcal{P}(X)$:

$$\{\{\mathrm{a}_i, \mathrm{b}_j\} \subseteq X \mid i, j \in \{0, 1\}\}. \tag{11}$$

The Bell scenario is a matching family for the presheaf $D(O^{(-)}) : \mathcal{P}(X)^{\mathrm{op}} \to \mathbf{Set}$ which does not have an amalgamation.

Following Section 5.1, we can rephrase this sheaf-theoretic analysis in terms of a missing factorization in the category of presheaves $\mathcal{P}(X)^{\mathrm{op}} \to \mathbf{Set}$:

$$\begin{tikzcd} \mathcal{M} \arrow[rr] \arrow[dr] & & D(O^{(-)}) \\ & 1 \arrow[ur, "|" marking] & \end{tikzcd} \tag{12}$$

Here 1 is the terminal presheaf. The 'measurement cover' $\mathcal{M} \subseteq 1$ is defined by $\mathcal{M}(U) = \emptyset$ if $\{\mathrm{a}_0, \mathrm{a}_1\} \subseteq U$ or $\{\mathrm{b}_0, \mathrm{b}_1\} \subseteq U$, and $\mathcal{M}(U) = \{*\}$ otherwise. In general, $\mathcal{M}(U)$ is inhabited, i.e., non-empty, if the measurement context U is allowed in the Bell situation.

We now relate this diagram (12) with our diagram (2) from the introduction by using an adjunction between $\mathbf{EA}$ and $\mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}$. We construct this adjunction as the following composite:

$$\mathbf{EA} \underset{\longleftarrow}{\overset{T}{\underset{\top}{\longrightarrow}}} \mathbf{Set}^{\mathbb{N}} \underset{\underset{\Sigma_{O^X}}{\longleftarrow}}{\overset{\Delta_{O^X}}{\underset{\top}{\longrightarrow}}} \mathbf{Set}^{\mathbb{N}}/\mathbb{N}(O^X, -) \;\simeq\; \mathbf{Set}^{(\mathbb{N}^{\mathrm{op}}/(O^X))^{\mathrm{op}}} \underset{\underset{I_!}{\longleftarrow}}{\overset{I^*}{\underset{\top}{\longrightarrow}}} \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}} \tag{13}$$

The first two adjunctions in this composite have already been discussed. The categorical equivalence $\mathbf{Set}^{\mathbb{N}}/\mathbb{N}(O^X, -) \simeq \mathbf{Set}^{(\mathbb{N}^{\mathrm{op}}/(O^X))^{\mathrm{op}}}$ is an instance of a general fact about slices by representable presheaves (e.g. [28, Prop. A.1.1.7, Lem. C2.2.17]): in general, $\mathbf{Set}^{\mathbb{D}^{\mathrm{op}}}/\mathbb{D}(-, d) \simeq \mathbf{Set}^{(\mathbb{D}/d)^{\mathrm{op}}}$.

It remains to explain $I_! \dashv I^*$. The functor $I^* : \mathbf{Set}^{(\mathbb{N}^{\mathrm{op}}/(O^X))^{\mathrm{op}}} \to \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}$ is induced by precomposing with the functor $I : \mathcal{P}(X) \to \mathbb{N}^{\mathrm{op}}/O^X$ that takes a subset $U \subseteq X$ to the pair $(O^U, O^{i_U} : O^X \to O^U)$ where $i_U : U \to X$ is the set inclusion function. It has a left adjoint, $I_!$, for general reasons (see e.g. [28, Prop. A.4.1.4]): the left adjoint is given by left Kan extensions along I.

Proposition 29. *The right adjoint in* (13) *takes the effect algebra* $[0, 1]$ *to the presheaf* $D(O^{(-)}) : \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}$. *The left adjoint in* (13) *takes the measurement cover* $\mathcal{M} \subseteq 1$ *to the effect algebra* $E_{\mathrm{A}} \otimes E_{\mathrm{B}} \subseteq B_{\mathrm{A}} \otimes B_{\mathrm{B}}$.

Proof. Denote the left adjoint of the chain of adjunctions by L and the right adjoint by R. Reading the chain of adjunctions from left to right, starting with an effect algebra A gives the presheaf $RA = T(A)(O^{(-)}) : \mathcal{P}(X)^{\mathrm{op}} \to \mathbf{Set}$. A special case gives $R[0,1] = D(O^{(-)})$. For any effect algebra X, we then have

$$\mathbf{EA}(L\mathcal{M}, X) \cong \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}(\mathcal{M}, RX) = \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}(\mathcal{M}, T(X)(O^{(-)})).$$

If we can now show $\mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}(\mathcal{M}, T(X)(O^{(-)})) \cong \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}(E_{\mathrm{A}} \otimes E_{\mathrm{B}}, X)$, natural in X, we can conclude that $L\mathcal{M} \cong E_{\mathrm{A}} \otimes E_{\mathrm{B}}$, by uniqueness of left adjoints. By Proposition 19, it suffices to show that $\mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}(\mathcal{M}, T(X)(O^{(-)}))$ is in natural bijection with the X-tables, in other words, that a matching family $(d_{\mathrm{a}_i,\mathrm{b}_j})_{i,j}$ for (11) in $(T(X))(O^{(-)})$ is the same thing as an X-table. This comes immediately from expanding the definition of matching family. Note that each $d_{\mathrm{a}_i,\mathrm{b}_j}$ by definition a 4-tuple $(x_{\mathrm{a}_i:0,\mathrm{b}_j:0}, x_{\mathrm{a}_i:0,\mathrm{b}_j:1}, x_{\mathrm{a}_i:1,\mathrm{b}_j:0}, x_{\mathrm{a}_i:1,\mathrm{b}_j:1})$ such that $\bigovee_{o,o'} x_{\mathrm{a}_i:o,\mathrm{b}_j:o'} = 1$. We define a table by $t(\mathrm{a}_i : o, \mathrm{b}_j : o') = (x_{\mathrm{a}_i:o,\mathrm{b}_j:o'})$. The first condition on tables amounts to requiring that each 4-tuple is a test, and the second condition on tables amounts to the compatibility condition for matching families.

Corollary 30. *The adjunction* (13) *relates the effect algebra formulation of Bell's paradox* (2)*, with the formulation of Abramsky and Brandenburger* (12).

5.3. Relationship with techniques for memory locality

The techniques used to study quantum non-locality in this paper are also used in computer science to study locality in computer memory, that is, to analyze which areas of memory a program uses. We now explore this connection. In the simplest model of memory there is a fixed set of all memory locations, L, and a fixed set V of storable values. The entire memory is described by a function $L \to V$, assigning a value to each memory location. This is analogous to the way that the classical sample space of a Bell scenario is a function $X \to O$, assigning an outcome to each measurement. So the set L of memory locations is analogous to the set X of measurements in quantum non-locality, and the set V of storable values is analogous to the set O of outcomes in quantum non-locality. There are various further analogies to be made, but the most striking analogy is between marginalizing some of the measurements in quantum non-locality, and hiding public global memory into private local memory (so-called block structure). To be more precise, we now briefly consider three abstract techniques for memory locality and relate them with techniques for quantum non-locality.

- Typically there is a class of program configurations which includes values for public memory locations as well as some private memory and other structure. One can organize the class of all program configurations as a functor $C : \mathcal{P}(L)^{\mathrm{op}} \to \mathbf{Set}$ (e.g. [32, 38]). The idea is that $C(U)$ is the set of configurations that involve the public locations U. The functorial action hides global memory into local memory. This is analogous to the way that Abramsky and Brandenburger model the measurement cover and the distributions as functors $\mathcal{P}(X)^{\mathrm{op}} \to \mathbf{Set}$. Hiding public memory is thus analogous to marginalization of measurements.

- Partial monoids also play a role in the study of memory locations. A partial commutative monoid with cancellation is often called a 'separation algebra' [10]. These are not always effect algebras. For example, the prototypical separation algebra is the partial monoid of partial functions, $h : L \to V$, which does not have a top element. However, some examples *do* form effect algebras. For example, the unit interval $[0, 1]$ is used as a monoid of permissions (e.g. [10, 11]), and in 'classical bunched implications' a set of top elements is actually postulated [9]. Techniques from effect algebras have been rediscovered, for example, the Riesz decomposition property of effect algebras [12] has been rediscovered as the 'cross-split' property for separation algebras [11].

- Presheaves on sets and functions also play a role in theories of memory locality (e.g. [24, 34, 38]). The idea is to think of a set as a 'store shape'. For example, any set of memory locations $U \subseteq L$ induces a set V^U of possible value assignments to those locations. It is convenient to forget that the sets have the form V^U, and instead work with a category of *all* sets and functions. Roughly speaking, the class of system configurations can be organized into a functor $C : \mathbb{N} \to \mathbf{Set}$, with $C(n)$ the set of configurations when the memory is allowed to take n possible values (see e.g. [30]). This is analogous to the presheaves of tests in quantum non-locality.

  (This is a rough overview; in practice it is useful to vary the objects and morphisms of $\mathbb{N}$ when studying memory locality.)

  When the store shape is of the form $m \times n$, this suggests that the memory splits in two parts, one part with m possible values and the other with n possible values. There is a projection function $m \times n \to m$, and the functorial action $C(m \times n) \to C(m)$ describes how to hide the second part of the memory. This is analogous to marginalization in quantum non-locality.

  Moreover, the convolution tensor product of presheaves, which plays an important role in memory locality (e.g. [33]), appears to be closely related to the tensor product of effect algebras (§4.2.3).

There are limitations to these analogies. It remains to be seen whether the analogies extend to Bell scenarios. It could be argued that programmers typically assume that programs run in a consistent global memory state, i.e. a function $L \to V$, even if they only use part of the memory or a particular store shape. However, this assumption has been challenged in recent work relating Bell scenarios to database consistency [5]. It is possible that relaxed memory models also challenge the assumption.

6. Other paradoxes

6.1. Different kinds of values in tables: paradoxes of possibility

In this section we move from probability to possibility. As far as the authors are aware, this originated from [23]. Let N be a finite set, considered as a sample space; a *possibility distribution* on N is a non-empty subset S of N; the elements of S are the events of N that are possible. Equivalently, a possibility distribution is a function $p : N \to \{0, 1\}$ such that $\bigvee_{i \in N} p(i) = 1$, with $p(i) = 1$ meaning 'i is possible'.

We can move away from the classical situation by replacing the set N by an effect algebra E. We say that a possibility distribution on an effect algebra E is a morphism of PPCMs $E \to (\{0, 1\}, \vee, 0, 1)$ into the pointed monoid.

The other direction of generalization begins by using the Yoneda lemma to conclude that a possibility distribution on N is a natural transformation $\mathbb{N}(N, -) \to \mathcal{P}^+$, where $\mathcal{P}^+ : \mathbb{N} \to \mathbf{Set}$ is the non-empty powerset functor. We can thus say that a possibility distribution on a functor $F : \mathbb{N} \to \mathbf{Set}$ is a natural transformation $F \to \mathcal{P}^+$. The two approaches are related because $T(\{0, 1\}, \vee, 0, 1) \cong \mathcal{P}^+$.

Possibilities are related to probabilities by the map $s : ([0, 1], +, 0, 1) \to (\{0, 1\}, \vee, 0, 1)$ given by $s(0) = 0, s(x) = 1$ for $x \neq 0$. This takes a probability distribution to its support, and by composing this with a probability distribution we get a possibility distribution.

Hardy's paradox. Abramsky and Brandenburger consider the following possibilistic table

$$\tau : \{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1\} \times \{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1\} \to (\{0, 1\}, \vee, 0, 1)$$

following Hardy's work [23]:

τ	a_0:0	a_0:1	a_1:0	a_1:1
b_0:0	1	1	0	1
b_0:1	1	1	1	1
b_1:0	0	1	1	1
b_1:1	1	1	1	0

(14)

This has a quantum realization but no classical realization.

We can relate our effect algebraic formulation of this paradox with the analysis of Abramsky and Brandenburger [2], by again using the chain of adjunctions in (13).

Corollary 31. *The right adjoint in* (13) *takes the effect algebra* $(\{0, 1\}, \vee, 0, 1)$ *to the presheaf* $\mathcal{P}^+(O^{(-)}) : \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}$. *Thus the adjunction* (13) *relates the effect algebra formulation of Hardy's paradox with the formulation of Abramsky and Brandenburger.*

6.2. Kochen-Specker systems

Recall that $\mathit{Proj}(\mathcal{H})$ are the projections on a Hilbert space $\mathcal{H}$. A reformulation of the well known *Kochen-Specker theorem* [29] is that there are no effect algebra morphisms $\mathit{Proj}(\mathcal{H}) \to (\{0, 1\}, \otimes, 0, 1)$ if $\dim \mathcal{H} \geq 3$. To prove this, it suffices to to find a subalgebra E of $\mathit{Proj}(\mathcal{H})$ such that there are no morphism $E \to \{0, 1\}$.

A Kochen-Specker system is represented by a sub-effect algebra E of $\mathit{Proj}(\mathcal{H})$ such that there is no effect algebra morphism $E \to (\{0,1\}, \oslash, 0, 1)$. This means we cannot assign a value 0 or 1 to every element of E in such a way that whenever $p_1, \ldots p_n \in E$ with $p_1 + \ldots p_n = 1$, exactly one of the p_i is assigned 1 and this assignment does not depend on the other $p_j, j \neq i$. (NB here we use partial join $\oslash$, with $1 \oslash 1$ undefined, whereas we used the total join $\vee$ in §6.1.)

We now view this in the presheaf category $\mathbf{Set}^{\mathbb{N}}$. Since there is no morphism $\mathit{Proj}(\mathcal{H}) \to (\{0,1\}, \oslash, 0, 1)$, there is no natural transformation $T(\mathit{Proj}(\mathcal{H})) \to T(\{0,1\}, \oslash, 0, 1)$, by Corollary 10. We now explore this more explicitly.

The bounded operators on $\mathcal{H}$ form a C*-algebra, $B(\mathcal{H})$. An n-test in the effect algebra $\mathit{Proj}(\mathcal{H})$ can be identified with a unital *-homomorphism $\mathbb{C}^n \to B(\mathcal{H})$ from the commutative C*-algebra $\mathbb{C}^n$, by looking at the images of the characteristic functions on single points. So $T(\mathit{Proj}(\mathcal{H})) \cong \mathbf{C}^*(\mathbb{C}^-, B(\mathcal{H}))$. On the other hand, $T(\{0,1\}, \oslash, 0, 1)(N) = N$.

There is another way to view this, via a restricted Gelfand duality. Let $\mathbf{CC}^*_\mathrm{f}$ be the category of finite dimensional commutative C*-algebras. The functor $\mathbb{C}^- : \mathbb{N}^{\mathrm{op}} \to \mathbf{CC}^*_\mathrm{f}$ is an equivalence of categories. Under this equivalence we have presheaves $T(\mathit{Proj}(\mathcal{H})), T(\{0,1\}, \oslash, 0, 1) \in \mathbf{Set}^{\mathbf{CC}^{*\,\mathrm{op}}_\mathrm{f}}$ with

$$T(\mathit{Proj}(\mathcal{H}))(A) = \mathbf{C}^*(A, B(\mathcal{H})) \qquad T(\{0,1\}, \oslash, 0, 1)(A) = \mathrm{Spec}(A)$$

where $\mathrm{Spec}(A)$ is the Gelfand spectrum of A. That is,

$$\mathrm{Spec}(A) = \{\phi : A \to \mathbb{C} \mid \phi \text{ a non-zero }^*\text{-homomorphism}\}.$$

The elements of $\mathrm{Spec}(A)$ are called the characters. Thus the Kochen-Specker paradox says:

$$\text{There is no natural transformation } \mathbf{C}^*(-, B(\mathcal{H})) \to \mathrm{Spec} \text{ in } \mathbf{Set}^{\mathbf{CC}^{*\,\mathrm{op}}_\mathrm{f}}. \quad (15)$$

(See also [35], Theorem 1.2.)

We can use adjunctions to transport this statement to other categories. If a functor $R : \mathbf{Set}^{\mathbf{CC}^{*\,\mathrm{op}}_\mathrm{f}} \to \mathcal{C}$ has a left adjoint $L : \mathcal{C} \to \mathbf{Set}^{\mathbf{CC}^{*\,\mathrm{op}}_\mathrm{f}}$ and $L(X) = \mathbf{C}^*(-, B(\mathcal{H}))$ then the paradox says there is no morphism $X \to R(\mathrm{Spec})$ in $\mathcal{C}$.

Most notably, we transport the paradox to the setting of Hamilton et al. [22], who were concerned with presheaves on the poset $C(B(\mathcal{H}))$ of commutative subalgebras of $B(\mathcal{H})$. The spectral presheaf on $C(B(\mathcal{H}))$ assigns to every commutative sub-algebra A its spectrum as above. A natural transformation from the terminal presheaf to the spectral presheaf now assigns to every such sub-algebra A a particular character. A character in turn assigns to every self-adjoint element in A an element of $\sigma(a)$, the spectrum of a,[2] which in our finite dimensional setting is just an eigenvalue of a. As a *-homomorphism, a character respects sums and products, while naturality implies this assignment of eigenvalues is independent of the surrounding algebra. Thus a natural transformation from

[2]The spectrum of an operator is not to be confused with the spectrum of an algebra.

the terminal presheaf to the spectral presheaf is a global assignment of outcomes for experiments, which by the Kochen-Specker theorem does not exist.

We transport this using the following composite adjunction:

$$\mathbf{Set}^{\mathbf{CC}_{\mathrm{f}}^{*\,\mathrm{op}}} \underset{\Sigma_{\mathbf{C}^*(-,B(\mathcal{H}))}}{\overset{\Delta_{\mathbf{C}^*(-,B(\mathcal{H}))}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{Set}^{\mathbf{CC}_{\mathrm{f}}^{*\,\mathrm{op}}}/\mathbf{C}^*(-,B(\mathcal{H})) \;\simeq\; \mathbf{Set}^{(\mathbf{CC}_{\mathrm{f}}^* \downarrow B(\mathcal{H}))^{\mathrm{op}}} \underset{J_!}{\overset{J^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{Set}^{C(B(\mathcal{H}))^{\mathrm{op}}} \tag{16}$$

The first adjunction between slice categories is as in Section 5.2. The middle equivalence is standard (e.g. [28, Prop. A.1.1.7]); here $(\mathbf{CC}_{\mathrm{f}}^* \downarrow B(\mathcal{H}))$ is the category whose objects are pairs $(A, f : A \to B(\mathcal{H}))$ where A is a finite-dimensional commutative C*-algebra and f is a *-homomorphism. The adjunction $J_! \dashv J^*$ is induced by the evident embedding $J : C(B(\mathcal{H})) \to (\mathbf{CC}_{\mathrm{f}}^* \downarrow B(\mathcal{H}))$, where a commutative C^*-subalgebra A of $B(\mathcal{H})$ is mapped to $(A \hookrightarrow B(\mathcal{H}))$.

Proposition 32. *The right adjoint of* (16) *takes the spectral presheaf on* $\mathbf{CC}_{\mathrm{f}}^*$ *to the spectral presheaf on* $C(B(\mathcal{H}))$*. The left adjoint of the composite* (16) *takes the terminal presheaf on* $C(B(\mathcal{H}))$ *to the presheaf* $\mathbf{C}^*(-, B(\mathcal{H}))$ *on* $\mathbf{CC}_{\mathrm{f}}^*$*.*

Proof. Let $K : C(B(\mathcal{H})) \to \mathbf{CC}_{\mathrm{f}}^*$ be the inclusion functor. Reading the adjunction from left to right sends a presheaf F on $\mathbf{CC}_{\mathrm{f}}^*$ to $F \circ K$ on $C(B(\mathcal{H}))$. In particular, the spectral presheaf gets mapped to the spectral presheaf.

To show the second half of the statement we show, similar to Proposition 29, that natural transformations $\sigma : 1 \to G \circ K$ are in natural correspondence to natural transformations $\alpha : \mathbf{C}^*(-, B(\mathcal{H})) \to G$. This bijection is given as follows: given $\sigma : 1 \to G \circ K$, define $\alpha : \mathbf{C}^*(-, B(\mathcal{H})) \to G$ as $\alpha_A(f) = \sigma_{f(A)}(*)$ and given $\alpha : \mathbf{C}^*(-, B(\mathcal{H})) \to G$, define $\sigma : 1 \to G \circ K$ as $\sigma_A(*) = \alpha_A(i_A)$ where $i_A : A \hookrightarrow B(\mathcal{H})$.

Corollary 33. *The paradox (15) is equivalent to the statement of [22]: the spectral presheaf has no global section.*

7. Test spaces

In this section we want to consider another approach to non-locality and contextuality via *test spaces*. However, test spaces come in different guises in the literature. Here we want to present a small overview of these different approaches to test spaces and make a link with effect algebras. Currently, the following definition, which comes from [18], is probably the most common.

Definition 34 (Test space). *A test space* (X, Σ) *consists of a set* X *together with a set of subsets* $\Sigma \subset 2^X$ *such that the members of* Σ *cover* X*, i.e.,* $\bigcup_{T \in \Sigma} T = X$*. A probability measure on a test space* (X, Σ) *is a function* $\mu : X \to \mathbb{R}_{\geq 0}$ *such that* $\sum_{x \in T} \mu(x) = 1$ *for every test* $T \in \Sigma$*.*

More details about these test spaces can be found in [14]. The version of test spaces we will use is more general in the sense that tests can include multiple instances of the same element. This will be comparable to how it might be possible to add an element in an effect algebra to itself. The following definitions are from [20] and [21].

Definition 35. *An* effect test space $(X, \mathcal{T})$ *consists of a set* X *and a collection* $\mathcal{T} \subset \mathbb{N}^X$ *such that*

- *For any* $x \in X$ *there exists some* $t \in \mathcal{T}$ *such that* $t(x) \neq 0$*.*
- *If* $s, t \in \mathcal{T}$ *with* $s(x) \leq t(x)$ *for all* $x \in X$*, then* $s = t$*.*

An important notion in the theory of test spaces is that of *perspectivity*.

Definition 36. *Given an effect test space* $(X, \mathcal{T})$*, any function* $f \leq t \in \mathcal{T}$ *is called an* event*. Two events* f, g *are* orthogonal *if* $f + g$ *is again an event and* complementary *if* $f + g \in \mathcal{T}$*. Two events* f, g *are* perspective *if there exists an event* h *such that* f, h *and* g, h *are complementary. We write* $f \approx g$ *if* f *and* g *are perspective.*

Definition 37. *An effect test space* $(X, \mathcal{T})$ *is* algebraic *if every* $t \in \mathcal{T}$ *has finite support and if for events* f, g, h*, if* $f \approx g$ *and* $h + f \in \mathcal{T}$ *then* $h + g \in \mathcal{T}$*. That is to say, if two events share a complement, they share all complements.*

Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be algebraic effect test spaces. Any (partial) function $\psi : X \to Y$ defines a function $\hat{\psi} : \{f \in \mathbb{N}^X \mid f \text{ has finite support}\} \to \mathbb{N}^Y$ by $\hat{\psi}(f)(y) = \sum\{f(x) \mid \phi(x) = y\}$. We understand the empty sum to be zero. In particular, if $\mathbf{1}_x$ is the characteristic function of $x \in X$, then $\hat{\psi}(\mathbf{1}_x)(y) = \sum\{\mathbf{1}_x(x') \mid \psi(x') = y\}$, which is 1 only if $x' = x$, that is, $y = \psi(x)$, so $\hat{\psi}(\mathbf{1}_x) = \mathbf{1}_{\psi(x)}$.

We obtain a category **AEtest** of algebraic effect test spaces whose morphisms $(X, \mathcal{T}) \to (Y, \mathcal{S})$ are partial functions $\psi : X \to Y$ such that $\hat{\psi}(t) \in \mathcal{S}$ if $t \in \mathcal{T}$. The reason to consider partial functions becomes clear when we consider $E = 0$, the terminal effect algebra, in the adjunction below.

Example 38. *Let* $\mathcal{I} = \big((0,1], \{f : (0,1] \to \mathbb{N} \mid supp(f) \text{ is finite}, \sum_x (f(x)) \cdot x = 1\}\big)$*. Then* $\mathcal{I}$ *is an algebraic effect test space describing the unit interval. Here* $(0, 1]$ *is the half-open unit interval, which we need because of the second point of Definition 35. Let* $(X, \mathcal{T})$ *be any algebraic effect test space. A morphism* $\mu : (X, \mathcal{T}) \to \mathcal{I}$ *corresponds to a probability measure* $\tilde{\mu}$ *on* X*, where* $\tilde{\mu}(x) = 0$ *if* $\mu(x)$ *is undefined.*

By slight modification of the ideas in [21] we now distill an adjunction between algebraic effect test spaces and effect algebras, which will allow us to transpose paradoxes. (We note that Jacobs and Mandemaker [26, §3] also extracted a similar adjunction from [21], but for a modified notion of test space called 'test perspective'; we contend that our use of bona fide effect test spaces and partial maps stands more closely to test space literature.)

Let f be an event in an algebraic effect space $(X, \mathcal{T})$. Denote by $\pi(f)$ the set $\pi(f) = \{g \mid g \approx f\}$ and let $\Pi(X) = \{\pi(f) \mid f \text{ an event}\}$. In [21] it is shown that that $\Pi(X)$ can be given the structure of an effect algebra in a straightforward way. That is, $\pi(f) \ovee \pi(g) = \pi(f+g)$ whenever this makes sense and $\pi(f)^\perp = \pi(h)$ if h is a complement of f. We extend Π to a functor $\Pi : \mathbf{AEtest} \to \mathbf{EA}$ by $\Pi(\psi)(f) = \pi(\hat{\psi}(f))$.

There is also a functor in the other direction, which we denote by S. Let E be an effect algebra. We obtain an algebraic test space $S(E) = (X, \mathcal{T})$ where $X = E\backslash\{0\}$ and $\mathcal{T} = \{f : X \to \mathbb{N} \mid \text{supp}(f) \text{ is finite}, \bigovee_x (f(x)) \cdot x = 1\}$. If $\varphi : E \to A$ is an effect algebra morphism, we obtain an **AEtest** morphism $S(\varphi)$ by restricting to $E\backslash\{0\}$. Note that $\mathcal{I} = S([0,1])$.

Lemma 39. *Let E be an effect algebra. The map $\phi : E \to \Pi S(E)$, $e \mapsto \pi(\mathbf{1}_e)$ is an isomorphism.*

Proof. An inverse to ϕ is given as follows: let f be an event in $\Pi S(E)$, then $f = \sum_e f(e)\mathbf{1}_e$. Now $\phi^{-1}(\pi(f)) = \bigovee_e f(e)e$. See [21] for details.

Proposition 40. *The functors Π and S form an adjoint pair $\Pi \dashv S$. The map ϕ^{-1} is the counit of this adjunction.*

Proof. We want to show $Hom(\Pi(X, \mathcal{T}), E) \cong Hom((X, \mathcal{T}), S(E))$. Given $\varphi : \Pi(X, \mathcal{T}) \to E$, define $\bar{\varphi} : (X, \mathcal{T}) \to S(E)$ by $\bar{\varphi}(x) = \varphi(\pi(\mathbf{1}_x))$. Given $\psi : (X, \mathcal{T}) \to S(E)$ define $\bar{\psi} : \Pi(X, \mathcal{T}) \to E$ by $\bar{\psi}(\pi(f)) = \phi^{-1}(\pi(\hat{\psi}(f)))$. Notice that any event $f : X \to \mathbb{N}$ can be written as $f = \sum_x f(x)\mathbf{1}_x$. We then have

$$\begin{aligned}
\bar{\bar{\varphi}}(\pi(f)) &= \phi^{-1}(\pi(\hat{\bar{\varphi}}(f))) \\
&= \phi^{-1}(\pi(\hat{\bar{\varphi}}(\sum_x f(x)\mathbf{1}_x))) \\
&= \phi^{-1}(\pi(\sum_x f(x)\mathbf{1}_{\bar{\varphi}(x)})) \\
&= \sum_x f(x)\bar{\varphi}(x) \\
&= \sum_x f(x)\varphi(\pi(\mathbf{1}_x)) \\
&= \varphi(\pi(f)),
\end{aligned}$$

and

$$\begin{aligned}
\bar{\bar{\psi}}(x) &= \bar{\psi}(\pi(\mathbf{1}_x)) \\
&= \phi^{-1}(\pi(\hat{\psi}(\mathbf{1}_x))) \\
&= \phi^{-1}(\pi(\mathbf{1}_{\psi(x)})) \\
&= \psi(x).
\end{aligned}$$

In order to show the relation between test spaces and effect algebras, we take a look at two non-locality scenarios.

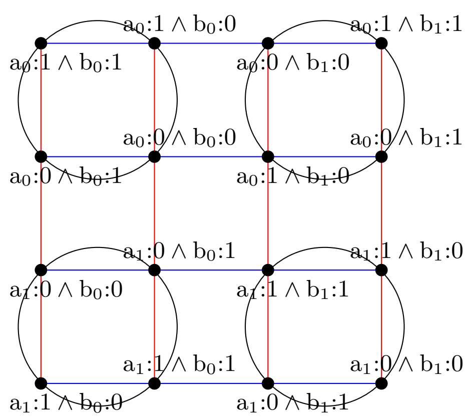
7.1. Bell scenario

We shall take a look on how to transfer the Bell paradox to test spaces. It follows from Lemma 39 and Proposition 40 that S is fully faithful. Hence we can easily transfer the Bell paradox by applying S to the non-factoring triangle as follows:

$$\begin{tikzcd} S(E_{\mathrm{A}} \otimes E_{\mathrm{B}}) \arrow[rr] \arrow[dr] & & S([0,1]) \\ & S(B_{\mathrm{A}} \otimes B_{\mathrm{B}}) \arrow[ur, "|" marking] & \end{tikzcd}$$

In fact, whenever we have a morphism $\phi : (X, \mathcal{T}) \to (Y, \mathcal{S})$ such that $\Pi(\phi) : \Pi(X, \mathcal{T}) \to \Pi(Y, \mathcal{S})$ is the inclusion $i : E_{\mathrm{A}} \otimes E_{\mathrm{B}} \to B_{\mathrm{A}} \otimes B_{\mathrm{B}}$, the method of Diagram 9 allows us to transfer the paradox to **AETest**. This might be relevant as the space $S(E_{\mathrm{A}} \otimes E_{\mathrm{B}})$ is quite involved and we could find a smaller space.

Example 41. *Consider figure 17 below. This is also depicted in [6, Fig. 7], where it is called a hyper-graph, but we understand it as a test space by identifying vertices and hyper-edges of a graph with points and tests of a test space. Let Z be the set of points in it and for every line or circle define a function from Z to $\mathbb{N}$, which sends the points on this line or circle to 1 and the rest to 0. We call the set of these functions $\mathcal{Q}$. Then $(Z, \mathcal{Q})$ is an algebraic effect test space.*



(17)

We have conveniently labelled the points of this space. In the terminology of non-locality, the circles correspond to fixed measurement settings and the lines correspond to the no-signalling conditions. Hence we see that applying the functor Π to this test space gives an effect algebra isomorphic to $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$. The Bell table thus describes a distribution $(Z, \mathcal{Q}) \to \mathcal{I}$ which does not factor though the canonical map $(Z, \mathcal{Q}) \to S(B_{\mathrm{A}} \otimes B_{\mathrm{B}})$.

N.B. Since the functor Π is not full, we cannot just take any test space $(X, \mathcal{T})$ for which $\Pi(X, \mathcal{T}) \cong B_{\mathrm{A}} \otimes B_{\mathrm{B}}$ in order to find a non-factorization of the Bell scenario. In particular there is no map $(Z, \mathcal{Q}) \to (16, \{f\})$ where 16 is a 16 element set and $f : 16 \to \mathbb{N}$ is the map $f(i) = 1 \, \forall i \in 16$.

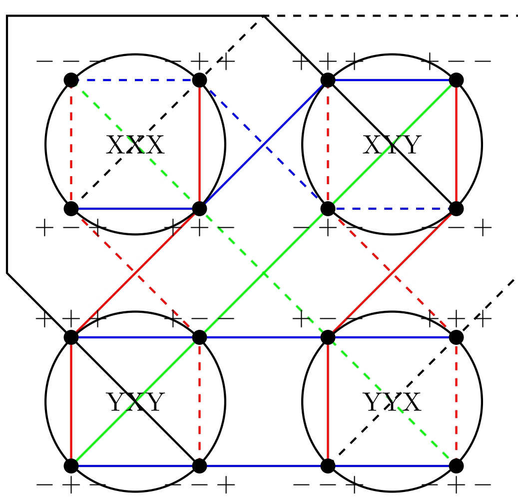
7.2. GHZ scenario

The second example we will look at is the GHZ scenario [19]. We will use the adjunction to explore the scenario from the perspective of both test spaces and effect algebras. Like the Bell scenario, the GHZ scenario involves separate observers, each with measurement settings and possible outcomes. But like the Kochen-Specker scenario the 'paradox' here is absolute and not probabilistic.

There are three separate observers, Alice, Bob and Charlie, each with two measurement settings (x and y) and two possible outcomes (-1 and $+1$). The quantum realization of the scenario is as follows. Alice, Bob and Charlie share a quantum state of the form $\Psi_{GHZ} = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow> - |\downarrow\downarrow\downarrow>)$. They each have the choice to perform the Pauli-x operator, σ_x, which sends $\uparrow$ to $\downarrow$ and $\downarrow$ to $\uparrow$ or Pauli-y operator, σ_y, which sends $\uparrow$ to $i\cdot\downarrow$ and $\downarrow$ to $-i\cdot\uparrow$. The crux is that certain combinations of Pauli operators have the GHZ state as an eigenvector. Indeed,

$$\sigma_x\sigma_x\sigma_x\Psi_{GHZ} = -\Psi_{GHZ}, \tag{18}$$
$$\sigma_x\sigma_y\sigma_y\Psi_{GHZ} = \Psi_{GHZ}, \tag{19}$$
$$\sigma_y\sigma_x\sigma_y\Psi_{GHZ} = \Psi_{GHZ}, \tag{20}$$
$$\sigma_y\sigma_y\sigma_x\Psi_{GHZ} = \Psi_{GHZ}. \tag{21}$$

Now in a local non-contextual setting we should be able to assign eigenvalues, $+1$ or -1 to the Pauli operators in such a way that it respects the above products, but this is impossible as we can see from a parity argument: every Pauli operator occurs twice on the left hand sides, hence the total product is $+1$, while the product of the right hand side is -1. By the methods of [6] we can write down a test space, $(X_{GHZ}, \mathcal{T}_{GHZ})$, for this scenario (Figure 22). The vertices on the circles correspond to the outcome of measurements whose settings are written inside the circle. The remaining lines correspond to tests coming from no-signalling.



(22)

The statement of the paradox is now that there are no **AETest** morphisms from $(X_{GHZ}, \mathcal{T}_{GHZ})$ to the test space $(\{*\}, \{!\})$ with one point and one test $! : * \mapsto 1$. Translated to effect algebras this statement becomes: there is no effect algebra map $\Pi(X_{GHZ}, \mathcal{T}_{GHZ})$ to $(\{0, 1\}, \otimes)$, which is exactly a Kochen-Specker type theorem.

8. Concluding remarks

We have exhibited a crucial adjunction between two general approaches to finite probability theory: effect algebras and presheaves (Corollary 10). We have used this to analyze paradoxes of non-locality and contextuality (Section 4). There are simple algebraic statements of these paradoxes in terms of partial commutative monoids, but these transport across the adjunction to statements about presheaves on $\mathbb{N}$. By taking slice categories of the presheaf category, we recover earlier analyses of the paradoxes (e.g. Corollary 31). Finally we investigated the transportation of non-locality and contextuality paradoxes between test spaces and effect algebras.

Cohomology

To conclude, we mention some work that comes from the presheaf formalism. Informally, presheaves have to do with "gluing together" local information. Now non-locality is precisely a statement about how different local pieces of information fail to be glued together. Cohomology tries to capture the reason why this local information cannot be glued together and can therefore be used to study non-locality. We refer to [3] and [1]. A connection with effect algebras has been developed in [36].

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