Counting Problems for Parikh Images

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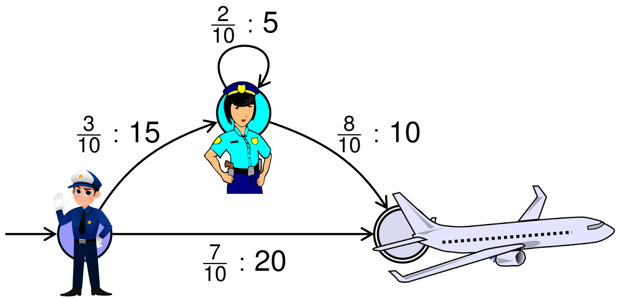
The Cost Problem



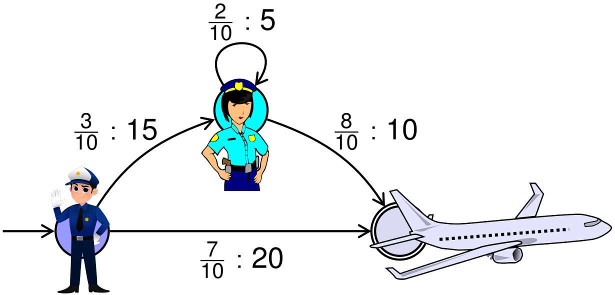
What is the probability to reach the gate in 25–45min?
Quantiles?

Cost Problem :=

Input: Cost Markov chain
cost formula φ ← $25 \leq \text{cost} \leq 45$

Output: $\Pr(\varphi)$

The Cost Problem



What is the probability to reach the gate in 25–45min?
Quantiles?

Cost Problem :=

Input: Cost Markov chain
cost formula φ ← $25 \leq \text{cost} \leq 45$
threshold $\tau \in [0, 1]$

Output: Is $\Pr(\varphi) \geq \tau$?

Complexity of the Cost Problem

Theorem (Laroussinie, Sproston, FoSSaCS'05)

The cost problem is in EXPTIME.
The cost problem is NP-hard.

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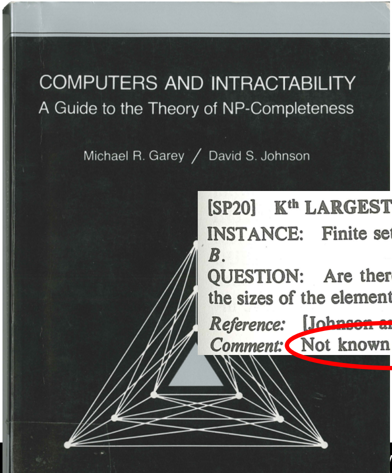


[SP20] **K^th LARGEST SUBSET (*)**

INSTANCE: Finite set A, size $s(a) \in Z^+$ for each $a \in A$, positive integers K and B.

QUESTION: Are there K or more distinct subsets $A' \subseteq A$ for which the sum of the sizes of the elements in A' does not exceed B?

Reference: [Johnson and Kashdan, 1976]. Transformation from SUBSET SUM.

Comment: Not known to be in NP. Solvable in pseudo-polynomial time (polyno-

Complexity of the Cost Problem

Theorem (Laroussinie, Sproston, FoSSaCS'05)

The cost problem is in EXPTIME.
The cost problem is NP-hard. ← by reduction from the Kth largest subset problem

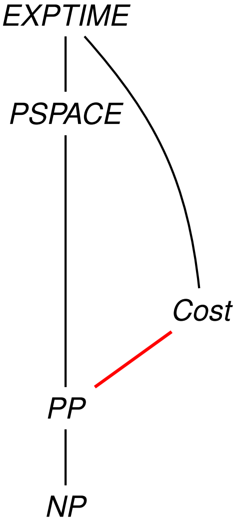
Theorem (HK, IPL'16)

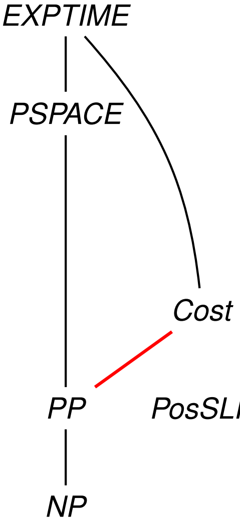
The Kth largest subset problem is PP-complete. ← PP: a superset of NP

Corollary

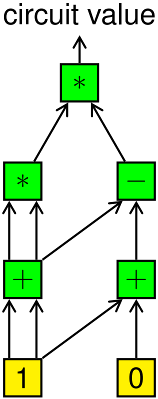
The cost problem is PP-hard.

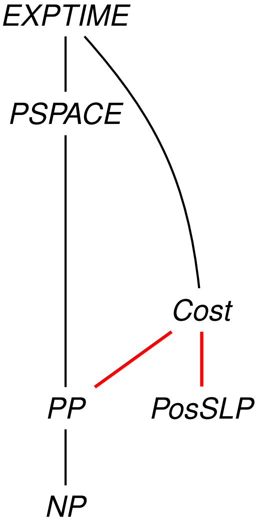
Complexity of the Cost Problem





PosSLP := Input: arithmetic circuit
Output: Is the value > 0 ?





Theorem (HK, ICALP'15)

The cost problem is PosSLP-hard.

PosSLP :=
Input: arithmetic circuit
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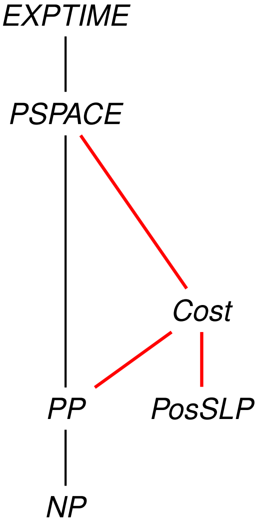
Complexity of the Cost Problem



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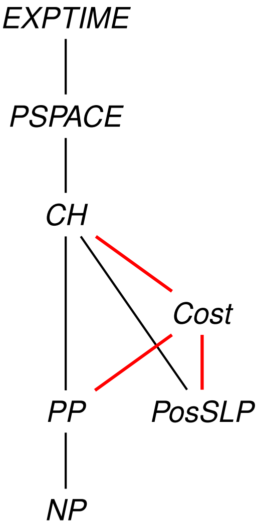

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The cost problem is PosSLP-hard.
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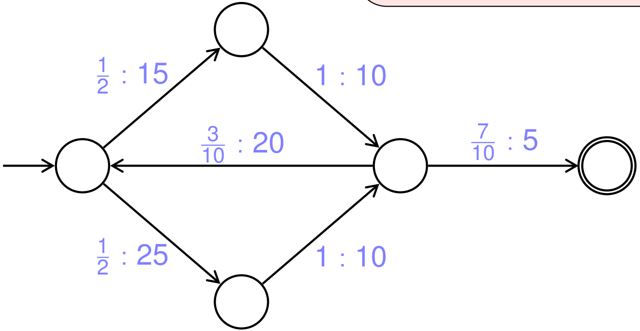
Theorem (HKL, LICS'17)

The cost problem is in CH.

Solving the Cost Problem

Cost Problem :=

Input:	Cost Markov chain cost formula φ
Output:	$\Pr(\varphi)$



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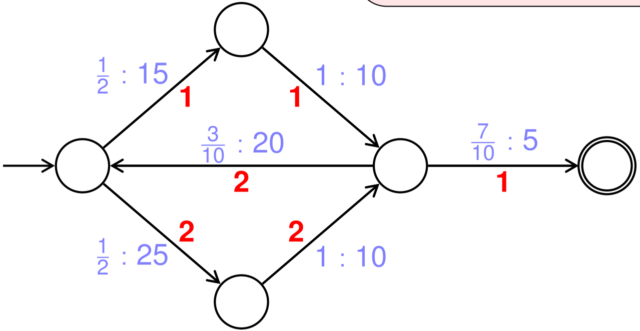

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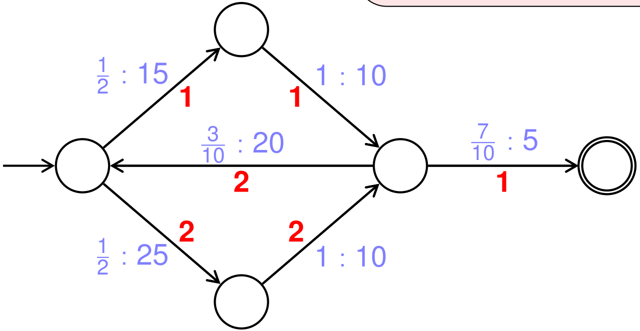

Enumerate the **Parikh images**.

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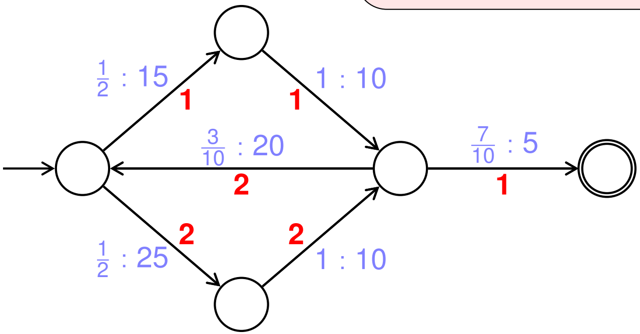

Enumerate the **Parikh images**.
Problem: there might be multiple paths per Parikh image.

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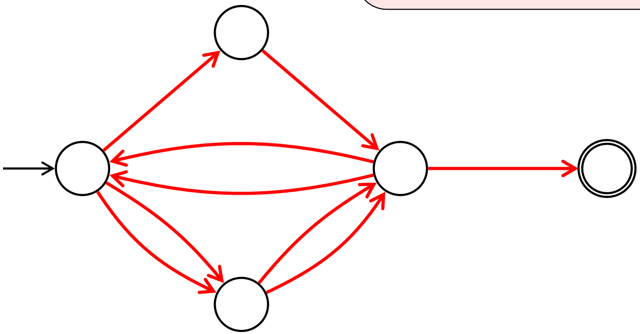

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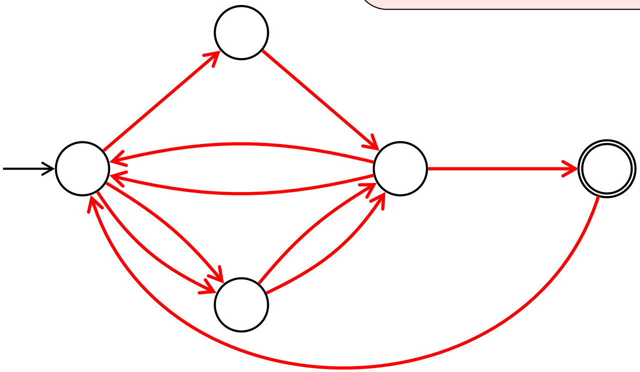

Enumerate the **Parikh images**.

Problem: there might be multiple paths per Parikh image.

The BEST Theorem

Theorem (de Bruijn, van Aardenne-Ehrenfest, Smith, Tutte)

The number of Eulerian cycles in an Eulerian graph G equals

$$t(G) \cdot \prod_{v \in V} (d(v) - 1)!$$

number of directed spanning trees in G

Theorem (Tutte's matrix-tree theorem)

$$t(G) = \det(L(G)_{11})$$

Laplacian of G

QUANT: A Tool for the Cost Problem

Theorem ([HK, ICALP'15], [HK, IPL'16], [HKL, LICS'17])

The cost problem is hard for PP and PosSLP.
The cost problem is in CH.

```
while solver.check() == sat:
    model = solver.model()

    for t in T:
        v = model[tv[t]].as_long()
        M.itemset((t[0], t[1]), v)

    minor = Decimal(round(np.linalg.det(laplacian(M)[1:,1:])))
    degs = Decimal(np.prod(map(math.factorial, degreeVector(M) -
                                   np.ones(m+1))))
    weights = Decimal(np.prod(map(math.factorial, [ model[tv[t]].as_long()
                                                 for t in T ])))
    probs = Decimal(np.prod([math.pow(P[t],
                                       model[tv[t]].as_long()) for t in T]))
    p += ((minor * degs) / weights) * probs

    if p > threshold:
        return p

    solver.add(Or([tv[t] != model[tv[t]] for t in T]))
return p
```

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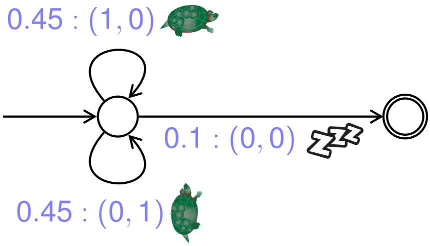
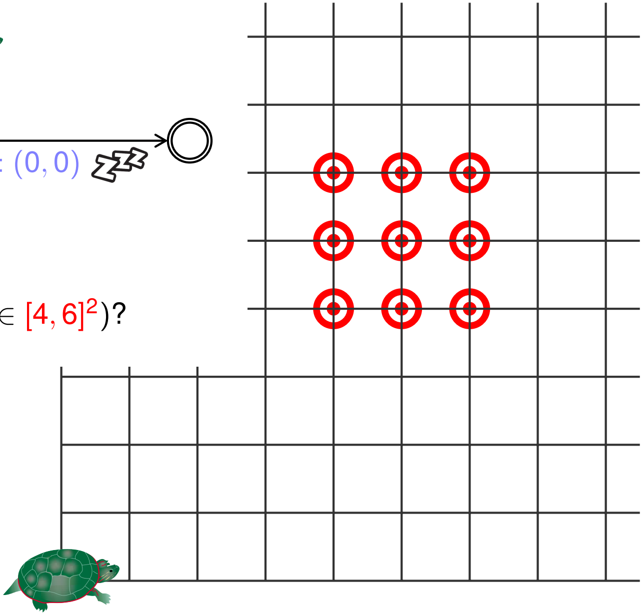


What is Pr(cost $\in [4, 6]^2$)?



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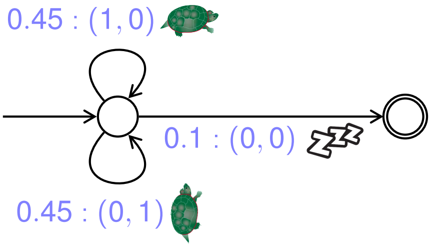

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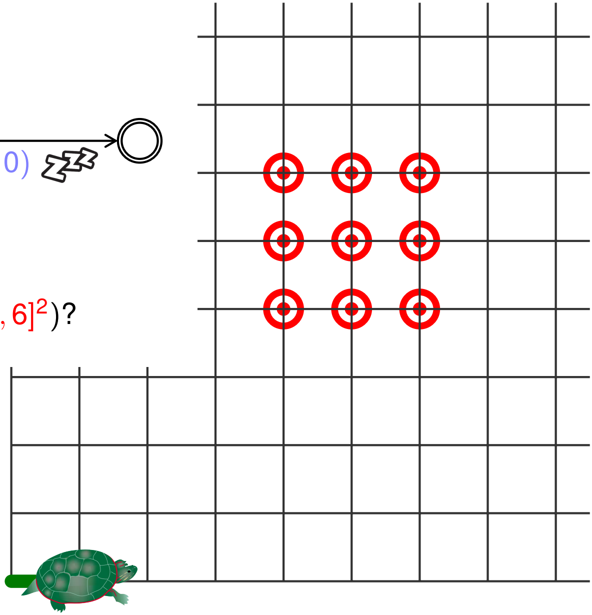

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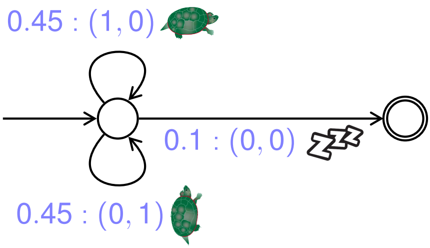

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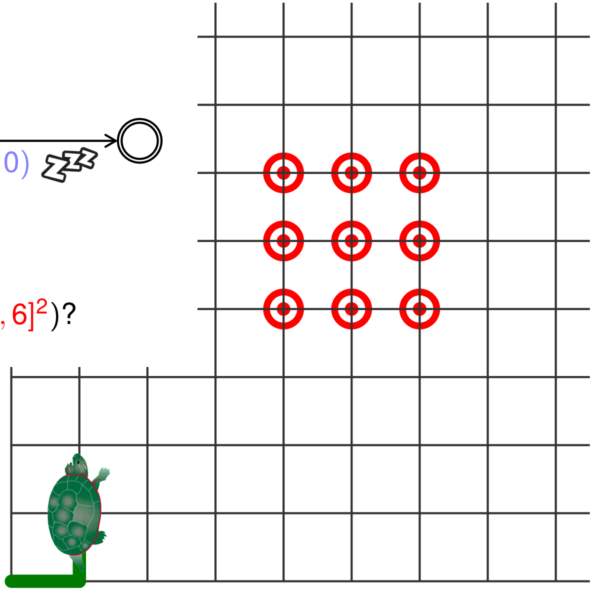

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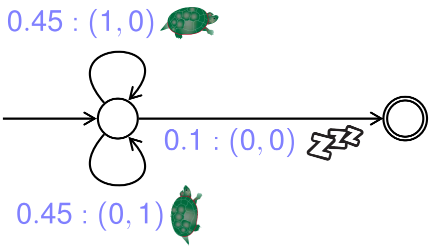

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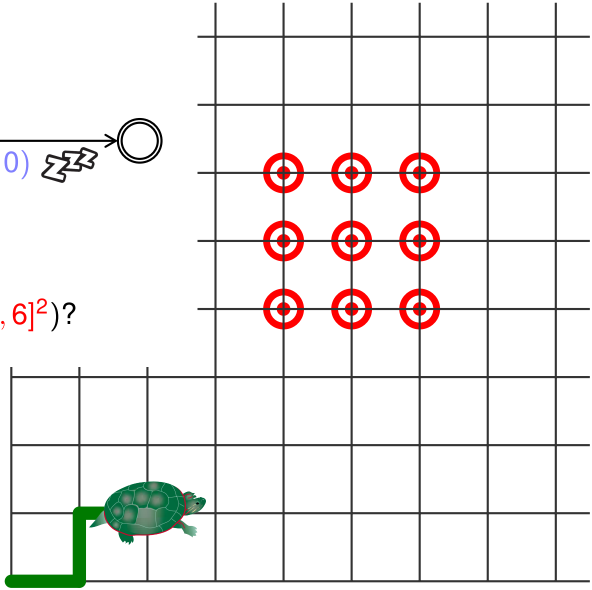

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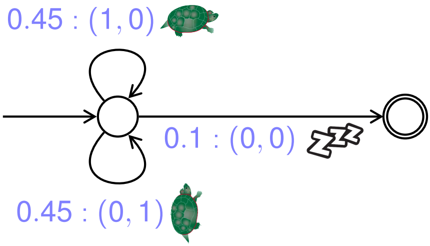

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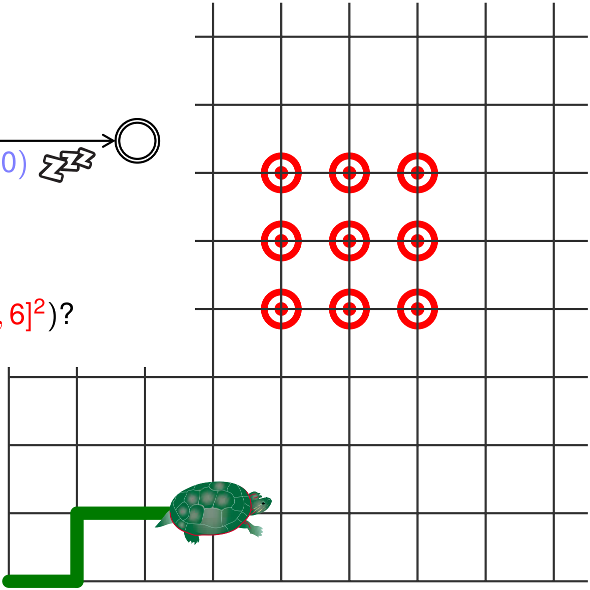

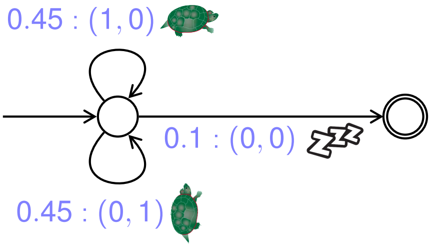

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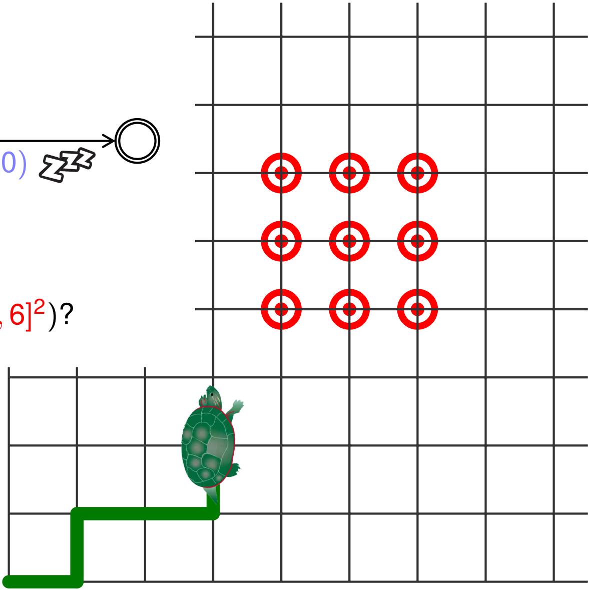

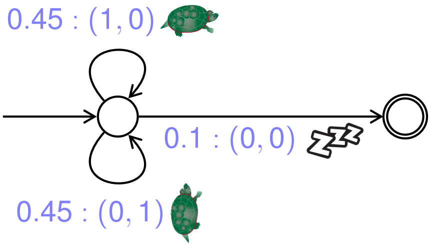

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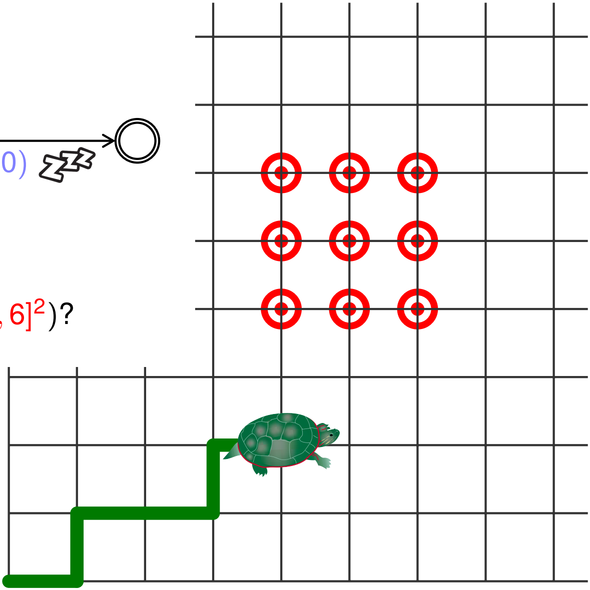

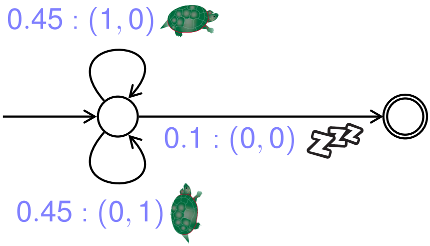

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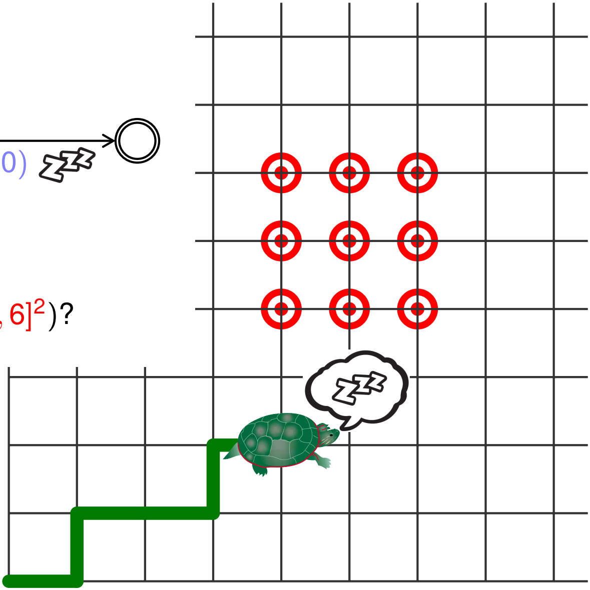

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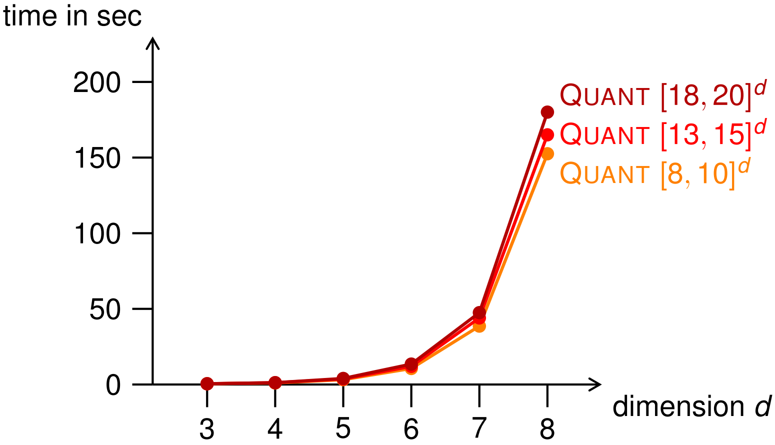

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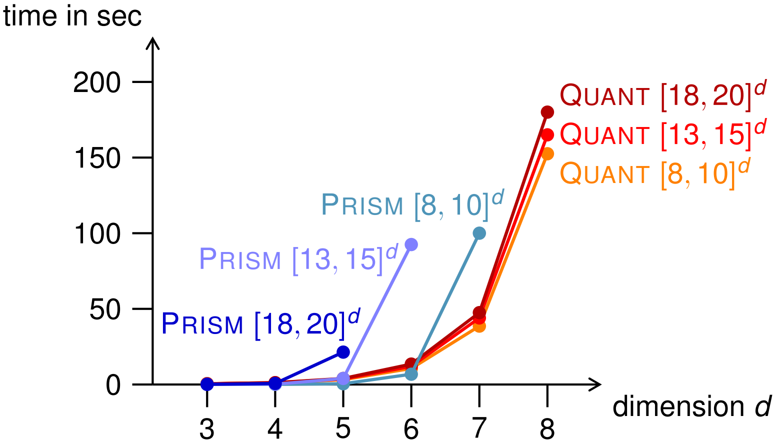
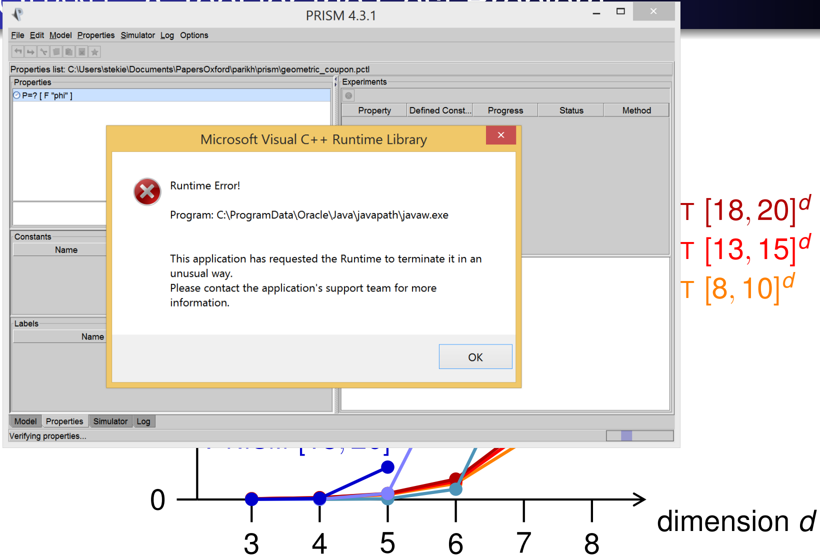


Quant: A Tool for the Cost Problem



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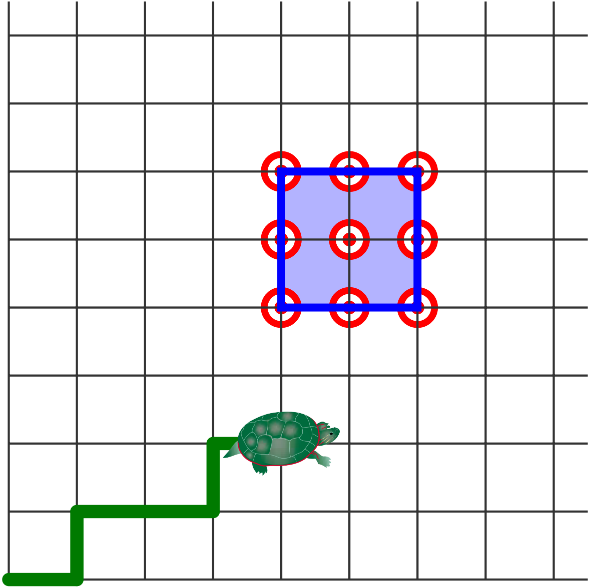

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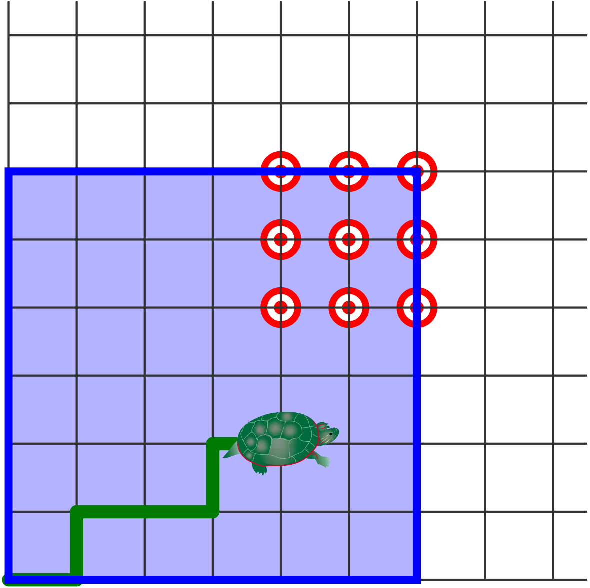

Counting Parikh Images

Σ: finite alphabet

$\boldsymbol{p} \in \mathbb{N}^{\Sigma}$: vector

$\mathcal{A}$: language generator (DFA, NFA, CFG)

$N(\mathcal{A}, \boldsymbol{p})$: number of words accepted by $\mathcal{A}$ with Parikh image $\boldsymbol{p}$

Example: for $\mathcal{A} = a^* b a^*$ and $\boldsymbol{p} = (2, 1)$: $N(\mathcal{A}, \boldsymbol{p}) = 3$

PosParikh :=

Input:	Language generators $\mathcal{A}, \mathcal{B}$
	vector $\boldsymbol{p} \in \mathbb{N}^{\Sigma}$
Output:	Is $N(\mathcal{A}, \boldsymbol{p}) > N(\mathcal{B}, \boldsymbol{p})$?

Different variants:

- language generator: DFA, NFA, CFG
- unary or binary encoding of $\boldsymbol{p}$
- fixed or variable alphabet Σ

Results of This Paper: Complexity of PosParikh

vector ***p***	size of Σ	DFA	NFA	CFG
unary encoding	unary	in L	NL-c.	P-c.
	fixed	PL-c.		
	variable	PP-c.		
binary encoding	unary	in L	NL-c.	DP-c.
	fixed	PosMatPow-hard, in CH	PSPACE-c.	PEXP-c.
	variable	PosSLP-hard, in CH		

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BitParikh $:=$

Input: Language generator $\mathcal{A}$
vector $\boldsymbol{p} \in \mathbb{N}^{\Sigma}$
number $i \in \mathbb{N}$ in binary

Output: Is the i-th bit of $N(\mathcal{A}, \boldsymbol{p})$ equal to 1 ?

PosMatPow

PosMatPow :=

Input: $m \times m$ integer matrix M (in binary)
number $n \in \mathbb{N}$ (in binary)

Output: Is $(M^n)_{1,m} \geq 0$?

The entries of M^n are of exponential size.

Theorem (Galby, Ouaknine, Worrell, 2015)

PosMatPow is in P for $m = 2$.
PosMatPow is in P for $m = 3$ and M given in unary.

PosMatPow reduces to PosSLP.

Theorem (HKL, 2017)

PosMatPow reduces to
*PosParikh with DFAs, binary encoding of **p**, even for $|\Sigma| = 2$.*

From Matrix Powers to Graphs

G: multi-graph with edges labelled by multiplicities

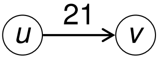
$N(G, u, v, n)$: number of paths from u to v of length n

Lemma

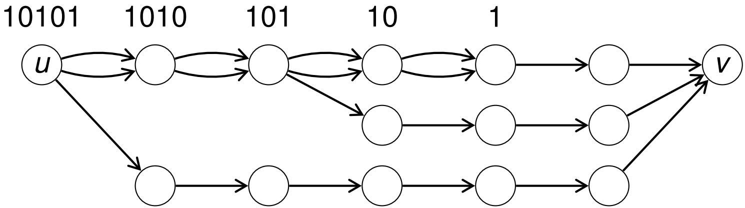
Given an integer matrix M and indices i, j, one can compute in logspace a multi-graph G with vertices u, v^+, v^- such that

$$(M^n)_{i,j} = N(G, u, v^+, n) - N(G, u, v^-, n) \qquad \text{for all } n \in \mathbb{N}.$$

Getting Rid of Edge Weights

Replace 

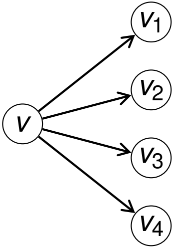
by



The edge weights are removed at the expense of longer paths.

From Graphs to DFAs

Replace



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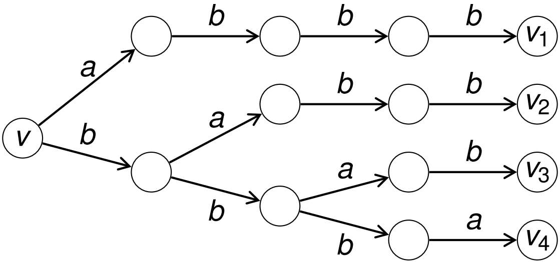

Summary

Counting Parikh images is closely related to performance analysis of probabilistic systems.

The complexity depends strongly on various parameters:

- language generator: DFA, NFA, CFG
- unary or binary encoding of $\boldsymbol{p}$
- fixed or variable alphabet Σ

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