Linkless and flat embeddings in 3-space and the Unknot problem

[Extended Abstract] *

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ABSTRACT

We consider piecewise linear embeddings of graphs in 3-space $\mathbb{R}^3$. Such an embeding is *linkless* if every pair of disjoint cycles forms a trivial link (in the sense of knot theory). Robertson, Seymour and Thomas [47] showed that a graph has a linkless embedding in $\mathbb{R}^3$ if, and only if, it does not contain as a minor any of seven graphs in Petersen's family (graphs obtained from K_6 by a series of $Y\Delta$ and ΔY operations). They also showed that a graph is linklessly embeddable in $\mathbb{R}^3$ if, and only if, it admits a *flat embedding* into $\mathbb{R}^3$, i.e. an embedding such that for every cycle C of G there exists a closed 2-disk $D \subseteq \mathbb{R}^3$ with $D \cap G = \partial D = C$. Clearly, every flat embeddings is linkless, but the converse is not true. We first consider the following algorithmic problem associated with embeddings in $\mathbb{R}^3$:

Flat Embedding: For a given graph G, either detect one of Petersen's family graphs as a minor in G or return a flat (and hence linkless) embedding in $\mathbb{R}^3$.

The first outcome is a certificate that G has no linkless and no flat embeddings. Our first main result is to give an $O(n^2)$ algorithm for this problem. While there is a known polynomial-time algorithm for constructing linkless embeddings [20], this is the first polynomial time algorithm for constructing flat embeddings in 3-space and we thereby settle a problem proposed by Lovász [29]. We also consider the following classical problem in topology.

The Unknot Problem: Decide if a given knot is trivial or not.

This is a fundamental problem in knot theory and low dimensional topology, whose time complexity is unresolved. It has been extensively studied by researchers working in computational geometry. A related problem is:

The Link Problem: Decide if two given knots form a link.

Hass, Lagarias and Pippenger [16] observed that a polynomial time algorithm for the link problem yields a polynomial time algorithm for the unknot problem. We relate the link problem to the following problem that was proposed independently by Lovász and by Robertson et al.

Conjecture. (Lovász [29]; Robertson, Seymour and Thomas [48]) *There is a polynomial time algorithm to decide whether a given embedding of a graph in the 3-space is linkless.*

Affirming this conjecture would clearly yield a polynomial-time solution for the link problem. We prove that the converse is also true by providing a polynomial-time solution for the above conjecture, if we are given a polynomial time oracle for the link problem.

General Terms

Algorithm, Theory

Categories and Subject Descriptors

G.2 [**Discrete Math**]: Combinatorics; G.2.2 [**Graph Theory**]: Combinatorics—*Graph algorithms,Computations on discrete structures*

Keywords

Linkless embedding, knot, unknot, flat embedding.

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1. INTRODUCTION

1.1 Embedding graphs in 3-space

A seminal result of Hopcroft and Tarjan [19] from 1974 is that there is a linear time algorithm for testing planarity of graphs. This is just one of a host of results on embedding graphs in surfaces. These problems are of both practical and theoretical interest. The practical issues arise, for instance, in problems concerning VLSI, and also in several other applications of "nearly" planar networks, as planar graphs and graphs embedded in low genus surfaces can be handled more easily. Theoretical interest comes from the importance of the genus parameter of graphs and from the fact that graphs of bounded genus naturally generalize the family of planar graphs and share many important properties with them. Recently, some apparently new and nontrivial linear time algorithms concerning graph embeddings have appeared [21, 22]. In addition, Mohar [33, 34] gave a linear time algorithm for testing embeddability of graphs in an arbitrary surface and constructing an embedding, if one exists. This is one of the deepest results in this area, and it generalizes linear time algorithms for testing planarity and constructing a planar embedding if one exists [5, 19, 54]. This algorithm is further simplified in [23].

In [41], Robertson and Seymour proved that for any fixed graph H there is a cubic-time algorithm for testing whether H is a minor of a given graph G. This implies that there is an $O(n^3)$ algorithm for deciding membership in any minor-closed family of graphs, because by their seminal result in [42], such a family can be characterized by a finite collection of excluded minors. However, this theorem is not explicit. More precisely, the graph minor theorem is not constructive; in general, we do not know how to obtain an excluded minor characterization of a given minor-closed family of graphs. In addition, Robertson and Seymour's algorithm solves the decision problem but it is not apparent how to construct, e.g., an embedding from their algorithm.

In this paper, we consider *embeddings of graphs* in $\mathbb{R}^3$, where all embeddings are piecewise linear. There are many minor-closed families of graphs that arise in the study of topological problems. An illustrative example is the class of linklessly embeddable graphs. We call a pair of vertex disjoint cycles drawn in $\mathbb{R}^3$ *un-linked* if there is a 2-dimensional disk[1] in $\mathbb{R}^3$ that contains the first cycle and is disjoint from the other one. Otherwise, the two cycles are *linked*. Intuitively, if two cycles in $\mathbb{R}^3$ are linked, we can not contract one into a single point without cutting the other. By a *linkless embedding* we mean an embedding of a graph in $\mathbb{R}^3$ in such a way that no two vertex-disjoint cycles are linked. Linkless embeddings were first studied by Conway and Gordon [9]. An algorithmic problem concerning linkless embeddings is studied by Motwani, Raghunathan and Saran [36], who gave a partial result for linkless embeddings and its algorithmic applications.

Robertson, Seymour and Thomas [47] proved that G is linklessly embeddable in the 3-space if, and only if, G does not have any graph in the Petersen family as a minor. By the *Petersen family* we mean the graphs that can be obtained from K_6 by a series of $Y\Delta$ and ΔY operations. See Figure 1 for drawings of these graphs on the projective plane; note that the third one, $K_{4,4} - e$, cannot be embedded in the projective plane. In the same paper [47] it is shown that a graph is linklessly embeddable in $\mathbb{R}^3$ if, and only if, it admits a *flat embedding* in $\mathbb{R}^3$, i.e. an embedding such that for every cycle C of G, there exists a closed 2-dimensional disk $D \subseteq \mathbb{R}^3$ with $D \cap G = \partial D = C$. Clearly, every flat embedding is linkless, but the converse is not true. In many ways flat embeddings are nicer to work with and in this paper we will work with flat rather than linkless embeddings.

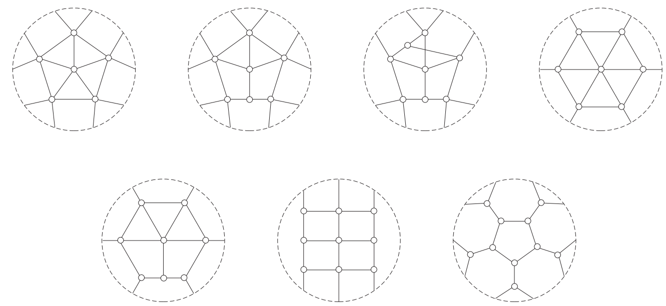

Figure 1: Petersen's family

Linkless embeddings have drawn attention of many researchers. Besides those working on knot theory, many researchers working in discrete mathematics are also interested in this topic. For example, Lovász and Schrijver [30, 31] proved that two well-known invariants given by Colin de Verdière [6, 7, 8] are closely related to linklessly embeddable graphs. The invariants are based on spectral properties of matrices associated with a graph G.

Flat embeddings in the 3-space are a generalization of embeddings in the plane. These two embedding problems share an interesting property. The famous theorem by Whitney which says that every 3-connected planar graph has a unique planar embedding, can be generalized to the flat embedding case, i.e., every flatly embeddable 4-connected graph G has an "essentially unique" flat embedding in $\mathbb{R}^3$, see [47]. Here, "essentially unique" means embeddings up to ambient isotopy, which we define later. In [29], Lovász conjectured that there is a polynomial time algorithm for the following problem:

Flat and Linkless Embedding

Input: A graph G.

Output: Either (1) detect one of Petersen's family graphs as a minor in G or (2) return a flat (and hence linkless) embedding of G in $\mathbb{R}^3$.

Let us observe that if the output is (1), then we have one of the excluded minors for linklessly embeddable graphs and hence this is a certificate that G has no linkless (and no flat) embeddings. As mentioned above, by Robertson and Seymour's results [41], we can test whether or not an input graph has one of Petersen's family graphs as a minor, but if it does not contain any of them, that algorithm does not give the second conclusion. In [20], van der Holst gives the first polynomial-time algorithm for deciding whether a given graph has a linkless embedding. In this paper we give a different algorithm for computing linkless embeddings which runs in quadratic time. Furthermore, our algorithm not only computes linkless embeddings but flat embeddings of linklessly embeddable graphs. As far as we are aware, no polynomial-time algorithm for constructing flat embeddings

[1] In this paper, by a 2-dimensional disk we always mean a 2-dimensional topological disk.

was known prior to our work. See below for a more detailed comparison of our algorithm and van der Holst's.

1.2 Testing linklessness of an embedding

We also consider some classical problems in knot theory and low dimensional topology, specifically the topology of 1-dimensional curves in the 3-space, with the objective of determining their computational complexity. Historically, determining whether a given knot is trivial or not is one of the central questions in low dimensional topology. The problem of finding an algorithm to decide about triviality of a knot was posed already by Dehn [11]. Let us formulate the problem in the following way:

The unknot problem

Input:	A cycle C of order n and an embedding of C in $\mathbb{R}^3$.
Output:	Decide whether or not C is knotted.

This problem has been extensively studied by researchers working on computational geometry as well. Let us observe that topologists study this problem at several levels, with varying meanings given to the terms *embedded* and *deformed*. The level that is most appropriate for studying computational questions is that of *piecewise-linear* embeddings. At this level, a closed curve is embedded in $\mathbb{R}^3$ as a simple non-self-intersecting polygonal curve, which is composed of finitely many straight line segments.

The first algorithm for the unknot problem was given by Haken [17]. But as observed by Hass, Lagarias and Pippenger [16], the algorithm clearly does not run in polynomial time. In fact, the question of how easy or difficult it might be to recognize the unknot was in the air immediately following Haken's 1954 announcement that the problem was decidable (the paper was published in 1961, see [17]). It is stated explicitly as an open question by Welsh in [53]. He actually traces the general idea of looking for efficient procedure for solving this problem back farther than Haken, to Tait.

As mentioned above, Hass, Lagarias and Pippenger [16] observed that Haken's algorithm runs in time $2^{O(n^2)}$. They also proved that the unknot problem is in co-NP, and conjectured that it is in NP $\cap$ co-NP. Some recent progress was made in [1, 18]. In particular, the result in [18] predicts that the unknot problem would be in P.

We now consider the following problem:

The link problem

Input:	Two cycles C_1, C_2 of order n and an embedding of $C_1 \cup C_2$ in $\mathbb{R}^3$.
Output:	Decide whether or not $C_1 \cup C_2$ forms a link in $\mathbb{R}^3$.

Hass, Lagarias and Pippenger [16] observed that a polynomial time algorithm for the link problem yields a polynomial time algorithm for the unknot problem.

More than 15 years ago, the following conjecture was made independently by Lovász [29], and Robertson, Seymour and Thomas [48].

CONJECTURE 1.1. *There is a polynomial time algorithm to decide for a given graph G and its embedding in the 3-space, whether or not this embedding is linkesss.*

Let us observe that Conjecture 1.1 clearly implies a polynomial time solution for the link problem. As pointed out by Robertson, Seymour and Thomas, there is an algorithm to test whether or not a given embedding of a graph G is linkless, using the results in [47, 49], but this does not give a polynomial time algorithm.

In this paper, we shall prove that Conjecture 1.1 is actually equivalent to polynomial-time solvability of the link problem, see Theorem 1.3 below.

1.3 Our results

Our first contribution in this paper is to give a polynomial-time algorithm for the flat and linkless embedding problem, which proves the conjecture by Lovász [29] mentioned above.

THEOREM 1.2. *There is an $O(n^2)$-time algorithm for the flat and linkless embedding problem.*

Clearly, every flat embedding is linkless but the converse is false, as a graph that consists of one vertex with two loops shows. However, Robertson, Seymour and Thomas proved that a graph admits a linkless embedding in $\mathbb{R}^3$ if, and only if, it admits a flat embedding. In [20], van der Holst gives a polynomial-time algorithm to construct a linkless embedding if one exists. Our algorithm is different from van der Holst's algorithm in several respects:

1. Our algorithm finds a flat embedding in $\mathbb{R}^3$, while the algorithm from [20] finds linkless embeddings that are not necessarily flat.
2. We improve on the time complexity as van der Holst's algorithm needs at least $\Omega(n^5)$ steps whereas ours runs in $O(n^2)$.
3. In order to give a polynomial time algorithm for the linkless embedding problem, [20] uses deep results of Robertson and Seymour [41] algorithmically, while our algorithm does not. To be precise, the proof of correctness for our algorithm relies on several results from [41] (recently, a shorter proof is found in [25]), but the algorithm itself does not use Robertson and Seymour's algorithm. So together with the arguments in [25, 41], our full proof can be given within 50 pages (in contrast with this, Robertson and Seymour's algorithm needs more than 300 pages for the correctness of the algorithm).
4. The algorithm and the proof method in [20] are very algebraic and are completely different from our paper, which is more combinatorial and geometric.

There are many NP-hard problems which can be solved in polynomial time (often, even linear time) when considering planar graphs or "nearly" planar graphs. Even for problems that remain NP-hard on planar graphs, we often have efficient approximation algorithms, e.g., INDEPENDENT SET, TSP, Weighted TSP, VERTEX COVER, DOMINATION SET, etc. [3, 15, 26, 28]. We expect that most of these fast algorithms for planar graphs can be generalized to linklessly embeddable graphs as well, using our algorithm in Theorem 1.2 and a flat embedding. In addition, linklessly embeddable graphs have an interesting property for the well-known feedback arc set problem. Seymour [51] proved that the minimum size of a feedback arc set in an eulerian linklessly embeddable digraph (i.e, the underlying undirected graph is linklessly embeddable) is equal to the maximum number of arc-disjoint directed cycles. This result can be compared to the well-known result by Lucchesi and Younger [32], who proved that the same conclusion holds for any directed planar graph. The proof given in [51] implies that, given a flat embedding of a given Eulerian digraph G, there is a polynomial time algorithm to find the minimum size of a feedback arc set, which gives rise to the maximum number of arc-disjoint directed cycles. Thus by Theorem 1.2, we can find such sets in polynomial time for any Eulerian linklessly embeddable graph.

Our second main result is an oracle algorithm that shows that Conjecture 1.1 is equivalent to polynomial-time solvability of the unknot problem.

THEOREM 1.3. *There is an oracle polynomial-time algorithm to decide whether a given embedding of a graph in the 3-space is linkless by using an oracle for the link problem.*

1.4 Overview of our algorithms

We first sketch the algorithm for Theorem 1.2. At a high level of description, our algorithm for Theorem 1.2 proceeds as follows: the algorithm first iteratively reduces the size of the input graph until it is 4-connected and reaches a graph of bounded tree-width. Then the algorithm solves the problem on this graph of bounded tree-width.

Bounded tree-width case. This second step needs two deep results in [47]. The first ingredient is that any Kuratowski graph, i.e, a subdivision of K_5 or $K_{3,3}$, has a unique flat embedding in the 3-space $\mathbb{R}^3$, where uniqueness means "up to an ambient isotopy". The second ingredient is that if G is 4-connected and has a flat embedding in $\mathbb{R}^3$, then G has a unique flat embedding. By combining these two results, we get the following strong fact:

> Fix a flat embedding of a Kuratowski subgraph K of a 4-connected graph G in $\mathbb{R}^3$. Then the rest of the graph is uniquely attached to K, if G has a flat embedding.

In general, this fact is not enough to derive a polynomial time algorithm for constructing flat embeddings. However, if the tree-width of a 4-connected graph is bounded, we can construct a flat embedding in polynomial time (even in linear time) using dynamic programming, whenever one exists.

Reduction step. For the reduction step, the algorithm uses the excluded grid theorem [13, 37, 39, 44]: if the tree-with of G is big enough, G contains a huge grid minor. By combining the results in [41] and [24], if an input graph does not contain a K_6-minor, then, after deleting at most one vertex, we can find a grid minor which is planarly drawn, i.e., up to 3-separations, the grid minor induces a planar embedding. In fact, if there is a separation (A, B) of order at most three in this planarly drawn grid, then this gives us a reduction (for details, we refer to Section 3). Otherwise, this grid minor induces a 2-cell embedding in a plane (and hence it is a planar subgraph).

A deep theorem in [41] tells us that every vertex "deep inside" this grid minor is irrelevant with respect to all excluded minors in the Petersen family (A shorter proof of this fact is given in [25]). In addition we will prove that such a vertex does not affect our flat embedding in the 3-space. Hence, we can remove this vertex without affecting flat embeddability of G. Note that the difficulty here is only in the proof of the existence of this vertex. Once we have proved that there is an irrelevant vertex, algorithmically such a vertex can easily be found in linear time. This concludes the reduction step for the proof of Theorem 1.2.

We now sketch the proof of Theorem 1.3. Suppose an embedding σ of a given graph G in $\mathbb{R}^3$ is given. In order to prove Theorem 1.3, we have to deal with the following two technical difficulties.

(a) We need to test, for any cycle, whether or not the embedding induced by this cycle is unknotted.

(b) We need to test, for any two disjoint cycles, whether or not the embedding induced by these two disjoint cycles contains a link.

Assuming the embedding induced by each cycle in G is unknotted and the embedding induced by any two disjoint cycles does not contain a link, it is possible to prove that σ is linkless.

Task (a) can be completed in polynomial time, having a (polynomial-time) oracle for the unknot problem, which, as explained above, can be obtained from an oracle for the link problem (see [16]). Also task (b) can also be completed in polynomial time, having a (polynomial-time) oracle for the link problem.

So by considering all the cycles in G, and then testing (a) and (b) for them, we can figure out whether or not σ is linkless. But the problem here is that G may contain exponentially many cycles. So, to obtain a polynomial time algorithm for Theorem 1.3, we cannot look at all cycles.

In order to overcome this problem, we first test whether or not G has small tree-width. If G has small tree-width, it can be shown that we only have to look at polynomially many disjoint cycles to test (a) and (b) for them.

So, suppose G has large tree-width. Then G contains a huge grid minor. As described above, after deleting at most one vertex, we can find a grid minor T which is planarly drawn, i.e, up to 3-separations, T induces a planar embedding.

We now construct an optimal flat embedding by Theorem 1.2. In this optimal flat embedding, the embedding induced by T must be an embedding in the sphere. So, our first task is to figure out whether or not the embedding induced by T satisfies this property. To do so, we need to figure out all faces in a planar embedding σ' of T. As it turns out, to test whether or not the embedding induced by the grid minor T is linkless, we only need to test (a) and (b) for all faces of σ'. Since there are linearly many faces in the graph induced by T, we can test, in polynomial time, whether or not the embedding induced by T is linkless.

The following two facts are our key observations:

1. Let G, T be as above, and suppose an embedding of G in $\mathbb{R}^3$ is given. If the embeddings induced by $G - T$ and by T are linkless, then there is a polynomial time algorithm to test whether or not σ is linkless.
2. If one of the embeddings induced by $G - T$ and by T is not linkless, then σ is not linkless either.

These two observations imply that it remains to look at the embedding induced by $G - T$. Thus this allows us to make a reduction. We just recurse the algorithm to $G - T$. Due to space restrictions we refer to the full version of this paper for a proof of Theorem 1.3.

1.5 Basic definitions

Before proceeding, we review basic definitions. For basic graph theory notions, we refer the reader to the book by Diestel [12], for topological graph theory we refer to the monograph by Mohar and Thomassen [35].

A *separation* of a graph G is a pair (A, B) of subgraphs of G with $A \cup B = G$ such that there is no edge between $A - B$ and $B - A$. The *order* of the separation is $|V(A) \cap V(B)|$. If (A, B) is a separation of G of order k, we write A^+ for the graph obtained from A by adding edges joining every pair of nonadjacent vertices in $V(A) \cap V(B)$. We define B^+ analogously.

A *tree-decomposition* of a graph G is a pair (T, B), where T is a tree and B is a family $\{B_t \mid t \in V(T)\}$ of vertex sets $B_t \subseteq V(G)$, such that the following two properties hold:

1. $\bigcup_{t \in V(T)} B_t = V(G)$, and every edge of G has both ends in some B_t.

2. If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $B_t \cap B_{t''} \subseteq B_{t'}$.

The *width* of a tree-decomposition (T, B) is $\max\{|B_t| : t \in V(T)\} - 1$. The *tree-width* of G is defined as the minimum width taken over all tree-decompositions of G. Let (T, B) be a tree-decomposition of a graph G. By fixing a root r of T we give T an orientation. For $t \in V(T)$ we define T_t to be the subtree of T rooted at t, i.e., the subtree of T induced by the set of nodes $s \in V(T)$ such that the unique path from s to r contains t. We define $B(T_t) := \bigcup_{s \in V(T_t)} B_s$.

One of the most important results about graphs, whose tree-width is large, is the existence of a large grid minor or, equivalently, a large wall. Let us recall that an *r-wall* or a *wall of height r* is a graph which is isomorphic to a subdivision of the graph W_r with vertex set $V(W_r) = \{(i, j) \mid 1 \leq i \leq r,\ 1 \leq j \leq r\}$ in which two vertices (i, j) and (i', j') are adjacent if, and only if, one of the following possibilities holds: (1) $i' = i$ and $j' \in \{j - 1, j + 1\}$ or (2) $j' = j$ and $i' = i + (-1)^{i+j}$. We can also define an $(a \times b)$-wall in a natural way, so that the r-wall is the same as the $(r \times r)$-wall. It is easy to see that if a graph G contains an $(a \times b)$-wall as a subgraph, then it has an $(\lfloor \frac{1}{2} a \rfloor \times b)$-grid minor, and conversely, if G has an $(a \times b)$-grid minor, then it contains an $(a \times b)$-wall. Let us recall that the $(a \times b)$-grid is the Cartesian product of paths, $P_a \,\square\, P_b$. An (8×5)-wall is shown in Figure 2.

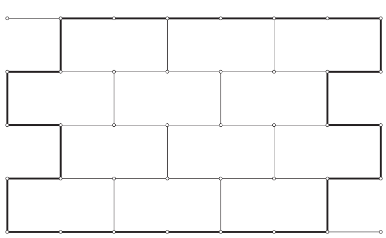

Figure 2: An (8×5)-wall and its outer cycle

THEOREM 1.4. *There is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that if a graph G has tree-width at least $f(r)$, then G contains an r-wall.*

The best known upper bound for $f(r)$ is given in [13, 37, 44]. It is $20^{2^{r^5}}$. The best known lower bound is $\Theta(r^2 \log r)$, see [44].

2. FLATLY AND LINKLESSLY EMBEDDABLE GRAPHS

In this section we recall some results about linklessly embeddable graphs used later on. Due to space restrictions, we can only state the most relevant results and almost no background.

2.1 Construction of flat embeddings

Recall that a piecewise-linear embedding of a graph in the 3-space $\mathbb{R}^3$ is *flat* if every cycle of the graph bounds a 2-dimensional disk disjoint from the rest of the graph. If C, C' are disjoint simple closed curves in $\mathbb{R}^3$, then their *linking number* is the number of times (modulo 2) that C crosses C' in a regular projection of $C \cup C'$ onto some hyperplane. It is easy to see that the linking number (modulo 2) is the same for every such projection. Hereafter, all linking numbers discussed in this paper will be mod 2. The proof of the following result is easy and the details are left to the reader.

LEMMA 2.1. *Let G be a graph, and σ be an embedding of G in $\mathbb{R}^3$. Let C_1, C_2 be disjoint cycles in G, and let P be a path in G that is disjoint from $C_1 \cup C_2$, except that its endvertices are in $V(C_1)$. Let C_1', C_1'' be the two cycles in $V(C_1 \cup P)$ with $V(C_1') \cap V(C_1'') = V(P)$. If the linking number of C_1, C_2 is nonzero, then also one of the linking numbers of C_1', C_2 or C_1'', C_2 is nonzero.*

Let v be a vertex of degree 3 in a graph G, with its three neighbors v_1, v_2, v_3. Let H be a graph obtained from $G - v$ by adding three new edges v_1v_2, v_2v_3, v_3v_1. We say that H is obtained from G by a $Y\Delta$ change (at v), and G is obtained from H by a ΔY change (at v_1, v_2, v_3). If G' can be obtained from G by a series of $Y\Delta$ or ΔY changes, we say that G and G' are *$Y\Delta$-equivalent*. The *Petersen family* is the set of the seven graphs (up to isomorphism) that are $Y\Delta$-equivalent to K_6. One of these is the Petersen graph. The following result was proved in [47].

THEOREM 2.2. *Let G be a graph obtained from H by a ΔY operation. Then G has a flat embedding in $\mathbb{R}^3$ if, and only if, H has.*

It is easy to see that degree 1 vertices can be deleted and degree 2 vertices can be suppressed without affecting linkless embeddability. Thus it follows from Theorem 2.2 that we may assume that each vertex has degree at least 4. Furthermore, concerning vertices of degree 4, we have the following.

LEMMA 2.3. *Suppose that a graph G contains an edge uv, where $\deg_G(u) = 4$ and $\deg_G(v) = 5$. Suppose, moreover, that $N(u) = (N(v) \cup \{v\}) - \{u, a\}$ for some vertex a in $N(v)$. Then there is a separation (A, B) of order four such that $B - A$ consists of u and v only, and G has a flat embedding in $\mathbb{R}^3$ if, and only if, for every vertex $b \in N(v) - \{u, a\}$, the graph G' obtained from G by contracting the edge ub has a flat embedding.*

We sketch the proof of Lemma 2.3. If G' has a flat embedding in 3-space $\mathbb{R}^3$, then $A \cap B \cup \{v\} - \{a\}$ is contained in a disk. Then we can easily put the vertex u back to this disk so that the resulting embedding of G (that extends the flat embedding of G') is still flat.

The following is (6.5) in [47]. Recall from above the definition of the graphs A^+, B^+ for a separation (A, B) of a graph G.

THEOREM 2.4. *Suppose G has a separation (G_1, G_2) with $|V(G_1) \cap V(G_2)| \leq 4$. Suppose furthermore, G contains both G_1^+ and G_2^+ as a minor. Then G has a flat embedding in 3-space $\mathbb{R}^3$ if, and only if, both G_1^+ and G_2^+ have one.*

2.2 Spatial embeddings of 4-connected graphs

The aim of this section is to show that 4-connected linklessly embeddable graphs essentially have a unique flat embedding in 3-space $\mathbb{R}^3$. The following results are proved in [47] along with its companion papers [45, 46]. Readers not familiar with these results may wish to consider the survey [48] which contains many of the results needed below. We refer to the graphs K_5 and $K_{3,3}$ as the *Kuratowski graphs*. A *Kuratowski subgraph* of a graph G is a subgraph of G isomorphic to a subdivision of a Kuratowski graph. For embeddings ϕ_1, ϕ_2 of a graph G we write $\phi_1 \cong_{a.i.} \phi_2$ to denote that they are ambient isotopic. Let us recall that ϕ_1 and ϕ_2 are *ambient isotopic*, if there exists an orientation preserving homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ mapping ϕ_1 onto ϕ_2.

LEMMA 2.5. *1. Any two flat embeddings of a planar graph in $\mathbb{R}^3$ are ambient isotopic.*

2. The graphs K_5 and $K_{3,3}$ have exactly two non-ambient isotopic flat embeddings.

3. Let ϕ_1, ϕ_2 be two flat embeddings of a graph G which are not ambient isotopic. Then there is a Kuratowski subgraph H of G for which $\phi_{1|H}$ and $\phi_{2|H}$ are not ambient isotopic. Here $\phi_{i|H}$ denotes the restriction of ϕ_i to H.

We now define a neighbourhood relation between Kuratowski subgraphs of a graph G. Let $H_1, H_2 \subseteq G$ be Kuratowski subgraphs of G such that $H_1 \neq H_2$. H_1 and H_2 are 1-*adjacent* if there exists a path $P \subseteq G$ and an $i \in \{1, 2\}$ such that P has only its endpoints in common with H_i and such that $H_{3-i} \subseteq H_i \cup P$.

H_1 and H_2 are 2-*adjacent* if there are distinct vertices $v_1, \ldots, v_7 \in V(G)$ and pairwise internally vertex disjoint paths $L_{i,j}$, for $(i, j) \in \{1, \ldots, 4\} \times \{5, 6, 7\} \cup \{(3, 4)\}$ linking v_i and v_j such that $H_1 = \bigcup \{L_{i,j} \mid (i, j) \in \{2, 3, 4\} \times \{5, 6, 7\}\}$ and $H_2 = \bigcup \{L_{i,j} \mid (i, j) \in \{1, 3, 4\} \times \{5, 6, 7\}\}$. The path $L_{3,4}$ is not used here but is required to exist. Note that if H_1, H_2 are 2-adjacent then they are both isomorphic to subdivisions of $K_{3,3}$. H_1 and H_2 are *adjacent* if they are 1- or 2-adjacent. Let H, H' be Kuratowski subgraphs of G. We say that H and H' *communicate* if there are Kuratowski subgraphs $H = H_1, \ldots, H_k = H'$ of G such that H_i and H_{i+1} are adjacent for all $1 \leq i < k$.

LEMMA 2.6. *1. Let ϕ_1, ϕ_2 be flat embeddings of G and let H, H' be adjacent Kuratowski subgraphs of G. If $\phi_{1|H} \cong_{a.i.} \phi_{2|H}$ then $\phi_{1|H'} \cong_{a.i.} \phi_{2|H'}$.*

2. If G is 4-connected, then all pairs H, H' of Kuratowski subgraphs of G communicate.

3. If ϕ, ϕ' are flat embeddings of a 4-connected graph G, then $\phi \cong_{a.i.} \phi'$ or $\phi \cong_{a.i.} -\phi'$.

Note that (*iii*) follows easily from (*i*) and (*ii*) and the previous lemma. For the purpose of this paper this suggests the following algorithm for computing a flat embedding of a graph G. We first choose a Kuratowski subgraph H of G and compute a flat embedding of H. We then choose an adjacent Kuratowski subgraph H' of G and extend the flat embedding of H to $H \cup H' \cup L$ (where $L = L_{3,4}$ is the additional path in the case of 2-adjacency, and $L = \emptyset$ in case of 1-adjacency). This is essentially unique. For, suppose there were two non-ambient-isotopic flat embeddings ϕ_1, ϕ_2 of $H \cup H' \cup L$ extending the embedding of H. As ϕ_1, ϕ_2 agree (up to ambient isotopy) on H, and H and H' are adjacent in $H \cup H' \cup L$, Part 1 of Lemma 2.6 implies that they also agree on H'. But then, by Part 3 of Lemma 2.5, as they agree on all Kuratowski subgraphs, ϕ_1, ϕ_2 agree on $H \cup H' \cup L$. Note that Part 1 of Lemma 2.6 and Part 3 of Lemma 2.5 do not require the graph to be 4-connected.

However, if G is 4-connected, then by starting from one Kuratowski subgraph whose embedding we fix and iteratively proceeding to adjacent Kuratowski subgraphs, we can embed all Kuratowski subgraphs of G. This embedding then has a unique extension to the complete graph. We will employ this idea in Section 4 below.

3. BOUNDING THE TREE-WIDTH

In this section we will present the reduction step of the general algorithm for solving the Linkless Embedding problem presented in Section 5 below. Let us observe that by Theorem 2.2 and Lemma 2.3 (and remarks just after Lemma 2.3), we may assume that every vertex in a given graph G has minimum degree at least 4, and any vertex of degree 4 does not satisfy the assumption of Lemma 2.3.

Let us define that the *nails* of a wall are the vertices of degree three within it. The *perimeter* of a wall W, denoted $per(W)$, is the unique face in this embedding which contains more than 6 nails. The *bricks* of a wall are the faces containing 6 nails. As walls are essentially 3-connected, Whitney's theorem implies that any wall has a unique planar embedding. For any wall W in H, there is a unique component U of $H - per(W)$ containing $W - per(W)$. The *compass* of W, denoted $comp(W)$, consists of the graph with vertex set $V(U) \cup V(per(W))$ and edge set $E(U) \cup E(per(W)) \cup \{xy | x \in V(U), y \in V(per(W))\}$, where $xy \in E(G)$. A *subwall* W' of a wall W is a wall which is a subgraph of W. A h-subwall of W is *proper* if it consists of h consecutive bricks from each of h consecutive rows of W. The *exterior* of W' is $W - W'$. A proper subwall is *dividing* if its compass is disjoint from its exterior. We say a proper subwall W' is dividing in a subgraph H of F if $W' \subseteq H$ and the compass of W' in H is disjoint from $(W - W') \cap H$.

A wall is *planarly drawn* if its compass does not contain two vertex disjoint paths connecting the diagonally opposite corners. Note that if the compass of W has a planar embedding whose infinite face is bounded by the perimeter of W then W is clearly planarly drawn.

Seymour [50], Thomassen [52], and others have characterized that if the wall W is planarly drawn, then its compass $comp(W)$ can be embedded into a plane, up to 3-separations, such that its perimeter $per(W)$ is the outer face boundary.

It is easy to see that any subwall of a planarly drawn wall must be both planarly drawn and dividing. Furthermore, if x and y are two vertices of a planarly drawn wall W and there is a path between them which is internally disjoint from W then either x and y are both on $per(W)$ or some brick contains both of them. Robertson and Seymour [41] proved:

THEOREM 3.1. *For every pair of integers l and t there exist integers $w(l,t) > w'(l,t) > max(l,t)$ such that the following holds. Let K be a graph of order t. If the tree-width of a graph H is at least $w(l,t)$, and H has no K-minor, then there is a wall W of height $w'(l,t)$, and for some subset X of less than $\binom{t}{2}$ vertices of H there are t^{10} disjoint proper subwalls W' (of the wall W) of height l, which are disjoint from X and are planarly drawn and dividing in $H - X$. In addition, any of these disjoint proper subwalls of height l has face-distance at least t^{10} from any other in the wall W.*

In fact, we can give the explicit bound for $w(l,t)$. Combining the best known bound for the grid theorem in [44], the proof of Theorem 3.1 in [41] implies that $w(l,t) \leq 10^{10^{10^q}}$, where $q = l^t$.

A vertex v in G is called *irrelevant* with respect to a given minor M in G if G has an M-minor if, and only if, $G - v$ has. The following result was proved in [41] (a shorter proof was given in [25]).

THEOREM 3.2. *There is a computable function $f : \mathbb{N} \to \mathbb{N}$ satisfying the following: let $l \geq f(t)$ and let $H, W, X, K, w(l,t)$, $w'(l,t)$ be as in Theorem 3.1. Let W' be one of the proper subwalls (of the wall W) of height l which is disjoint from X, and is planarly drawn and dividing in $H - X$. Suppose furthermore that the comp(W') has a 2-cell embedding in a plane with per(W') in the outer face boundary. Then the unique vertex v which has distance exactly $l/2$ from the per(W') in the wall W' is irrelevant with respect to a K-minor.*

We now give an algorithmic result of Theorems 3.1 and 3.2 in [41].

THEOREM 3.3. *Let $t, l, f, w'(l,t)$ be as in Theorem 3.2. There is an $O(m)$ time algorithm, where m is the number of edges, which, given H, W, K as in Theorem 3.2 (thus the wall W is of height $w'(t,l)$), constructs one of the following.*

1. *A K-minor in H, or*
2. *for some subset X of less than $\binom{t}{2}$ vertices of H there are t^{10} disjoint proper subwalls W' (of the wall W) of height l, which are disjoint from X, and are dividing in $H - X$. In addition, any of these disjoint proper subwalls of height l has face-distance at least t^{10} from any other in the wall W.*

It is easy to see that if G has at least $2^t|V(G)|$ edges, then one can easily find a K_t-minor in linear time (see [38]). If we can find a K_6-minor in a given graph G in linear time, we are done. So a flatly embeddable graph G has at most $2^6|V(G)|$ edges, which, hereafter, we assume. Thus the time complexity of Theorem 3.3 can be improved to $O(n)$, where n is the number of vertices of an input graph.

If $K = K_6$ in Theorem 3.1, then the following stronger version of Theorem 3.1 is proved in [24].

THEOREM 3.4. *Let $f(t)$ be as in Theorem 3.2 with $t = 6$. For any $l \geq f(6)$, there are integers $w(l) > w'(l)$ satisfying the following: If the tree-width of a graph H is at least $w(l)$, then there is a wall W of height $w'(l)$, and one of the following holds: (1) a K_6-minor in H, or (2) for some subset X of at most one vertex of H, there are at least 10 disjoint proper subwalls W' (of the wall W) of height l, which are disjoint from X and dividing in $H - X$. In addition, any of these disjoint proper subwalls of height l has face-distance at least 2 from any other in the wall W. Moreover,*

- *(a) if $|X| = 0$ and three of the disjoint proper subwalls of height l are not planarly drawn, then H has a K_6-minor, and*
- *(b) if $|X| = 1$ and there is a proper subwall of height l that is not planarly drawn, then H has a K_6-minor.*

Furthermore, given the wall W, there is a linear time algorithm to compute one of 1 and 2 above.

Note that the last algorithmic conclusion follows from planarity testing [5, 19, 54].

We now prove the following result for the linkless embedding case. Recall from the beginning of this section that we assume that a graph G has minimum degree at least 4.

THEOREM 3.5. *There is a constant f satisfying the following: Let G be a given input graph with minimum degree 4, and $W, w(l), w'(l)$ be as in Theorem 3.4 with $l \geq f$. Then one of the following holds:*

1. *G has a K_6-minor, or*
2. *for some subset X of at most one vertex of G, there is a proper subwall W' (of the wall W) of height l, which is disjoint from X and dividing in $G - X$, and has a 2-cell embedding with per(W') in the outer face boundary in $G - X$. Moreover, the unique vertex v which has distance exactly $l/2$ from the per(W') in the wall W' is irrelevant with respect to a K-minor, where K is any graph in the Petersen family. Furthermore, no matter how we give a flat embedding of the graph $G - v$ in 3-space $\mathbb{R}^3$, the embedding of $G - v$ can be changed so that, after putting the vertex v back to the resulting embedding, the resulting embedding of G is flat, or*
3. *either there is a reduction as in Lemma 2.3, or there is a separation (A, B) of order at most three in $G - X$ such that $V(A) \cap V(B)$ only involves the vertices in the comp(W) in $G - X$, and B contains all the vertices of the per(W). Moreover, if the second happens, G contains both $(A \cup X)^+$ and $(B \cup X)^+$ as minors. Hence it follows that this separation $(A \cup X, B \cup X)$ is a reduction as in Theorem 2.4. Note that it is a separation of order at most 4 in G.*

Due to space restrictions we refer to the full version of this paper for a proof of Theorem 3.5 and the following main result of this section.

THEOREM 3.6. *Let $G, W, f, l, w(l), w'(l)$ be as in Theorem 3.5. Suppose the wall W is given. Then there is an $O(n)$ time algorithm, where n is the number of vertices of G, to construct one of the following:*

1. *one of the graphs in the Petersen family in G as a minor, or*
2. *for some subset X of at most one vertex of G there is a proper subwall W' (of the wall W) of height l, which is disjoint from X, dividing in $G - X$, and has a 2-cell embedding with per(W') in the outer face boundary in $G - X$. Moreover, we can find an irrelevant vertex as in the second conclusion of Theorem 3.5, or*

3. *either there is a reduction as in Lemma 2.3, or there is a separation (A, B) of order at most three in $G - X$ such that $V(A) \cap V(B)$ only involves the vertices in the comp(W) in $G - X$, and B contains all the vertices of the per(W'). Moreover, if the second happens, G contains both $(A \cup X)^+$ and $(B \cup X)^+$ as minors. Hence it follows that this separation $(A \cup X, B \cup X)$ is a reduction as in Theorem 2.4. Note that it is a separation of order at most 4 in G.*

4. LINKLESS EMBEDDINGS ON GRAPH CLASSES OF BOUNDED TREE-WIDTH

In this section we present a linear time algorithm for the linkless embedding problem on graph classes of bounded tree-width. The algorithm is based on various results about flat embeddings presented in Section 2.2. We first consider the case of 4-connected graphs.

LEMMA 4.1. *There is an algorithm which for a given 4-connected graph G either returns a flat embedding of G or produces a minor $H \preceq G$ of the Petersen family, in time $f(\mathrm{tw}(G)) \cdot |G|$, for some computable function $f : \mathbb{N} \to \mathbb{N}$.*

PROOF. Due to space restrictions we only prove the existence of an $f(\mathrm{tw}(G)) \cdot |G|^2$-algorithm here (which is enough for the application in Section 5). With some further effort this can be improved to linear time. Details of the linear time algorithm can be found in the full version of this paper.

Given G, we first test in linear time (for instance using Courcelle's theorem [10]) if any graph in the Petersen family is a minor of G. If no such minor is found, we compute a flat embedding of G. The lemmas outlined in Section 2.2 and the discussion at the end of that section suggest the following simple algorithm for constructing a flat embedding of a 4-connected graph in $\mathbb{R}^3$.

1. If G is planar, compute the unique embedding of G into $\mathbb{R}^3$. Otherwise, choose a Kuratowski subgraph of G and embed it flat into $\mathbb{R}^3$.

2. While there is a Kuratowski subgraph K of G that is not yet embedded, compute two adjacent Kuratowski subgraphs H, H' so that H is already embedded but at least one edge of H' is not yet embedded. As discussed in Section 2.2, there is a unique way of extending the embedding of H to an embedding of $H \cup H'$ which can easily be computed.

3. Once all Kuratowski subgraphs are embedded, the rest of the graph is planar and, by Lemma 3, there is an essentially unique extension to an embedding of G.

We claim that this algorithm can be implemented to run in time $f(\mathrm{tw}(G)) \cdot |G|^2$, where f is a computable function $f : \mathbb{N} \to \mathbb{N}$. Given a 4-connected graph G, first use Bodlaender's algorithm [4] to compute an optimal tree-decomposition of G in time $g(\mathrm{tw}(G)) \cdot |G|$, where $g : \mathbb{N} \to \mathbb{N}$ is some computable function. Using a planarity algorithm [5, 19, 54], we can find the Kuratowski subgraph K of G required in step 1 in linear time or conclude that G is planar, in which case we can easily compute the unique embedding into $\mathbb{R}^3$. Towards the second step, note that the while loop can take at most $||G||$ iterations as each iteration embeds at least one new vertex or edge. In each iteration we have to find the two Kuratowski subgraphs H and H'. This can easily be done in linear time, either by dynamic programming or by realizing that the condition "H, H' are Kuratowski subgraphs of which H is already embedded and H' is not" has a straightforward definition in monadic second-order logic (MSO). It follows from a result by Arnborg, Lagergren and Seese [2] that given the MSO-definition, the graphs H, H' can be computed in time $h(\mathrm{tw}(G)) \cdot ||G||$. As for all graphs $||G|| \leq \mathrm{tw}(G) \cdot |G|$, the result follows. □

We are now ready to present the complete linear time algorithm for computing flat embeddings of linklessly embeddable graphs of bounded tree-width.

LEMMA 4.2. *There is an algorithm which, on input G, solves the* Flat and Linkless Embedding *problem for G in time $f(\mathrm{tw}(G)) \cdot |G|$, for some computable function $f : \mathbb{N} \to \mathbb{N}$.*

Due to space restrictions, we only sketch the proof here. A complete proof can be found in the full version of this paper. Let G be given. Using Bodlaender's algorithm [4], we first compute a tree-decomposition of G of width $\mathrm{tw}(G)$ in time $g(\mathrm{tw}(G)) \cdot |G|$. The next step is to test if any graph of the Petersen family is a minor of G. This can be done in time $g'(\mathrm{tw}(G)) \cdot |G|$. If G contains a Petersen family minor, we can conclude that G has no linkless embeddings and return the minor. Otherwise, we split the graph into its 4-connected components using dynamic programming over the tree-decomposition. After some preprocessing in which we use $Y\Delta$-transformations to ensure that 3-separations are joined at triangles, we then use Lemma 4.1 to find flat embeddings of 4-connected components. The individual embeddings can then be glued together along the disks bounding 3-separations.

5. ALGORITHM

Finally, we are ready to present the complete algorithm.

Flat and Linkless Embedding

Input: A graph G.

Output: Either detect one of Petersen's family graphs as a minor in G or return a flat (and hence also linkless) embedding of G in $\mathbb{R}^3$.

Running time: $O(n^2)$.

Description: Initially, we delete all vertices of degree at most 1. Also, if there is a vertex v of degree 2, then we just contract vu, where u is one of the two neighbors of v. Thus we may assume that minimum degree is at least 3.

Step 1. If there is a vertex v of degree 3, then we perform $Y\Delta$ operation at v. We repeat doing this as long as there are some vertices of degree 3. Hereafter, we may assume that the minimum degree of the resulting graph G is at least 4.

Step 2. Test if the tree-width of the current graph G is small or not, say smaller than some value f, where f comes from Theorem 3.5. This can be done in linear time by the algorithm of Bodlaender [4]. If the tree-width is at least f, then go to Step 3. Otherwise we use the algorithm described in Lemma 4.2 to compute a flat embedding in linear time or certify that no such embedding exist by computing a minor of G in the Petersen family.

Step 3. It is easy to see that if the current graph G has at least $2^6|V(G)|$ edges, then one can easily find a K_6-minor in linear time (see [38]). So we may assume that the current graph G has at most $2^6|V(G)|$ edges.

At this moment, the tree-width of the current graph G is at least f. Use the algorithm of Bodlaender [4] (or the algorithm of Robertson and Seymour [41]) to construct a wall W of height at least $w'(l)$, where $w'(l)$ is as in Theorem 3.4. Perform the algorithm of Theorem 3.6 to find a separation or reduction as in the third conclusion of Theorem 3.6, or an irrelevant vertex v, or a minor of a graph in the Petersen family. If the third outcome occurs, then output the minor. If the second one happens, then we recurse this algorithm to $G - v$. If the first one happens, then we reduce the size of the the current graph G as in the proof of Theorem 3.6. This completes the description of the algorithm. □

The correctness of Steps 2 and 3 follow from Sections 3 and 4. It is easy to see that degree 1 vertices can be deleted and degree 2 vertices can be contracted. Thus the correctness of Step 1 follows.

The time complexity of the algorithm can be estimated as follows. All individual steps in the above algorithm can be done in linear time. Applying the recursion results in another factor of n. Thus the time complexity is $O(n^2)$.

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