## Problem sheet 1.

→ Problem 4 better solved after Monday's lecture.

# PAC Learning (Take II)

Let $C_n$ be a representation class over $X_n$. Let $C = \bigcup_{n \geq 1} C_n$, $X = \bigcup_{n \geq 1} X_n$.

We say that $C$ is PAC learnable if there exists a learning algorithm $L$, s.t. $\forall n \geq 1$, $\forall c \in C_n$, $\forall$ distribution $D$ over $X_n$, $\forall\, 0 < \varepsilon < 1/2$ and $\forall\, 0 < \delta < 1/2$, if $L$ is given access to $EX(c, D)$ and inputs $\varepsilon, \delta$, size$(c)$, with probability at least $1-\delta$, it outputs a hypothesis $h \in C_n$ satisfying $err(h) \leq \varepsilon$.

We say that $C$ is efficiently PAC learnable if the running time of $L$ is polynomial in $n$, size$(c)$, $1/\varepsilon$, $1/\delta$, when learning $c \in C_n$.

Theorem The class of CONJUNCTIONS is efficiently PAC-learnable.

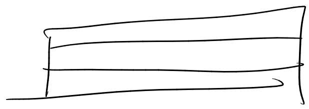
# Intractability of Learning 3-term DNF formulae

what is a 3-term DNF formula?

$$\text{3-term-DNF}_n = \{ T_1 \vee T_2 \vee T_3 \mid T_i \text{ a conjunction over } x_1, \ldots, x_n \}$$

$$\underbrace{(x_1 \wedge \bar{x}_3 \wedge \ldots \wedge x_7)}_{\text{term / conjunction}} \vee (x_2 \wedge x_4 \wedge x_8 \wedge \ldots) \vee (x_1 \wedge x_3 \wedge \bar{x}_7 \wedge \ldots)$$

$\text{size}(c) \leq 6n$



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**Theorem**: 3-term-DNF formulae are not efficiently PAC-learnable unless $RP = NP$.

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What is NP?

A language $L \in NP$,
if there exists a polynomial time algorithm:
if $x \in L$, then $\exists y$, $|y| \leq \text{poly}(|x|)$, $A(x, y) = 1$
if $x \notin L$ then $\forall y$, $|y| \leq \text{poly}(|x|)$, $\underbrace{A}_{\text{verifier}}(x, y) = 0$

A language $L \in RP$
if there exists a randomize polytime algorithm $A$, s.t.

if $x \notin L$, then $A(x) = 0$ with probability 1
if $x \in L$ then $A(x) = 1$ with probability $\geq 2/3$.

Widely believed that $RP \neq NP$.

PROBLEM : Graph 3-COLOURING

Given an undirected unweighted graph $G$, is $G$ 3-colourable? Is there an assignment from $V(G) \longrightarrow \{R, B, G\}$, s.t. for every edge $e \in G$ the end-points of $G$ are assigned different colours.

3-COLOURABLE = $\{ \langle G \rangle \mid G$ is 3-colourable$\}$

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Graph $G$ $\longrightarrow$ Sample of positively & negatively labelled example $\underline{S}$

- size of the sample is at most poly($|G|$).

STEP 2

$G$ is 3-colourable $\Longleftrightarrow$ $\exists\, \varphi$ - 3termDNF that is consistent with the sample

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$EX(c, D)$ : Pick a point at random from the sample $S$.

$D$ is uniform on the sample $S$, & 0 everywhere else

$\delta = 1/3$ , $\varepsilon = 1/2|S|$.

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Algorithm:

- Use the PAC learning algorithm for 3-term DNF using $EX(c, D)$, $\delta = 1/3$, $\varepsilon = 1/2|S|$.
- Let $h$ the 3-term DNF returned by the algorithm
- Say $\underline{YES}$ if $h$ is $\underline{\text{consistent with } S}$
  correct on every example in $S$
  $\underline{NO}$ otherwise.

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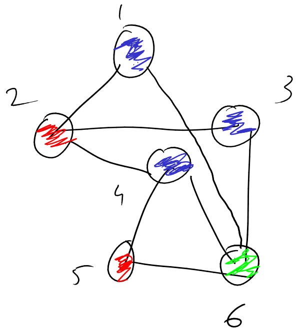
Distribution : uniform on a set of size $|S|$, if $\varepsilon = 1/2|S|$, the $err(h) \leq \varepsilon$, $h$ is correct on every example in $S$.

G be some graph $\quad X_n = \{0,1\}^n$



n - nodes

m - edges

S positive examples

$v_i \quad ((1, \ldots, 1, 0, 1 \ldots 1), +1)$

$i^{th}$ pos

$((0,1,1,1,1,1),+1)$

$((1,0,1,1,1,1)+1)$

$\vdots$

$((1,1,1,1,1,0)+1)$

S negative examples

$e_{ij} : ((1,1..,1,0,1,1,0,1,1..), -0)$

i $\quad$ j

$((0,0,1,1,1,1),-0)$

$\vdots$

$((1,1,1,1,0,0),-0)$

$|S| = m+n$

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(suppose G is 3-colourable)

$\varphi: T_R \vee T_B \vee T_G$

$T_R = (z_1 \wedge z_3 \wedge z_4 \wedge z_6)$ (include variables (positive literals) for vertices not coloured red)

$T_B = (z_2 \wedge z_5 \wedge z_6)$

$T_G = (z_1 \wedge z_2 \wedge z_3 \wedge z_4 \wedge z_5)$

$v_2$: red $(1, 0, 1 \ldots, 1)$, $T_R$ satisfies all red vertices

$T_B$ " " blue "

$T_G$ " " green "

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$e = (i, j)$ different colours

if colour i is red, $T_B$ & $T_G$ don't satisfy $e_{ij}$

because j is not red, $T_R$ also doesn't satisfy $e_{ij}$

None of the 3-terms $T_B, T_G, T_R$ can satisfy the example that we constructed for edge $e = (i,j)$

$\varphi = T_R \vee T_B \vee T_G$ correctly has all +ve examples satisfying $\varphi$

& none of the negative examples satisfying $\varphi$.

$(\underline{\Leftarrow})$ Suppose there exists a 3-term DNF formula that is consistent with the sample, then G must be 3-colourable.

$T_R \vee T_B \vee T_G$

For a vertex $v_i$: if $T_R$ satisfies $(1,1,1\ldots,\underset{i^{th}}{0}\cdots1)$ colour $v$ red

else if $T_B$ " " colour $v$ blue

else colour $v$ green.

Suppose $e=(i,j)$ where both endpoints colour $c \in \{R,B,G\}$

$(1,\ldots,1,\underset{i^{th}}{0},1\ldots1)$ satisfies $\underline{T_c}$ $\Rightarrow z_i \notin T_c$, $\bar{z}_k \notin T_c$ for all $k \neq i$

$(1,1\ldots,1\ \underset{j^{th}}{0}\ 1\ldots1)$ " $T_c$ $\Rightarrow z_j \notin T_c$ $\bar{z}_k \notin T_c$ for $k \neq j$

$\Rightarrow$ No $\bar{z}_k$ appears in $T_c$

$z_i, z_j$ don't appear in $T_c$

$(111\ \underset{i}{0}11\ \underset{j}{0}111)$ satisfy $T_c$ and can't have a negative label.

$\blacksquare$

Distributive Law:

$(a \wedge b) \vee (c \wedge d) \equiv (a \vee c) \wedge (a \vee d) \wedge (b \vee c) \wedge (b \vee d)$

$$\varphi \equiv T_1 \vee T_2 \vee T_3 \equiv \bigwedge_{\substack{l_1 \in T_1 \\ l_2 \in T_2 \\ l_3 \in T_3}} \underbrace{(l_1 \vee l_2 \vee l_3)}_{\text{clause/ disjunction}} \equiv \psi$$

$T_i$: conjunctions/term

3-term DNF formula

3-CNF formula
formula in conjunctive normal form, with every clause having at most literals.

3-TERM-FORMULA
$\varphi \in$

3-CNF (concept class)
$\ni \psi$

$\varphi$ & $\psi$ represent exactly the same boolean function.

Q: Is 3-CNF learnable?

$EX(\varphi, D) \equiv EX(\psi, D)$

Output hypothesis is 3-CNF formula

$h$, $err(h) = \mathbb{P}_{x \sim D}[\varphi(x) \neq h(x)] \leq \varepsilon$

<u>Hypothesis Class</u>: $H = \bigcup_{n \geq 1} H_n$, $H_n$ is a class of boolean functions defined over $X_n$.

<u>Def</u>: <u>Polynomially evaluatable hypothesis class</u>. We say that it is polynomially evaluatable, if there exists an algorithm, that ~~$\forall$~~, $\forall x \in X_n$, $\forall h \in H_n$, when given $x$ and a representation of $h$, outputs $h(x)$ in time that is polynomial in $|x|$, $size(h)$.

<u>PAC Learning</u>: For $n \geq 1$, let $C_n$ be a concept class over $X_n$. Let $C = \bigcup_{n \geq 1} C_n$, $X = \bigcup_{n \geq 1} X_n$. We say that $C$ is $\boxed{\text{efficiently}}$ PAC learnable if $\exists$ learning algorithm $L$ & a polynomially evaluatable hypothesis class $H$, s.t. $\forall n \geq 1$, $\forall c \in C_n$, $\forall D$ over $X_n$, $\forall 0 < \varepsilon < \frac{1}{2}$, $\forall 0 < \delta < \frac{1}{2}$, if $L$ is given access to $EX(c, D)$ and inputs $size(c), \varepsilon, \delta$, it outputs w.p. $\geq 1 - \delta$, $h \in H_n$, s.t. $err(h) \leq \varepsilon$. <u>$L$ runs in time polynomial</u> in $n$, $size(c)$, $1/\varepsilon$, $1/\delta$.

$\square$