Problem Sheet 2

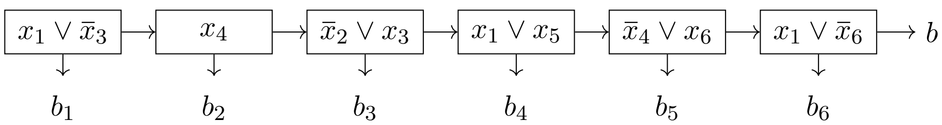
Instructions: The problem sheets are designed to increase your understanding of the material taught in the lectures, as well as to prepare you for the final exam. You should attempt to solve the problems on your own after reading the lecture notes and other posted material, where applicable. Problems marked with an asterisk are optional. Once you have given sufficient thought to a problem, if you are stuck, you are encouraged to discuss with others in the course and with the lecturer during office hours. You are *not permitted* to search for solutions online.

1 Learning Decision Lists

A k-decision list over n boolean variables $x_1, \ldots, x_n$, is defined by an ordered list

$$L = (c_1, b_1), (c_2, b_2), \ldots, (c_l, b_l),$$

and a bit b, where each c_i is a clause (disjunction) of at most k literals (positive or negative) and each $b_i \in \{0, 1\}$. For $a \in \{0, 1\}^n$ the value $L(a)$ is defined to be b_j, where j is the smallest index satisfying $c_j(a) = 1$ and $L(a) = b$ if no such index exists. Pictorially, a decision list can be depicted as shown below. As we move from left to right, the first time a clause is satisfied, the corresponding b_j is output, if none of the clauses is satisfied the default bit b is output.



Give an *efficient* consistent learner for the class of decision lists. As a first step, argue that it is enough to just consider the case where all the clauses have length 1, *i.e.,* in fact they are just literals.

2 Learning Convex Sets in $[0, 1]^2$

In this question we will consider the learnability of convex sets. Let us consider the domain to be $X = [0, 1]^2$, the unit square in the plane. For $S \subset X$ a convex set, let $c_S : X \to \{0, 1\}$, where $c_S(x) = 1$ if $x \in S$ and 0 otherwise. Let $C = \{c_S \mid S \text{ convex subset of } X\}$ be the concept class defined by convex sets of $[0, 1]^2$.

1. Show that the VC dimension of C is ∞. This shows that the concept class of convex sets of $[0, 1]^2$ is not PAC-learnable (efficiently or otherwise).

2. We will consider a restriction of PAC-learning where the learning algorithm is only required to work for a specific distribution D over X. Show that if D is the uniform distribution over $[0,1]^2$, then the concept class of convex sets is PAC-learnable. (*Hint*: Consider the algorithm that simply outputs the convex hull of positive points as the output hypothesis. You may use the fact that the perimeter of any convex set in the unit square can be at most 4.)

3 Growth Function

Prove that for any $d \in \mathbb{N}$, there is a concept class C such that $\mathsf{VCD}(C) = d$, and that for any $m \in \mathbb{N}$, $\Pi_C(m) = \Phi_d(m)$.

4 VC Dimension of Linear Halfspaces in $\mathbb{R}^n$

We will show that the concept class of linear halfspaces in $\mathbb{R}^n$ has VC-dimension $n+1$.

1. Give a set of $n+1$ points in $\mathbb{R}^n$ that is shattered by the class of linear halfspaces.

2. We want to show that no set of $m = n+2$ points in $\mathbb{R}^n$ can be shattered by the class of linear halfspaces. For this you can use what is called as Radon's theorem, described below.

3.* Prove Radon's theorem.

Given a set $S = \{x_1, \ldots, x_m\} \subset \mathbb{R}^n$, the convex hull of S is the set,

$$\{z \in \mathbb{R}^n \mid \exists \lambda_1, \ldots, \lambda_m \in [0,1], \sum_{i=1}^{m} \lambda_i = 1, z = \sum_{i=1}^{m} \lambda_i x_i\}$$

Radon's Theorem: Let $m \geq n+2$, then S must have two disjoint subsets S_1 and S_2 whose convex hulls intersect.